O.R. Applications

Innovation diffusion uncertainty, advertising and pricing policies

Bardia Kamrad a,*, Shreevardhan S. Lele b,1, Akhtar Siddique c,2, Robert J. Thomas a

a McDonough School of Business, Georgetown University, 37th and O. Street N.W., Washington, DC 20057, USA
b R.H. Smith School of Business and Management, University of Maryland, College Park, MD 20742-1815, USA
c Office of the Comptroller of the Currency, Risk Analysis Division, Mail Stop 2-1, 250 E Street SW, Washington, DC 20219, USA

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Abstract

We develop and analyze a normative and structurally stochastic model of innovation diffusion by depicting the market at an aggregate level. Model dynamics are defined through the flow pattern of individuals that move from the innovation unaware stage, to the innovation aware, and ultimately to the adopter stages. The stochastic evolution of this stage-wise transition unfolds according to tractable stochastic processes and is influenced by such factors as price, word of mouth, and advertisement efforts. In this environment, techniques of contingent claims analysis and stochastic control theory are employed to obtain optimal pricing or advertising policies that maximize the value of the innovation. To account for their optimal adjustment over time, these policies are modeled as positive real-valued adapted processes. Given this setting, policy adjustments over time (i.e. advertising or pricing) are viewed as a value additive sequence of nested real options. We present closed-form analytic results regarding the optimal policies. Simulations provide a numeric insight to the models' behavior.

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1. Introduction

Uncertainty in the models of innovation diffusion may be integrated in the model's parameters, its structure, or both. The resulting models may reflect a monopolistic or competitive market condition, depict
the market at the aggregate or disaggregate level and, be used for predictive or normative purposes. 3 In this paper, we assume a market monopoly and develop a normative and structurally stochastic innovation diffusion model. In this context, model uncertainty is portrayed through Ito processes characterizing the stochastic dynamics of the underlying transition process from the innovation unaware to the adopter stages. These dynamics reflect the random changes in the population size of potential adopters whose act of purchasing is triggered by such factors as advertising, price, word-of-mouth effect, and sales promotion. As such, adjustment to advertising expenditure rates are viewed as a sequence of nested real options that provide flexibility value. We derive a model for an optimal sequence of expenditure rate adjustments over a pre-established time horizon, by modeling expenditure rates as an adapted positive real-valued process. Accordingly, we use techniques of stochastic control theory and contingent claims analysis (CCA) to obtain value-maximizing policies.

Our approach extends the stochastic development of market diffusion process models in a number of ways. The optimization framework adapted here is similar to the Raman and Chatterjee (1995) model. Given a monopolistic setting, risk neutrality, and Wiener processes to depict uncertainty in (cumulative) sales, they obtain a dynamic model of optimal price policies. This approach to establishing optimal price adjustments is also considered in our analysis as a special case. In addition, and similar to Muller (1983) and Dodson and Muller (1978), in our approach we depict the total market as an aggregate of subpopulations or segments involving the “unaware”, “aware”, and “adopter” stages. In light of this characterization, we allow advertising, price, and word-of-mouth to affect the transition of individuals from one segment to the next according to an Ito (Wiener) process. That is, we approximate the stage-wise transition of individuals from the “unaware” to the “aware”, and to the “adopter” stages by diffusion (Wiener) processes.

In our model, the inclusion of price as a variable that influences the stage-wise transition of adopters parallels the models of Kalish (1983) and Raman and Chatterjee (1995). In particular, innovation-aware individuals can be price sensitive in their adoption process. In a comprehensive review of stochastic models for innovation, Eliashberg and Chatterjee (1986) recognize that numerous sources of uncertainty affect new product diffusion and emphasize the need for related research. They note, however, that the literature on innovation diffusion models in marketing has ignored stochastic considerations. In a review of dynamic optimal control models for advertising, Feichtinger et al. (1994) also indicate that of the existing cumulative sales or market growth models, none includes stochastic effects. They also reveal that none of the models with more than one state variable in the advertising process are stochastic. In general, most models of market innovation follow the deterministic diffusion model formulation of Bass (1969). 4 While the literature including the work of Dodson and Muller’s (1978) and Muller’s (1983) offers important intuitive insight to the stage-wise progression of individuals in their adoption process, it ignores some important facets of market diffusion. First, is the total neglect of uncertainty in the structure of the adoption process. Second, it omits the word-of-mouth effect as an important link between the unaware, the aware, and the adopter segments. Consequently, Mahajan et al. (1984) extend the Dodson and Muller (1978) model by including the effects of positive and negative word of mouth. 5

In this paper, by explicitly incorporating the aforementioned factors we extend the stochastic development of market diffusion process models in a number of ways. In our setup, the total market is taken as

3 For related literature see for instance Eliashberg and Chatterjee (1986); Eliashberg et al. (1985); Jain et al. (1991); Mahajan et al. (1984); Albright and Winston (1979); Deshmukh and Winston (1977); Monahan’s (1984); Tapiero (1983) and Holthausen and Assmus (1982).


5 An empirical test of their model (on movie attendance) predicted awareness well, but not actual attendance. This was attributed to external factors, thereby revealing in part a weakness of such deterministic models and an argument in favor of modeling uncertainty in the structure of innovation diffusion processes.
an aggregate of subpopulations or segments involving the unaware, aware, and adopter stages. For transition between these stages, we allow word-of-mouth, price and advertising efforts to affect the movement of individuals from one stage to the next. In our approach, transitions from the unaware to aware and ultimately to adopter stages are approximated by Wiener processes and are influenced by price, word-of-mouth and advertising. Given this setting, and by assuming that the innovator is a monopolist, techniques of contingent claims analysis and stochastic control theory are employed to obtain optimal advertising or pricing policies that maximize the value of the innovation. In our approach, policy adjustments over time (i.e. advertising or pricing) are viewed as a value additive sequence of nested real options. To that end, advertising and pricing policies are modeled as positive real-valued adapted processes so that their optimal adjustments over time maximize the value of the innovation. Generalizations of the model produce results for population (or segment) specific advertising with advertising and sales promotion as control variables. In all cases, normative results obtained from the model are consistent with innovative behavior. In addition to normative results, model simulations provide further results and insight to the behavior of the control variables over time. For instance, how the innovation’s value changes as a function of advertising expenditures for a given price and the how the innovation’s value changes as a function of price for a given level of advertising effort.

This paper is organized as follows. In the next section we provide the necessary assumptions and introduce a framework for modeling purposes. In Section 3, by defining the cash flow process, we develop the model as a stochastic control problem, manifest as a Bellman valuation/optimization equation. In Sections 4 and 5, optimal policies to the problem are obtained in closed form. As a special case, we also provide the concurrent optimal price and advertising policies. Model simulation and simulation established pricing and advertising results along with the innovations value as a function of simulated policies are shown in Section 6. The notion of segment specific advertising, intended for awareness and sales promotion respectively, is considered in Section 7. Here too, and in the context of the definitions considered, optimal policies are rendered in closed form and their economic implications discussed. Additional findings and insights are provided and reviewed in Section 8. Section 9 concludes the paper with a few closing remarks.

2. Assumptions and model framework

Our model is developed in a continuous time framework. Let the total market for the innovation be \( N^* \), a quantity that is assumed to remain constant across the pre-established advertising horizon, \( [0, T] \). Similar to models by Urban (1970); Dodson and Muller (1978) and Muller (1983) we also assume that the total market at time \( t \in [0, T] \) is comprised of three mutually exclusive and collectively exhaustive segments. The unaware segment, \( U(t) \) represents individuals who are unaware of the innovation. This defines an “untapped” market for the firm. The remaining \( N^* - U(t) \) individuals constitute the aware segment. This (aware) segment is comprised of individuals who have not purchased the product as of time \( t \), and those who have purchased (adopted) the innovation by time \( t \). The former segment in this population is referred to as the “potential” market while the latter is labeled the “current” market. At any time \( t \in [0, T] \), their corresponding size is depicted by \( N(t) \) and \( Q(t) \), respectively. We note that each segment is bounded: each segment, at various points in time, can be bounded below by zero and above by, \( N^* \). In our analysis, the potential market is considered as a contingent asset and thereby a potential source of future value to the firm. On the other hand, the current market’s value to the firm arises from the cash flows generated by the innovation’s sales and through a “word-of-mouth” effect exerted by those who have purchased the product on the aware population.

