AN INTERVAL METHOD FOR SOLVING THE ONE-DIMENSIONAL WAVE EQUATION

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Abstract. This paper presents path of construction of the interval methods of second order for solving the wave equation. Taken into consideration is the finite difference interval method for one dimensional Partial Differential Equation. It is shown that the exact solution belongs to the obtained interval solution. Chosen numerical results are made in floating-point interval arithmetic.
1 INTRODUCTION

In our previous papers we presented interval methods for solving the initial value problem in Ordinary Differential Equations [3, 4, 5, 12 and 13]. In this case the interval methods for solving Partial Differential Equations have been considered.

This paper studies how to solve the one dimensional wave equation using an interval method of the second order, in floating-point interval arithmetic. An interval method for solving Wave Equation was presented at the 80th Annual Meeting of the International Association of Applied Mathematics and Mechanics GAMM 2009 [11]. In this paper we present an interval method called finite-difference method. In numerical experiments we compare the solutions, which are obtained while using conventional and interval methods.

2 THE WAVE EQUATION

Taken into consideration is the wave equation, which is an example of the hyperbolic partial differential equations [2, 6, 8 and 19]. We present the wave equation as an example of the string.

We assume that the string is well flexible and homogeneous (mass of string per unit length $\rho$ is a constant). The tension $T$ of the string is constant and larger than the force of gravity (no other external forces act on the string). Provided that, damping effects are neglected and the amplitude is not to large. If the string is stretched between two points (see Fig. 1) where $x=0$ and $x=L$ and the amplitude of the displacement of the string denotes $u=u(x,t)$ satisfies the one-dimensional wave equation

\[ v^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} = 0, \]

in the region where $0<x<L$ and time $t>0$, where $v^2 = \frac{T}{\rho} = \text{const}.$

The initial position and velocity of the string are given by the initial conditions

\[ \begin{align*}
    u(x,0) &= \phi(x) \\
    \frac{\partial u(x,t)}{\partial t} \bigg|_{t=0} &= \psi(x)
\end{align*} \quad 0 \leq x \leq L, \quad (2) \]

where $\phi(x)$ and $\psi(x)$ are given functions.

Since the string is tied down at the ends, $u$ must also satisfy the Dirichlet boundary conditions, given by

\[ \begin{align*}
    u(0,t) &= 0 \\
    u(L,t) &= 0
\end{align*} \quad t > 0. \quad (3) \]

![Figure 1: The stretched string.](image-url)
3 THE FINITE DIFFERENCE METHOD

To set up the finite difference method (see [1]) we select the positive numbers \( n \) and \( m \), which adequately describe number of points \( x \) and \( t \).

We receive space-time grid with the mesh points \( (x_i, t_j) \) where \( \Delta x = h > 0 \) and \( \Delta t = k > 0 \) such that

\[
x_i = i \cdot h, \quad h = \frac{L}{n}, \quad i = 0, 1, \ldots, n, \tag{4}
\]
\[
t_j = j \cdot k, \quad j = 0, 1, \ldots, m.
\]

For the mesh points \( (x_i, t_j) \) we expand the second partial derivatives of function \( u(x_i, t_j) \) into the Taylor series

\[
\frac{\partial^2 u(x_i, t_j)}{\partial x^2} = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} + \frac{h^2 \partial^4 u(\xi_i, t_j)}{12 \partial x^4}, \tag{5}
\]

where \( \xi_i \in (x_{i-1}, x_{i+1}) \) and

\[
\frac{\partial^2 u(x_i, t_j)}{\partial t^2} = \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2} + \frac{k^2 \partial^4 u(x_i, \eta_j)}{12 \partial t^4}, \tag{6}
\]

where \( \eta_j \in (t_{j-1}, t_{j+1}) \).