The stage-wise transition begins at the untapped market and terminates at the current market, as depicted in Fig. 1. At time \( t = 0 \), all customers are in the unaware segment so that, \( U(0) = N^* \). We assume that the rate of flow from this segment into the next segment (i.e. potential market) is influenced by a word-of-mouth
effect as well as an advertising effect. Since the potential market evolves stochastically over time, a number of individuals proceed with their purchases and move into the purchasers’ pool. This transition is also affected by the firm’s advertising efforts as well as a word-of-mouth effect. In addition, we also allow for the price sensitivity of the potential purchasers. Hence, over time, there is a continuous flow of potential customers from $U(t)$ to $N(t)$, and concurrently from $N(t)$ to $Q(t)$, $t \in [0, T]$. This stage-wise transition is triggered by such market mechanisms as price and both the internal (i.e. word-of-mouth) and external (promotion) influence factors. We regulate this transition by imposing two constraints. First, we preclude impulse purchasing in our model by disallowing direct transition from the unaware (or untapped) into the purchased (or current) segments. Second, by preventing repeat purchases, at least within the time horizon considered. That is, we close off the possibility of a transition from the current back to the potential market. These constraints can be relaxed at the expense of added complexity. A schematic representation of the transition dynamics among the stages comprising $N^*$ is shown in Fig. 2.

Given the above, for the three mutually exclusive and collectively exhaustive subsets of $N^*$ it must be that for all $t \in [0, T]$,

$$N^* = U(t) + N(t) + Q(t)$$

Fig. 1. Basic model framework with three population segments.

To allow for uncertainty in the transition process among the three pools, we posit that each of the three stages behaves as a continuous time Markov process belonging to the class of Ito processes (also known as generalized Wiener processes). We model the instantaneous increments, at time $t$, of each of the three subsets of $N^*$ as:  

$$dU(t) = \alpha_U(U, N, Q, t) dt + \sigma_U(U, N, Q, t) dW_U(t),$$

$$dN(t) = \alpha_N(U, N, Q, t) dt + \sigma(U, N, Q, t) dW_N(t),$$

There is a non-zero probability that $dU(t) > 0$ and $dQ(t) < 0$ in our modeling choices. Yet, we choose the drifts, $\alpha_U$ and $\alpha_Q$ large enough (relative to the diffusion coefficients) so that those probabilities are negligibly small. To model, $U(t)$ and $Q(t)$, in our setting, as strictly decreasing and strictly increasing processes would lend our modeling intractable.
\[
\begin{align*}
\text{d}Q(t) &= \alpha_{Q}(U, N, Q, t) \text{d}t + \sigma_{Q}(U, N, Q, t) \text{d}W_{Q}(t). \\
\end{align*}
\] (5)

In the above expressions, \( \alpha_{i}(.) \) and \( \sigma_{i}(.) \) denote the instantaneous drift and volatility functions of the respective processes, \( i = U, N, Q \). In addition, \( \text{d}W_{i}(t) \) is an increment to the standard Brownian motion, \( W_{i}(t) \) which may be viewed as a Gaussian random variable in the limit form. \footnote{In defining a Brownian motion consider an arbitrary partition, \( t_{0} < t_{1} < t_{2} < \cdots \) of the interval \( [0, \infty) \). The random variables \( W(t_{k}) - W(t_{k-1}) \) are independent normal with zero mean and variance, \( (t_{k} - t_{k-1}), k = 1, 2, \ldots \).}

In particular,
\[
\text{d}W_{i}(t) = \lim_{\Delta t \to 0} \sqrt{\Delta t} \tilde{Z}_{i}, \quad i = U, Q, N;
\] (6)

where \( \tilde{Z}_{i} \) is a standard normal deviate. Of further interest is the correlation structure to the above Brownian increments. Moreover, by assuming various functional forms for \( \alpha_{i}(.) \) and \( \sigma_{i}(.) \), we can capture a rich set of behavioral and qualitative properties that are reflective of the true sales diffusion dynamics.

Let \( A(t) \) be the advertising expenditure rate at time \( t \), so that the amount spent on advertising over the interval \( (t, t + \text{d}t) \) is \( A(t) \text{d}t \). The market price of the innovation \( P(t) \) is assumed to be known over the advertising time horizon of \( [0, T] \). Furthermore, we assume the innovating firm is a monopolist. Our model will consider both the “awareness-specific” optimal advertising policy case, as well as “awareness-specific” and “purchase-specific” advertising policies case. Awareness-specific advertising targets the unaware individuals, and in that sense helps their transition into the aware segment. The purchase-specific advertising is an attempt to move the aware individual to the adopter stage. As a special case, we will also establish optimal prices to be followed when considering “awareness-specific” advertising only. In all, our approach defines the advertising expenditures, \( A = \{A(t) : t \in [0, T]\} \) and the price \( P = \{P(t) : t \in [0, T]\} \) as an adapted positive real-valued process. As such, the firm seeks an optimally established advertising (and/or price) policy that maximizes the value of its innovation. Of particular concern is the efficacy of advertising in making aware the unaware, and in influencing the aware to become a purchaser. Also, the sales response of the innovation aware individuals to the market price is also of interest. Let the efficacy of the advertising
effort in inducing awareness in an unaware individual be given by the response function, \( F(A(t)) \). Similarly, the efficacy of advertising toward inducing a sale is given by the response function, \( H(A(t)) \). That is, the potential purchasers’ response to advertising is characterized by \( H(\cdot) \). \(^8\) Concurrently, potential customers respond to the innovation’s price in a manner quantifiable by the function, \( G(P) \). The function, \( G(P) \) defines the demand function. As such, the drift or expected growth rates of the unaware segment, \( U(t) \) and the purchasers’ segment, \( Q(t) \) are defined by

\[
\dot{z}_U(U, N, Q, t) = -\mu_1 \frac{U(t)}{N^*} (N^* - U(t)) - \mu_2 U(t) F(A(t)),
\]

\[
\dot{z}_Q(U, N, Q, t) = \lambda_1 Q(t) \frac{N(t)}{N^*} G(P) + \lambda_2 N(t) H(A(t)) G(P).
\]

At time \( t \in (0, T] \) the size of the aware segment is \( (N^* - U(t)) \). Each individual in this segment communicates (word-of-mouth) their awareness with \( \mu_1 \) individuals on an average basis over the interval \( [t, t + dt] \). Of the \( \mu_1 (N^* - U(t)) \) individuals who are contacted only a fraction, \( U(t)/N^* \) belong to the unaware pool. Thus, the average number of individuals who have moved from the unaware to the aware segment, over time \( [t, t + dt] \), due to a word-of-mouth effect is \( \mu_1 (U(t)/N^*) (N^* - U(t)) \). We refer henceforth, to \( \mu_1 \) as the coefficient of external effects to signify transitions that emanate out of the unaware pool.

Among the unaware individuals \( U(t) \), a fraction \( \mu_2 U(t) \) are exposed to advertising efforts over the time interval \( [t, t + dt] \). Thereby, the average number of unaware individuals who become aware due to advertising over \( [t, t + dt] \) is given by \( \mu_2 U(t) F(A(t)) \). Here the parameter, \( \mu_2 \) is defined as the coefficient of external effects characterizing transitions out of the unaware pool due to advertising. As is customary for this literature, we assume the response function, \( F(A(t)) \) is increasing monotone and strictly concave in \( A(t) \): \( F'(A) > 0 \) and \( F''(A) < 0 \). Eq. (8) can be described analogously. Each of the \( Q(t) \) customers who have purchased the innovation by time \( t \in (0, T] \) communicates, on an average basis, with \( \lambda_1 \) individuals over time \( [t, t + dt] \) to induce a purchase. Of \( \lambda_1 Q(t) \) individuals, only a fraction, \( (N(t)/N^*) \) are in the aware pool. Therefore, the average number of potential customers affected through a word-of-mouth effect is \( \lambda_1 Q(t) (N(t)/N^*) G(P) \). As such, \( \lambda_1 \) is also a coefficient for the internal effects but only for the class of transitions that emanate form the aware segment and terminate into the purchased state. We make the assumption that the sales response to price, \( G(P) \) is a decreasing function in price so that \( G'(P) < 0 \). Furthermore, of the potential purchasers \( N(t) \), only a fraction, \( \lambda_2 N(t) \) are exposed to advertising whose sales-inducing efficacy is given by the response function, \( H(A(t)) \). Using a sales response function that is multiplicative in the effect of advertising \( H(A(t)) \) and price response \( G(P) \), the average number of sales generated over an instant \( [t, t + dt] \) is given by \( \lambda_2 N(t) H(A(t)) G(P) \). In the current context, the parameter \( \lambda_2 \) signifies the coefficient of external effects for transitions that terminate into the pool of purchasers. We also assume that \( H(A(t)) \) is increasing and strictly concave in the advertising effort, \( A(t) \) so that \( H'(A) > 0 \) and \( H''(A) < 0 \). In establishing an optimal advertising policy and for the purposes of analysis, we impose an additional constraint as a sufficient (but not necessary) condition for characterizing the policy. In particular,

\[
\frac{d}{dA(t)} \left( \frac{H'(A(t))}{F'(A(t))} \right) \geq 0,
\]

\(^8\) The distinction between the two function \( F(A(t)) \) and \( H(A(t)) \) is important. The advertising efficacy function, \( H(\cdot) \) measures the impact of advertising expenditures on sales, whereas, the function \( F(\cdot) \) is a gauge for the impact of advertising on the awareness level of the untapped market.
implying that the marginal sales response is to diminish at a relatively slower rate than the marginal awareness response. Observe too, that the above expression is the ratio of the marginal sales efficacy of advertising to the marginal awareness efficacy of advertising.

To characterize the diffusion dynamics, we define the volatility functions for \( dU(t) \) and \( dQ(t) \) by

\[
\sigma_U(U, N, Q, t) = \sigma_U U(t) N(t),
\]

\[
\sigma_Q(U, N, Q, t) = \sigma_Q Q(t) N(t)
\]

with both \( \sigma_U \) and \( \sigma_Q \) as constants. In effect, the above Eq. (10) signifies the uncertainty governing the transitions from \( U(t) \) into \( N(t) \). Similarly, \( \sigma_Q \) reflects transition uncertainty about \( N(t) \) to \( Q(t) \). Implicit is the supposition that this uncertainty is proportional to the size of each of the two segments involved in the transition: if either segment is empty, then no uncertainty exists. Also, as remarked earlier, each segment is bounded. That is, both \( U(t) \) and \( Q(t) \), at various points in time, are bounded below by zero and above by \( N^\ast \). While expressions (10) and (11) may be sufficient in that regard, their corresponding segments have to be explicitly curtailed as indicated. Note too, that the above expressions indicating our choice of the volatility functions, prohibit \( N(t) \) from becoming negative. In our setup, as \( N(t) \) approaches zero, both \( \sigma_U \) and \( \sigma_Q \) vanish: implying that when there are no aware customers, no buyers are generated. We assume further that the standard Brownian increments \( dW_U(t) \) and \( dW_Q(t) \) are instantaneously uncorrelated. In particular,

\[
E(dW_U(t) \cdot dW_Q(t)) = 0.
\]

The plausibility of the above assumption simply stems from the existence of a “buffer stage”, namely the pool of aware potential customers, \( N(t) \) that “indirectly” links \( U(t) \) and \( Q(t) \) through the mechanisms explained earlier. This intermediate stage essentially blocks any uncertainty at the \( U(t) - N(t) \) boundary from having a contemporaneous effect on the uncertainty at the \( N(t) - Q(t) \) boundary. It must be stressed that we are only assuming that the standard Brownian increments \( dW_U(t) \) and \( dW_Q(t) \) are uncorrelated. This, however does not in any way imply that the pools \( U(t) \) and \( Q(t) \) are uncorrelated. Indeed, the dynamics of the sales diffusion as characterized by Eqs. (7) and (8) guarantee that the unaware segment is negatively correlated with the purchasers’. In light of Eqs. (7)–(11), the sales diffusion dynamics can now be written as:

\[
dU(t) = -\left\{ \frac{\mu_1 U(t)}{N^\ast} (N^\ast - U(t)) + \mu_2 U(t) F(A(t)) \right\} dt + \frac{1}{N^\ast} [\sigma_U U(t) N(t) dW_U(t)],
\]

\[
dQ(t) = \left\{ \lambda_1 Q(t) \frac{N(t)}{N^\ast} G(P) + \lambda_2 N(t) H(A(t)) G(P) \right\} dt + \frac{1}{N^\ast} [\sigma_Q Q(t) N(t) dW_Q(t)].
\]

It follows from Eq. (2) that

\[
dN(t) = \left\{ \frac{\mu_1 U(t)}{N^\ast} (N^\ast - U(t)) + \mu_2 U(t) F(A(t)) - \lambda_1 Q(t) \frac{N(t)}{N^\ast} G(P) - \lambda_2 N(t) H(A(t)) G(P) \right\} dt
\]

\[
- \frac{1}{N^\ast} [\sigma_U U(t) N(t) dW_U(t) + \sigma_Q Q(t) N(t) dW_Q(t)].
\]

Given the above specifications, it follows that if \( N(t) \) is small, then \( U(t) \) is negligibly small. As a consequence of assuming that the Brownian increments \( dW_U(t) \) and \( dW_Q(t) \) are instantaneously uncorrelated (i.e. Eq. (12)), it follows that increments to the segment of unaware, \( dU(t) \), and the segment of purchasers, \( dQ(t) \), are also uncorrelated. Yet, it also follows directly from Eqs. (13)–(15) that \( dN(t) \) and \( dU(t) \) as well as \( dN(t) \) and \( dQ(t) \) are negatively correlated. That is,
This confirms our earlier postulate that the increments to any two adjacent pools must be negatively correlated as an increase in one pool reflects a decrease in the other.

Result 1. Let \( \rho_{i,j} \) be the instantaneous correlation between any two adjacent population increments. Then,

\[
\rho_{\Delta U, \Delta N}^2 + \rho_{\Delta N, \Delta Q}^2 = 1. \tag{17}
\]

Proof. See Appendix A.

Eqs. (1)–(17) provide the basic premise for developing the model. Our objective is to establish an advertising policy over the period \([0, T]\) so as to maximize the value of the innovation to the firm.