To obtain the difference method we put the formulas (5) and (6) into the wave equation (1)

\[
\nu^2 \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} = \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2} = \frac{v^2 h^2 \partial^4 u(\xi_i, t_j)}{12 \partial x^4} - \frac{k^2 \partial^4 u(x_i, \eta_j)}{12 \partial t^4}.
\]

Substituting \( \gamma = v \frac{k}{h} \) we have

\[
u^2 \gamma^2 u(x_i, t_{j+1}) + 2(1 - \gamma^2)u(x_i, t_j) + \gamma^2 u(x_{i+1}, t_j) - u(x_i, t_{j-1}) + E_{ij}.
\]

The truncation error \( e_{ij} \) of order \( O(k^2 - h^2 v^2) \) is in the form

\[
e_{ij} = \frac{k^2 h^2}{12} \left( \frac{\partial^4 u(x_i, \eta_j)}{\partial t^4} - v^2 \frac{\partial^4 u(\xi_i, t_j)}{\partial x^4} \right). \tag{7}
\]

Neglecting truncation error and substituting approximation \( u_{ij} \) for \( u(x_i, t_j) \) we can write a formula

\[
u_{i,j+1} = \gamma^2 u_{i-1,j} + 2(1 - \gamma^2)u_{i,j} + \gamma^2 u_{i+1,j} - u_{i,j-1}, \quad \text{for } i = 1, 2, \ldots, n-1, \quad j = 2, 3, \ldots, m.
\]

For \( i = 0 \) and \( i = n \) the boundary conditions (3) give

\[
u_{0j} = u_{n,j} = 0, \quad j = 1, 2, \ldots, m.
\]

For \( j = 0 \) the first initial condition (2) we obtain

\[
u_{i0} = \varphi_i = \varphi(x_i), \quad i = 1, 2, \ldots, n-1.
\]
For \( j = 1 \) we have to consider the second of the initial condition (2). The derivative \( \frac{\partial u(x,t)}{\partial t} \bigg|_{t=0} \) is replaced by a forward-difference approximation as follows

\[
\frac{\partial u(x,0)}{\partial t} = u(x, t_1) - u(x, 0) \cdot \frac{k}{2} \cdot \frac{\partial^2 u(x, \tilde{t}_i)}{\partial t^2}, \quad 0 < \tilde{t}_i < t_1, \quad i = 1, 2, \ldots, n - 1,
\]

for the points corresponding to the time \( t_1 = k \), we obtain

\[
u(x, t_1) = u(x, 0) + k \frac{\partial u(x, 0)}{\partial t} + \tilde{\epsilon}_g,
\]

for \( 0 < \tilde{t}_j < t_1, \quad i, j = 1, 2, \ldots, n - 1 \), where the truncation error of order \( O(k) \) is defined as

\[
\tilde{\epsilon}_j = \frac{k^2}{2} \cdot \frac{\partial^2 u(x, \tilde{t}_j)}{\partial t^2}
\]

the value \( \tilde{\epsilon}_j \) is neglected and it leads to the expression

\[
u_{ij} = \varphi_i + k \cdot \psi_j, \quad i = 1, 2, \ldots, n - 1.
\]

Consequently for mesh points, one of the conventional methods, called finite difference method is obtained as

\[
u_{i,j+1} = \gamma^2 u_{i-1,j} + 2(1 - \gamma^2)\nu_{i,j} + \gamma^2 u_{i+1,j} - u_{i,j-1}, \quad i = 1, 2, \ldots, n - 1, \quad j = 2, 3, \ldots, m,
\]

with the initial conditions

\[
u_{i,0} = \varphi_i, \quad \nu_{i,1} = \varphi_i + k \cdot \psi_j, \quad i = 1, 2, \ldots, n - 1,
\]

and boundary conditions

\[
u_{0,j} = u_{mj} = 0, \quad j = 1, 2, \ldots, m.
\]

On the Figure 2 is given the model of points, which is obtained by formula (9).

![Figure 2: The mesh points grid given by the formula (9).](image)

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4 AN INTERVAL METHOD OF THE SECOND ORDER

4.1 Finite difference interval method

In conventional difference methods the truncation errors are neglected. The idea of the interval methods is to involve errors of method into interval solution.

Construction of interval methods is analogously to [9].