3. The model

In this section we develop the valuation model within a stochastic control framework. To that end, let \( C(t) \) define the net cash flow rate; \( w \) to depict the marginal cost of a unit of the product; and \( A(t) \) the advertising expenditure rate. Accordingly, the cash accrued to the firm over an instantaneous time period, \([t, t + dt]\), is given by

\[
C(t) \, dt = (P - w)Q(t) - A(t) \, dt. \tag{18}
\]

That is, the marginal profit resulting from selling \( dQ(t) \) units of the product less the advertising expenditures. Therefore, using the drift function in Eq. (8) we may write:

\[
E_\psi(C(t)) = (P - w)\pi(t) - A(t). \tag{19}
\]

Upon commercialization, the firm receives an expected cash flow over the time interval \([0, T]\). Let \( R(U, Q, t; \psi) \) represent the expected value of the innovation at time \( t \) given an advertising policy \( \psi \). Define \( \pi(\psi) \) to be the set of admissible control policies. The time \( t \) value of the innovation is the expected net present value of the accrued profits. Therefore, conditional on a predetermined \( \psi \in \pi(\psi) \) where \( \psi = \{A(t') : t' \in [t, T]\} \)

\[
R(U, Q, t; \psi) = E_\psi\left( \int_t^T C(t')e^{-\gamma t'} \, dt' \right). \tag{20}
\]

In the above equation, \( \gamma \) represents the discount rate that is appropriately set with respect to the firm’s preference (or aversion) toward risk. It also may be the case that \( \gamma \) is action dependent so that \( \gamma = \gamma(A(t)) \).

To obtain \( V(0) \), let the Ito differentiable function \( V(U, Q, t) \) be the maximum expected value of the innovation given that the time \( t \) the unaware pool is \( U(t) \), the sales level is \( Q(t) \) and the advertising expenditure rate over the period \([t, T]\) is set at \( A(t) \). In particular, we have

\[
V(U, Q, t) = \max_{\psi} (R(U, Q, t; \psi)) \quad \text{where} \quad \psi = \{A(t') : t' \in [t, T]\}, \tag{21}
\]
subject to Eqs. (13) and (14) and appropriate boundary conditions. It follows from Ito’s lemma that the change in the value function is also an Ito process. That is,

$$\frac{dV(t)}{V(t)} = \alpha \, dt + S_U \, dW_U(t) + S_Q \, dW_Q(t)$$

(22)

with

$$\alpha = \frac{1}{V(t)} \left\{ \frac{\partial V}{\partial U} \alpha_U + \frac{\partial V}{\partial Q} \alpha_Q + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial U^2} \sigma_U^2 + \frac{1}{2} \frac{\partial^2 V}{\partial Q^2} \sigma_Q^2 \right\},$$

(23)

$$S_U = \frac{1}{V(t)} \sigma_U \frac{\partial V}{\partial U},$$

(24)

$$S_Q = \frac{1}{V(t)} \sigma_Q \frac{\partial V}{\partial Q}.$$  

(25)

Effectively, in the above expression, $\alpha$ defines the expected rate of change in the value of the innovation or simply the drift component of the value process, $dV(t)$. Eq. (22) represents a geometric Ito process with drift component $\alpha$. It’s volatility is decomposed into two distinct Brownian components. These components, given by $S_U$ and $S_Q$, reflect the risk of the innovation in terms of the sources of uncertainty. The expected total rate of return on the innovation must reflect the capital gains accrued from the commercialization of the innovation and is given by $\alpha + E[(C(t))]/V(t)^{-1}$.

Given Merton’s (1973) intertemporal capital asset pricing model (ICAPM), the equilibrium excess return on any asset in the economy is proportional to the covariance of that asset’s returns with economy wide returns. Consider the time $t=0$ set of options available to the innovating firm. It can choose to market its innovation and capture the resulting uncertain cash flows from its sales. Or, the firm may opt to forego the particular opportunity and otherwise invest in “other” market projects. The sum total of all “other” market projects can be represented by a financial market index. Such an index reflects foregone opportunities when the innovating firm selects to commercialize its innovation. The market index represents an investment option that is also uncertain. In that spirit, let $dZ_m(t)$ depict the uncertainty associated with returns from the market index. Viewed from an ICAPM perspective, the return from commercialization of the innovation is proportional to the covariance between $dV(t)$ and $dZ_m(t)$. In particular, let the instantaneous correlations between the unaware segment and the market and the purchasers’ segment and the market be depicted by

$$dW_U(t) \cdot dZ_m(t) = \rho_{U_m} \, dt,$$

(26)

$$dW_Q(t) \cdot dZ_m(t) = \rho_{Q_m} \, dt.$$  

(27)

Suppose further that the instantaneous expected return and the volatility on the market index are given by $\alpha_m$ and $\sigma_m$, respectively, and that the constant rate of return from a riskless asset is denoted by $r$. The market price of risk is given by

$$\theta = \frac{\alpha_m - r}{\sigma_m}.$$  

(28)

The ICAPM restriction implies that in equilibrium,

$$\alpha + E[(C(t))]/V(t) = r + \theta(S_U \rho_{U_m} + S_Q \rho_{Q_m}).$$

(29)
That is, the total expected return (as given by the LHS) in excess of the risk free rate of return is a linear combination of the correlations of the sources of uncertainty with that of the market. \(^9\) It should be noted that for a risk neutral firm the right hand side of Eq. (29) will include \(r\) as the only term. Substituting Eq. (23)–(25) into (29) obtains the “fundamental” valuation equation

\[
\frac{\partial V}{\partial U} (\sigma_U - \theta \rho_{Uq} \sigma_U) + \frac{\partial V}{\partial Q} (\sigma_Q - \theta \rho_{Qq} \sigma_Q) + \frac{\partial V}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 V}{\partial Q^2} \sigma_Q^2 + \frac{\partial^2 V}{\partial U^2} \sigma_U^2 \right) + (P - w) \sigma_Q - A(t) - rV = 0,
\]

which governs the innovation’s value as a function of the state and control variables. The valuation equation is fully specified once the parametric functions, as delineated by Eqs. (7)–(11), are substituted for. In light of such specification, it follows without loss of generality that the optimal advertising policy, \(\{A^*(t); t \in [0, T]\}\) is obtained from the following Bellman optimization equation, where \(A\) denotes the maximum feasible rate of advertising expenditures:

\[
\begin{align*}
\text{Max}_{A(t)} \left\{ \frac{\partial V}{\partial U} \left( -\mu_1 \frac{U}{N^*} (N^* - U) - \mu_2 U F(A(t)) - \theta \rho_{Uq} \sigma_U \frac{N}{N^*} \right) + \frac{1}{2} \frac{\partial^2 V}{\partial U^2} \sigma_U^2 \left( \frac{N}{N^*} \right)^2 
+ \frac{\partial V}{\partial Q} \left\{ \lambda_1 Q \frac{N}{N^*} G(P) + \lambda_2 NH(A(t))G(P) - \theta \rho_{Qq} \sigma_Q \frac{Q}{N^*} \right\} + \frac{1}{2} \frac{\partial^2 V}{\partial Q^2} \sigma_Q^2 \left( \frac{N}{N^*} \right)^2 
+ \frac{\partial V}{\partial t} + (P - w) \left\{ \lambda_1 Q \frac{N}{N^*} G(P) + \lambda_2 NH(A(t))G(P) \right\} - A(t) - rV \right\} = 0
\end{align*}
\]

with \(A(t) \in [0, \bar{A}]\). The Bellman equation (31), fully characterizes the evolution of \(V(t)\) for all \(t \in [0, T]\). In general, this second order PDE is not amenable to a closed form solution in its state and control variables regardless of the initial or boundary conditions imposed and the choice of response functions \(F(\cdot), H(\cdot)\) or the function \(G(P)\). Yet, rather than attempting to solve the PDE numerically to obtain a value for \(V(\cdot)\), in what follows we provide a set of qualitative results that highlight the structural properties of the optimal solution. To that end, the following conjectures, though not proven formally, are used to justify our findings.

**Conjecture 1**

The shadow price, \(\frac{\partial V(\cdot)}{\partial U(t)} < 0\).

When the unaware pool \(U(t)\) decrease by one unit, all else being constant, the aware pool \(N(t)\) increases by one unit. Since there is more value to an aware individual (potential purchaser) than an unaware one, the value (to-go) function increases. As a unit decrease in \(U(t)\) increases \(V(t)\), conjecture 1 is justifiable.

**Conjecture 2**

The shadow price, \(\frac{\partial V(\cdot)}{\partial Q(t)} > -(P - w)\).

\(^9\) See also Jagpal and Brick (1982) on CAPM, uncertainty and market mix.
Consider the effect of an increase in the “purchased” pool, \( Q(t) \) by one unit. This unit sale results in a profit of \( (P - w) \). If this sale had no long term ramifications, it would reduce the value-to-go-function, \( V(t) \) by exactly the profit amount, i.e. \( \frac{\partial V(.)}{\partial Q(t)} \) would be equal to \( (P - w) \). Yet the long-term or latent impact of each sale, which is due to the word-of-mouth effect, must be considered. Thus, the benefit of a sale should exceed the accrued short-term profit. Accordingly, the reduction in the value-to-go is less than the immediate profit from a sale, and therefore conjecture 2 is also plausible.

4. Optimal policies

In establishing optimal policies we consider two specific cases. In the first case, we develop an optimal dual policy for awareness advertising and price setting, concurrently. In the second case, we specialize the findings of Section 3 to reflect on “awareness” as well as “sales” specific advertising. In particular, the model is revised to admit targeted advertising policies. As such, segment specific advertising captures the impact of advertising in making aware the unaware individual, and in enticing an aware individual to make a purchase. In either case, essential inputs to the resulting Bellman optimization equation must be accounted for. These reflect our choice of the response functions, \( F(A(t)) \), \( H(A(t)) \), together with the demand function, \( G(P(t)) \). To that end, our choice of response and demand functions reflect

\[
F(A(t)) = a_0 + a_1A(t) - a_2A(t)^2 \quad \text{or} \quad F(A(t)) = 1 - e^{b_1-A(t)}, \tag{34a,b}
\]

\[
H(A(t)) = b_0 + b_1A(t) - b_2A(t)^2 \quad \text{or} \quad H(A(t)) = 1 - e^{b_2-A(t)}, \tag{34c,d}
\]

\[
G(P(t)) = C_0 - C_1P(t) \quad \text{or} \quad G(P(t)) = kP(t)^{-\eta}. \tag{35a,b}
\]

The constants \( \beta_j \leq 0 \), whereas \( a_i, b_i, C_j, k, \) and \( \eta \) (price elasticity of demand) are positive. The response functions shown above (i.e. 34a–d) indicate diminishing returns in advertising effort: additional expenditure levels increase response at a decreasing rate. The demand functions (35a,b) indicate that an increase in price decreases demand. In the following sections, we draw on the aforementioned conjectures and the Bellman valuation equation to obtain specific results.

5. Awareness advertising and pricing policies

We consider a particular case where advertising efforts are targeted for awareness purposes only. Thus, “awareness specific” advertising expenditures have no direct impact on moving an aware individual into the purchased pool. Our specification obtains both the optimal (awareness) advertising and price policies contemporaneously.

**Theorem 1.** Let the awareness response function be characterized by either a quadratic or an exponential function. The innovation’s value, \( V(.) \) is a concave function of the advertising expenditures, \( A(t), t \in 0, T \).

**Proof.** See Appendix B.

In light of the above theorem, for a quadratic response function as in expression (34a) the optimal awareness advertising policy \( A^*(t), t \in [0, T] \) is

\[
A^*(t) = \text{Max} \left\{ \frac{1}{2} \left( \frac{a_1}{a_2} + \frac{1}{a_2} \frac{1}{\mu_2} \right), 0 \right\}. \tag{36}
\]
For an exponential response function as in (34b) the optimal policy is given by:

$$A^*(t) = \text{Max}\left\{ \left( \beta_1 + \ln \left( -\frac{\partial V}{\partial Q} U \mu_2 \right) \right), 0 \right\}. \quad (37)$$

We remarked earlier that, in this particular case the monopolistic optimal price adjustments over time can be also established. Since advertising has only an awareness component, its influence does not directly impact the \( Q(t) \). As a consequence of the existence of a “buffer” segment, it becomes possible to establish \( P^*(t), t \in [0, T] \) independent of the path that \( A^*(t) \) takes on. As such, given Theorem 1, and irrespective of the functional choice for \( F(A^*(t)) \), we have for Eqs. (35a,b), respectively:

$$P^*(t) = \text{Max}\left\{ \frac{1}{2} \left( w - \frac{\partial V}{\partial Q} + \frac{C_0}{C_1} \right), 0 \right\}, \quad (38a)$$

$$P^*(t) = \text{Max}\left\{ \frac{\eta}{(\eta - 1)} \left( \frac{\partial V}{\partial Q} - w \right), 0 \right\}. \quad (38b)$$

Note too that, Eq. (38a) parallels the findings of Raman and Chatterjee (1995) exactly, while (38b) echoes Kalish (1983) in a deterministic setting. It follows without loss of generality that

**Corollary 1.** For a linear demand function \( G(P(t)) \), the innovation’s value, \( V(.) \) is a concave function of the price level, \( P(t), t \in [0, T] \).

**Proof.** Follows that of Theorem 1.

### 6. Numerically derived optimal policies

An alternative to the analytical solution of the partial differential equation is to compute the expectation of the optimal value directly (i.e. Eq. (20)). To that end, we simulate the processes given by Eqs. (13)–(15) and evaluate Eq. (17). Through a large set of simulations we can approximate the expectation in Eq. (21).10

<table>
<thead>
<tr>
<th>Parameters used for simulation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_1 )</td>
<td>0.005</td>
</tr>
<tr>
<td>( \lambda_1 )</td>
<td>0.500</td>
</tr>
<tr>
<td>( C_0 )</td>
<td>500</td>
</tr>
<tr>
<td>( \mu_2 )</td>
<td>0.002</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
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<tr>
<td>( C_1 )</td>
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<tr>
<td>( \sigma_U )</td>
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<tr>
<td>( P )</td>
<td>1.00</td>
</tr>
<tr>
<td>( k )</td>
<td>400</td>
</tr>
<tr>
<td>( \sigma_Q )</td>
<td>0.050</td>
</tr>
<tr>
<td>( w )</td>
<td>0.200</td>
</tr>
<tr>
<td>( \eta )</td>
<td>3</td>
</tr>
<tr>
<td>( r )</td>
<td>0.010</td>
</tr>
</tbody>
</table>

For the response functions to advertising by the aware and unaware segments (i.e. \( H(A) \) and \( F(A) \)) we use quadratic functional forms with the parameter vectors \( \left( a_0 = 0, a_1 = 0.05, a_2 = -0.02 \right) \) and \( \left( b_0 = 0, b_1 = 0.10, b_2 = -0.02 \right) \). For the demand function \( G(P(t)) \) we use an exponential function with elasticity, \( \eta = 3 \) or a linear demand function with \( C_0 \) equal to 500 and \( C_1 \) equal to 10. We simulate the

10 The approach of using expectations of diffusions to solve partial differential equations is Chapters 8 and 9 of Durrett (1984) (Chapter 8: PDE’s that can be solved by running a Brownian motion and Chapter 9: Stochastic differential equations).
stochastic process for 2000 periods and repeat the simulation 2000 times. A sample path for a single run of the simulation of movements from “unaware” to “aware” to “purchased” is shown in Fig. 3 where the dynamics of the processes are representative of the behavior of the three pools over time. By the end of the 2000 periods, the unaware pool gets exhausted. The maximum in the aware pool (but not yet purchasers) occurs when the purchased pool equals the unaware pool.