In order to obtain this method we consider the second derivatives of \( u(x,t) \). Adequate to (5) and (6) we have

\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} - \frac{h^2}{12} \frac{\partial^4 u(\xi,t)}{\partial x^4}, \quad \xi \in (x-h,x+h),
\]

and

\[
\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{u(x,t+k) - 2u(x,t) + u(x,t-k)}{k^2} - \frac{k^2}{12} \frac{\partial^4 u(x,\eta)}{\partial t^4}, \quad \eta \in (t-k,t+k).
\]

In order to estimate the truncation errors \( \frac{\partial^4 u(\xi,t)}{\partial x^4} \) and \( \frac{\partial^4 u(x,\eta)}{\partial t^4} \) we differentiate the wave equation (1) twice with respect to variables \( x \) and \( t \)

\[
\frac{\partial^4 u(x,t)}{\partial x^4} - \frac{1}{v^2} \frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2} = 0 \quad \Rightarrow \quad \frac{\partial^4 u(x,t)}{\partial x^4} = v^2 \frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2},
\]

\[
\frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2} - \frac{1}{v^2} \frac{\partial^4 u(x,t)}{\partial t^4} = 0 \quad \Rightarrow \quad \frac{\partial^4 u(x,t)}{\partial t^4} = v^2 \frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2}.
\]

Assuming that \( \left| \frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2} \right| \leq M \) for all \( 0 < x < L \) and \( 0 < t < t_m \). Thus \( \frac{\partial^4 u(x,t)}{\partial x^4} \in [-M,M] \).

We put it to the formulas (12) and we get \( \frac{\partial^4 u(x,t)}{\partial x^4} \in \frac{1}{v^2} [-M,M] \) and \( \frac{\partial^4 u(x,t)}{\partial t^4} \in v^2 [-M,M] \).

For \( \xi \in (x-h,x+h) \) and \( \eta \in (t-k,t+k) \) we have

\[
\frac{\partial^4 u(\xi,t)}{\partial x^4} \in \frac{1}{v^2} [-M,M] \quad \frac{\partial^4 u(x,\eta)}{\partial t^4} \in v^2 [-M,M].
\]

We put these terms to the truncation error

\[
E = \frac{k^2 h^2}{12} \left( \frac{k^2}{h^2} \frac{\partial^4 u(x,\eta)}{\partial t^4} - v^2 \frac{\partial^4 u(\xi,t)}{\partial x^4} \right) \in \frac{k^2 h^2}{12} \left( v^2 \frac{k^2}{h^2} [-M,M] - [-M,M] \right).
\]

Consequently for all \( 0 < x < L \), \( 0 < t < t_m \) and \( \gamma = v \frac{k}{h} \) we receive

\[
E \in \frac{k^2 h^2}{12} \left( v^2 - 1 \right) [-M,M].
\]
Analogously we estimate \( \frac{\partial^2 u(x, t)}{\partial t^2} \). For \( 0 < x < L \) and \( 0 < t < k \) we assume that
\[
\left| \frac{\partial^2 u(x, t)}{\partial t^2} \right| \leq M_1.
\]
Then \( \frac{\partial^2 u(x, t)}{\partial t^2} \in [-M_1, M_1] \). From wave equation for \( \tilde{\eta} \in (0, k) \) we have
\[
\frac{\partial^2 u(x, \tilde{\eta})}{\partial t^2} \in \nu^2 [-M_1, M_1].
\] (15)

We set (15) to the expanding error of \( \frac{\partial u(x, t)}{\partial t} \) at \( t = 0 \)
\[
\left( \tilde{E} = \tilde{E}_1 \in \frac{k^2 \nu^2}{2} \cdot [-M_1, M_1] \right)
\] (16)

As a result we can establish a theorem:

**Theorem 1.**
For the mesh points \((x_i, t_j) \in (X_1, T_1)\) the exact solution is included in the interval solution.

\[
u(x_i, t_j) \in U(X_1, T_1) = U_y,
\]

\( i = 0,1,\ldots, n \), \( j = 0,1,\ldots, m \).

We can write for \( i = 1,2,\ldots, n-1 \), \( j = 2,3,\ldots, m \) the interval finite difference method as
\[
U_{i,j+1} = \Gamma^2 U_{i-1,j} + 2(1-\Gamma^2)U_{i,j} + \Gamma^2 U_{i+1,j} - U_{i,j-1} = E,
\] (17)

with the initial conditions
\[
U_{i,0} = \Phi(X_i),
\]
\[
U_{i,1} = \Phi(X_i) + K \cdot \Psi(X_i) + E_i, \quad i = 1,2,\ldots, n-1,
\] (18)

and boundary conditions
\[
U_{0,j} = [0,0],
\]
\[
U_{n,j} = [0,0], \quad j = 1,2,\ldots, m.
\] (19)

These relations are true for the mesh points \((x_i, t_j) \in (X_1, T_1)\), where the functions \( \Phi(X), \Psi(X) \) and the values \( K, H, \Gamma \) are adequately interval extension (see [7, 14, 16]) of functions \( \phi(x), \psi(x) \) and values \( k, h, \gamma \). The errors \( E \) and \( E_1 \) are defined by formulas (14) and (16).