Along with the movement in the consumer population we also simulate advertising and pricing policies over the 2000 periods. We can choose from either of two approaches for the advertising/pricing policies. The first is to use “empirically” derived advertising policies such as a fixed advertising rate per period (in effect a “naïve” advertising strategy) or, an advertising expenditure that is a linear function of the size of the unaware population. The linear function approach is one where optimization allows us to obtain optimal $a$ and $b$ in the form, $A(t) = a + bU(t)$. In a similar manner, we can choose a price that is fixed or is allowed to vary based on the size of unaware pool. An alternative to the empirical advertising/pricing policies is to utilize the optimal advertising and pricing policies derived analytically in Eqs. (36)–(38b). Fig. 4 shows the optimal advertising policy derived for the linear demand function and with both quadratic response functions.

The optimal advertising policy shows an interesting variation over time with the highest advertising at the beginning and with the advertising declining with time. Eventually, the optimal advertising expenditure reaches 0 and the movement from ‘unaware’ to ‘aware’ and also from ‘aware’ to ‘purchased’ occurs without any further advertising (through sheer momentum). This implies that, once advertising ceases and the aware and purchased pools are sufficiently large, $\mu_1$ and $\lambda_1$ in Eq. (15) are responsible for movement of the unaware pool to aware and thence on to the purchased pool.

Fig. 5 shows the optimal pricing policy for the linear demand function given that the price is a linear function of the size of the unaware pool remaining. The optimal pricing policy suggests that very low prices
are charged in the very beginning and eventually as the aware pool, $N(t)$, becomes sufficiently large, optimal price stabilizes. As with the previous case, this is intuitively appealing. We can also compute the expected profit as the sample average over the 2000 simulations (with given advertising and pricing policies). We vary advertising policy (keeping the price constant) to assess how different advertising policies affect expected profits. In this manner, we can understand the properties of the optimal advertising policy. This approach also lets us examine how consistent our analytically obtained optimal policies are with the numerical results. In a similar manner, we examine how the expected profit varies as we alter the pricing policy, keeping the advertising policy constant.

To that end, first, we consider an advertising policy with a fixed amount per period (with the proviso that advertising stops if the entire unaware/aware pool is emptied; i.e., has become purchasers). The price is fixed at $1.00 (scaled). As stated before, this can be viewed as a naïve advertising strategy. Fig. 6 shows how the expected profit varies as the constant advertising rate varies. Everything else being equal, greater advertising induces earlier conversion to the purchased pool, translating into higher expected profits. At the same time, once the aware and purchased pools reach a sufficient size, conversion to the purchased pool occurs even without any advertising as shown in Fig. 4. Therefore, the optimal advertising rate is one in which the greater cost of advertising is balanced by the higher expected profits from the earlier conversion while also ensuring that the conversion without advertising (through $\mu_1$ and $\lambda_1$ in Eq. (15)) is utilized optimally. We find that the expected profit is maximized at an advertising rate of $3.00$ per period. Advertising less does not induce enough of the unaware/aware pool to move to the purchaser. On the other hand, advertising more exhausts the entire unaware/aware pool very quickly, before the end of the 2000 periods.

Next, we analyze “non-naïve” advertising policies. In a “non-naïve” advertising policy (such as those obtained in Eqs. (36) and (37)) advertising rate depends on the size of the unaware pool remaining as well as

![Fig. 6. Value function at different advertising rates per period (exponential demand function).](image-url)
the shadow price for advertising, \( \partial V(t) / \partial U(t) \). The solution in Eq. (36) suggests that optimal advertising would be a function of with the coefficient given by \( \left[ \frac{\partial V(t)}{\partial U(t)} \right]^{-1} \). Since \( \mu_2 \) is positive and the shadow price is negative (assumption 1), we should expect the optimal advertising rate to be inversely related to \( \frac{1}{U(t)} \) if the analytical conditions for optimal advertising and pricing policies are in fact satisfied. Therefore, we carry out simulations where we use the functional form \( A(t) = A_0 + A_1 \left( \frac{1}{U(t)} \right) \) and vary \( A_1 \) (the coefficient on \( \frac{1}{U(t)} \)). We find that the expected profit is maximized when \( A_1 = -1.75 \). This suggests that the analytically derived optimal advertising policy is consistent with numerical simulations. This supports our assumptions for the analytically derived \( A^*(t) \). Our next analysis concerns on the impact of varying the pricing policy on the expected profit for both the exponential and demand functions. These results are summarized, respectively, in Figs. 7 and 8. Fig. 7 shows that, for the exponential demand function, increasing price per unit increases the expected profit much greater when the price is low than when it is high, i.e. the elasticity of expected profit with respect to price decreases with the price level. Fig. 7 also suggests that this elasticity decreases with increasing price levels. In other words, the optimal price is bounded. Fig. 8 also shows the same phenomenon of declining impact of increasing price for the linear demand function.

7. Segment specific advertising

We now modify our previously established dynamics to reflect segment specific advertising. In particular, the model is revised to admit targeted advertising policies intending to specifically affect the “unaware” and the “potential” markets, respectively. In that spirit, segment specific advertising captures the impact of
advertising in making aware the unaware individual and in further enticing the aware to become a purchaser. To that end, let $A_U(t)$ and $A_N(t)$ define the “awareness advertising” and the “sales advertising” expenditure rates, respectively. The general notion is illustrated in Fig. 9.

The dual policy $(A_U(t), A_N(t)), t \in [0, T]$ is defined as a positive adapted real-valued process. The maximum feasible rate for advertising expenditure is given by $(A_U, A_N)$. As in the previous case, here too, $V(U, Q; A_U, A_N)$ defines the value function. Furthermore, the impact of each specific advertising expenditure on the corresponding target population is gauged in accordance to the response functions $F(A_U(t))$ and $H(A_N(t))$. Without loss of generality, we can alter the Bellman optimization Eq. (31) to reflect upon the current situation, while allowing the original functional forms to be maintained. That is, $F(A_U(t))$ and/or $H(A_N(t))$ continue to be quadratic and exponential functions. The demand function $G(P(t))$ also maintains its form; however, the price level is no longer viewed as a control variable in obtaining the desired policies. In what follows, the optimal dual advertising policy $A_U^*(t), A_N^*(t), t \in [0, T]$, is obtained for a pre-established level of price and for different functional forms.

**Theorem 2.** Let the sales response function, $H(A_N(t))$ be defined by either a quadratic or an exponential function. The innovation’s value $V(\cdot)$ is a concave function of the sales advertising $A_N(t), t \in [0, T]$.

**Proof.** See Appendix C.

**Corollary 2.** If the awareness response function to advertising, $F(A_U(t))$ is either quadratic or exponential, the resulting value function, $V(\cdot)$ is a concave function of the expenditures $A_U(t), t \in [0, T]$.

**Proof.** Follows that of Theorem 2.

We provide some economic insight to our findings. Advertising has two concurrent effects: It induces the unaware individual to become aware, and the aware customers to buy the product. Over the time interval $[t, t + dt]$, the number of unaware individuals who become aware due to advertising (as opposed to those who learn about the product through word-of-mouth or other means) is given by $Q(t)$, where $Q(t)$ is the number of unaware individuals who become aware due to advertising. The internal or word-of-mouth effect of aware on the unaware pulls individuals into the potential market. The combined effect of price and advertising pushes the aware individuals into the adopter stage. The word of mouth effect of adopters pulls in the innovation-aware individuals. The external or the advertising effect on the untapped market pushes individuals into the potential market. The internal or word-of-mouth effect of aware on the unaware pulls individuals into the potential market.

**Fig. 9.** Model framework with awareness advertising and sales (promotion) advertising.
who become aware due to word-of-mouth) is \( \mu_2 UF(A_U) \). Hence, the marginal decrease in the unaware pool, \( U(t) \), (as a result of expending a marginal dollar in advertising) is \( \mu_2 UF'(A_U) \). A decrease in \( U(t) \) by one unit increases the value-to-go by its corresponding “shadow value”, \( -\partial V/\partial U \). Thus, the marginal increase in value that can be attributed to the awareness inducing aspect of advertising is \( -\partial V/U \mu_2 UF'(A_U) \). Next, (recall from Section 2 that) the number of sales made over the time \( |t, t + \delta t| \) that can be attributed to advertising is \( \lambda_2 NG(P)H'(A_N) \). Therefore, the marginal increase in sales (as a result of expending an additional dollar in advertising) is \( \lambda_2 NG(P)H'(A_N) \). The “benefit” resulting from each sale, we rate, has two effects: a short term and a long run impact. In the short run, each sale results in a profit of \( \partial V/\partial Q \). In the long run, the value-to-go is changed by \( \partial V/\partial Q \). Hence, the net effect of each sale is \( \partial V/\partial Q \). Consequently, conditional on a given price level, the marginal increase in value that can be attributed to the sales inducing aspect of advertising is given by \( \partial V/\partial Q \). The same line of reasoning can be adopted to furnish economic insight to the previous section’s findings when awareness advertising was the only concern. Given the above, from which our first order conditions are obtained, we can establish that for a quadratic response function, the optimal awareness advertising expenditures is

\[
A_U(t) = \max\left\{ \frac{1}{2} \left( \frac{a_1}{a_2} + \frac{1}{\mu_2} \frac{\partial V}{\partial Q} \right), 0 \right\}, \tag{39}
\]

and for a similar response function and a pre-established given price level, the optimal policy is

\[
A_N(t) = \begin{cases} 
\max\left\{ \frac{1}{2} \left( \frac{b_1}{\lambda_2 b_2 N} \frac{1}{(P - w)(C_0 - C_1 P) + \frac{\partial V}{\partial Q}} \right), 0 \right\} & \text{if } G(P(t)) = C_0 - C_1 P(t), \\
\max\left\{ \frac{1}{2} \left( \frac{b_1}{\lambda_2 b_2 N} \frac{1}{(P - w)kP^{-\eta} + \frac{\partial V}{\partial Q}} \right), 0 \right\} & \text{if } G(P(t)) = kP^{-\eta}(t). 
\end{cases} \tag{40}
\]

We can also consider the modified version of the above expression in the case where both the response functions are exponential. Other combinations are also possible, but we will not explicitly consider them here for brevity sake. Specifically, in light of Eqs. (34b and 34d) we have

\[
A_U(t) = \max\left\{ \frac{1}{2} \left( \frac{a_1}{a_2} + \frac{1}{\mu_2} \frac{\partial V}{\partial Q} \right), 0 \right\}. \tag{41}
\]

Without loss of generality,

\[
A_N(t) = \begin{cases} 
\max\left\{ b_2 + \ln \left( \frac{\lambda_2 N}{(P - w)(C_0 - C_1 P) + \frac{\partial V}{\partial Q}} \right), 0 \right\} & \text{if } G(P(t)) = C_0 - C_1 P(t), \\
\max\left\{ b_2 + \ln \left( \frac{\lambda_2 N}{(P - w)kP^{-\eta} + \frac{\partial V}{\partial Q}} \right), 0 \right\} & \text{if } G(P(t)) = kP^{-\eta}(t). 
\end{cases} \tag{42}
\]

For any choice of response and demand functions, the above expressions, (39)–(42), can be substituted back in the Bellman equation to obtain a concentrated PDE that must be numerically solved. In the following section additional insights to this model are provided.

8. Additional findings and results

In light of our findings in Sections 5 and 7, a number of structural results can be readily furnished. Recall that \( \mu_2 \) is the coefficient of external effects for the unaware pool \( U(t) \) while \( \lambda_2 \) is the coefficient of external
effects for the aware pool, $N(t)$. Each coefficient may be thought of as the fraction of the corresponding populations that are reached by an advertising program.

**Result 2.** For the unaware pool, $U(t)$, an increase in the coefficient of external effects increases the optimal level of “awareness advertising”.

**Proof.** See Appendix D.

**Result 3.** For the aware pool, $N(t)$, an increase in the coefficient of external effects increases the optimal level of “sales advertising”.

**Proof.** See Appendix D.

In some situations, advertising may not be segment specific. That is, advertising expenditures have the concurrent and dual affect of “awareness” and “sales” without being segment specific expenditures.

**Theorem 3.** Let $\{A(t), t \in [0, T]\}$ represent a positive adapted real-valued process. Assume further that the expenditure rate, $A(t)$ is not segment specific (in that advertising effort serves both in terms of awareness as well as sales). Let the functions $F(A(t))$ and $H(A(t))$ be defined by Eqs. (34). The resulting value function $V(\cdot)$ is concave in the advertising effort.

**Proof.** See Appendix D.

While Results 2 and 3 can also be applied to and also hold true in the case of Theorem 3, our next result examines the effect of price on the optimal advertising level. This result is counterintuitive and for the most part, depends on the price elasticity of demand, $\eta$. We provide this result in the context of Theorem 3, where its adaption to the findings of Sections 5 and 7 is straightforward.

**Result 4.** There exists a threshold price, $P(\eta)$ such that, (i) whenever the price of the innovation is below this threshold, the optimal advertising level increases with price and, (ii) whenever the price of the innovation is above the threshold, the optimal advertising level decreases with price.

**Proof.** See Appendix D.

The above results together with those in the previous sections provide insight to various aspects and the dynamics of innovation markets as captured by this model. The general notion of advertising and price adjustment in monopolistic markets has spanned a host of research. Ours is an attempt to probe such phenomena from a real options perspective and in light of uncertainty to furnish related results.

9. Conclusion

Uncertainty in the models of (market) innovation diffusion may arise in the model’s parameters, its structure, or both. Within this setting, models may reflect a monopolistic or competitive market, depict the market at aggregate or disaggregate levels, and be used for predictive or normative reasons. In this paper, we develop and analyze a normative and structurally stochastic innovation diffusion model depicting the market at an aggregate level. Model uncertainty is, effectively, portrayed by the population of potential adopters. In this environment, population dynamics reflect the random change in the level of potential purchases, which
is designed to be triggered by price, word-of-mouth and advertising effects. In light of this characterization, and by assuming the innovator is a monopolist, we obtain a model for the value of the innovation together with an optimal advertising and price policy that maximizes this value to the firm. Determination of the optimal policies result from the application of contingent claims analysis (CCA) and stochastic optimal control techniques. Generalizations of the model accounts for population (or segment) specific advertising, reflecting awareness advertising and sales promotion as control variables. In all cases, normative results obtained from the model are consistent with innovative behavior. The model presented in this paper provides additional insight to key strategic factors affecting the diffusion of an innovation in monopolistic markets and its management, complementing both the theoretical as well as practical findings in this arena.

Acknowledgements

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Appendix A

Proof of Result 1. By definition, we have for \( i \neq j \),
\[
\rho_{i,j} = \frac{\text{Cov}(i,j)}{\sigma(i)\sigma(j)}, \quad i, j = \text{d}U, \text{d}N, \text{d}Q. \tag{A.1}
\]
Moreover,
\[
\text{Var}(\text{d}U) = \sigma^2_U U^2 \left( \frac{N}{N^*} \right)^2 \text{dt}, \tag{A.2}
\]
\[
\text{Var}(\text{d}Q) = \sigma^2_Q Q^2 \left( \frac{N}{N^*} \right)^2 \text{dt}, \tag{A.3}
\]
and
\[
\text{Var}(\text{d}N) = \left( \frac{1}{N^*} \right)^2 \text{Var}(-\sigma_U UN \text{d}W_U(t) - \sigma_Q QN \text{d}W_Q(t)),
\]
\[
\text{Var}(\text{d}N) = \left( \frac{N}{N^*} \right)^2 (\sigma^2_U U^2 + \sigma^2_Q Q^2) \text{dt}. \tag{A.4}
\]
We can now substitute into (A.1). Specifically, from Eqs. (16a) and (16b), the covariance terms are obtained. Given the standard deviations, expression (A.1)–(A.3), it follows with some work that Result 1, Eq. (17), holds. \( \square \)

Appendix B

Proof of Theorem 1. Specializing the Bellman equation (31) to this situation obtains the first order condition,
\[ \frac{\partial V}{\partial A} = - \frac{\partial V}{\partial U} \mu_2 F'(A(t)) - 1. \]  
(B.1)

To see that the first order condition yields a maximum, and not a minimum, we examine the second order condition,

\[ \frac{\partial^2 V}{\partial A^2} = - \frac{\partial V}{\partial U} \mu_2 F''(A(t)). \]  
(B.2)

Given expression (32) and our choice of \( F(A(t)) \) as given by Eqs. (34a) or (34b), it follows that (B.2) is negative.

Appendix C

**Proof of Theorem 2.** Specializing Eqs. (31) to Section 7 results in the first derivative of the function (for first order condition)

\[ \frac{\partial V}{\partial Q} \dot{\lambda}_2 NH''(A_N(t)) + (P - w) \dot{\lambda}_2 NH'(A_N(t)) G(P) - 1. \]  
(C.1)

To show that the second derivative with respect to \( H(A_N(t)) \) is negative we have the second derivative as

\[ \dot{\lambda}_2 N \left( \frac{\partial V}{\partial Q} + (P - w) G(P) \right) H''(A_N(t)). \]  
(C.2)

Since \( H''(A_N(t)) < 0 \) given our choice of \( H(\cdot) \), and all other terms are positive, it follows that (C.2) is negative.

Appendix D

(For simplicity the subscripts “U” and “N” on the control variable \( A(t) \) have been omitted).

**Proof of Theorem 3.** The necessary (first order) condition for the optimal advertising policy is obtained by differentiating the fundamental PDE, Eq. (31), with respect to \( A(t) \). We obtain:

\[ \frac{r}{\partial A} \frac{\partial V}{\partial A} = - \frac{\partial Q}{\partial U} \mu_2 UF'(A) + \left( \frac{\partial V}{\partial Q} + P - w \right) \dot{\lambda}_2 NG(P) H'(A) - 1. \]  
(D.1)

To show that the first order condition provides a maximum, and not a minimum, we need to examine the second derivative of \( V \) at the turning points. From Eq. (D.1) above, it follows that at \( A^*(t) \),

\[ \frac{\partial V}{\partial Q} + P - w = \frac{1 + \frac{\partial V}{\partial U} \mu_2 UF'(A)}{\dot{\lambda}_2 NG(P) H'(A)}. \]  
(D.2)

Differentiating both sides of Eq. (D.1) with respect to \( A \), we obtain

\[ \frac{r}{\partial A^2} \frac{\partial^2 V}{\partial A^2} = - \frac{\partial V}{\partial U} \mu_2 UF''(A) + \left( \frac{\partial V}{\partial Q} + P - w \right) \dot{\lambda}_2 NG(P) H''(A). \]  
(D.3)
Substituting Eq. (D.2) into (D.3), we have
\[ r \frac{\partial^2 V}{\partial A^2} = - \frac{\partial V}{\partial U} \mu_2 UF(A) + \left( 1 + \frac{\partial V}{\partial U} \mu_2 UF'(A) \right) \frac{H''(A)}{H'(A)} \]
\[ = \frac{H''(A)}{H'(A)} + \frac{\partial V}{\partial U} \mu_2 U \frac{[H''F' - F'']}{H' \left( |F''| \right)} \]
\[ = \frac{H''(A)}{H'(A)} + \frac{\partial V}{\partial U} \mu_2 U \frac{F'^2}{H' \left( |F''| \right)} \frac{\partial}{\partial A} \left( \frac{H'}{F'} \right) . \]
Since by assumption the response functions \( F(\cdot) \) and \( H(\cdot) \) are both increasing and strictly concave in \( A(t) \) and in light of Eq. (9), it follows that each of the terms on the right hand side of the above equation is negative. Hence at \( \mu_2 \), \( r \frac{\partial^2 V}{\partial A^2} < 0 \), and thus the turning point yields a maximum. ∎

**Proof of Result 2.** From a basic result in comparative statics (Dixit, 1990), from Theorem 1, and from Eqs. (36) and (37), we have
\[ \frac{\partial \mu^*}{\partial \mu} = - \frac{\partial^2 V}{\partial \mu^2} \frac{\partial^2 V}{\partial \mu^2}. \quad (D.4) \]
In the proof of Theorem 3 it was shown that \( \frac{\partial^2 V}{\partial A^2} \) is negative. Hence, \( sgn(\frac{\partial \mu^*}{\partial \mu}) = sgn \left( \frac{\partial^2 V}{\partial \mu^2} \right) \).

Differentiating Eq. (D.1) with respect to \( \mu_2 \), obtains \( r \frac{\partial \mu^*}{\partial \mu_2} = - \frac{\partial V}{\partial U} UF'(A) \). From assumption 1 and the fact that \( F'(A) > 0 \), it follows that \( \frac{\partial^2 V}{\partial \mu^2} > 0 \) and therefore \( \frac{\partial \mu^*}{\partial \mu_2} > 0 \). ∎

**Proof of Result 3.** As in the proof of Result 2, we have
\[ \frac{\partial \mu^*}{\partial \mu_2} = - \frac{\partial^2 V}{\partial \mu_2}, \quad (D.5) \]
and since the denominator is negative, \( \frac{\partial \mu^*}{\partial \mu_2} \) takes on the sign of \( \frac{\partial^2 V}{\partial \mu_2} \). By differentiating (D.1) with respect to \( \mu_2 \), we have, \( r \frac{\partial^2 V}{\partial \mu_2} = \left( \frac{\partial V}{\partial Q} + P - w \right) NG(P)H'(A) \). From the assumptions and that \( H'(A) > 0 \), the right hand side is positive and thus so is \( \frac{\partial \mu^*}{\partial \mu_2} \). ∎

**Proof of Result 4.** Analogous to the previous two results,
\[ sgn \left( \frac{\partial \mu^*}{\partial P} \right) = sgn \left( \frac{\partial^2 V}{\partial \mu_2} \right) . \]
Differentiating Eq. (A.1) with respect to \( P \) we obtain
\[ r \frac{\partial^2 V}{\partial A \partial P} = \left( \frac{\partial V}{\partial Q} + P - w \right) \lambda_2 NG'(P)H'(A) + \lambda_2 NG(P)H'(A). \]
Then
\[
\frac{\partial A'}{\partial P} > 0 \iff \frac{\partial^2 V}{\partial A' \partial P} > 0 \\
\iff \left( \frac{\partial V}{\partial Q} + P - w \right) G'(P) + G(P) > 0 \\
\iff - \left( \frac{\partial V}{\partial Q} + P - w \right) \frac{G'(P)P}{G(P)} - P < 0 \\
\iff P < \frac{\eta}{\eta - 1} \left( w - \frac{\partial V}{\partial Q} \right),
\]
where \( \eta = - \frac{G'(P)P}{G(P)} \) is the price elasticity of demand. The quantity \( \frac{\eta}{\eta - 1} (w - \frac{\partial V}{\partial Q}) \) provides the threshold price, \( P(\eta) \).

References