To find the interval solution at the mesh points of the grid we have to solve the \((m-1)^x(n-1)^y\) system of interval linear equations for the \((m-1)^x(n-1)^y\) unknowns \( U_{i,j} \). The formula (17) leads to obtaining the system of interval linear equations in the form
Matrix $I_{(n-1)\times(m-1)}$ is identity matrix and $A$ is the tridiagonal matrix:

$$A = \begin{bmatrix}
2(\gamma^2-1) & \gamma^2 & 0 & \ldots & 0 & 0 \\
\gamma^2 & 2(\gamma^2-1) & \gamma^2 & \ldots & 0 & 0 \\
0 & \gamma^2 & 2(\gamma^2-1) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2(\gamma^2-1) & \gamma^2 \\
0 & 0 & 0 & \ldots & \gamma^2 & 2(\gamma^2-1)
\end{bmatrix}_{(n-1)\times(m-1)}$$

the vectors of the unknowns $U_j$ and constants $R_j$ ($j = 2,3,\ldots,m$) are as follows

$$U_j = \begin{bmatrix} U_{1,j} \\ U_{2,j} \\ \vdots \\ U_{n-1,j} \end{bmatrix}, \quad R_j = \begin{bmatrix} R_{1,j} \\ R_{2,j} \\ \vdots \\ R_{n-1,j} \end{bmatrix},$$

where

$$R_{1,2} = E + \gamma^2 U_{0,1} - 2(\gamma^2-1) U_{1,1} + \gamma^2 U_{2,1} - U_{1,0},$$
$$R_{i,2} = E + \gamma^2 U_{i,-1} - 2(\gamma^2-1) U_{i,1} + \gamma^2 U_{i+1,1} - U_{i,0}, \quad i = 2,3,\ldots,n-2;$$
$$R_{n-1,2} = E + \gamma^2 U_{n-2,1} - 2(\gamma^2-1) U_{n-1,1} + \gamma^2 U_{n,1} - U_{n-1,0},$$
$$R_{1,3} = E + \gamma^2 U_{0,2} - U_{1,1},$$
$$R_{i,3} = E - U_{i,1}, \quad i = 2,3,\ldots,n-2;$$
$$R_{n-1,3} = E + \gamma^2 U_{n-2,2} - U_{n-1,1},$$
$$R_{1,j} = E + \gamma^2 U_{0,j-1}, \quad j = 4,5,\ldots,m;$$
$$R_{i,j} = E, \quad i = 2,3,\ldots,n-2, \quad j = 4,5,\ldots,m;$$
$$R_{n-1,j} = E + \gamma^2 U_{n,j-1}, \quad j = 4,5,\ldots,m.$$
In this case the boundary conditions are zero. Then these formulas together with the conditions (18) give
\[ R_{i,2} = E - 2(\gamma^2 - 1)(\Phi(H) + K\Psi(H) + E_i) + \gamma^2(2H) + K\Psi(2H) + E_i - \Phi(H), \]
\[ R_{i,1} = E + \gamma^2(\Phi((i-1)H) + K\Psi((i-1)H) + E_i) - 2(\gamma^2 - 1)(\Phi(iH) + K\Psi(iH) + E_i) + \]
\[ + \gamma^2(\Phi((i+1)H) + K\Psi((i+1)H) + E_i) - \Phi(iH), \quad \text{for } i = 2, 3, \ldots, n - 2; \]
\[ R_{n-1,2} = E + \gamma^2(\Phi((n-2)H) + K\Psi((n-2)H) + E_i) - \]
\[ - 2(\gamma^2 - 1)(\Phi((n-1)H) + K\Psi((n-1)H) + E_i) - \Phi((n-1)H), \]
\[ R_{i,3} = E - (\Phi(iH) + K\Psi(iH) + E_i), \quad \text{for } i = 1, 2, \ldots, n - 1; \]
\[ R_{i,j} = E, \quad \text{for } i = 1, 2, 3, \ldots, n - 1, \ j = 4, 5, \ldots, m; \]

To solve this interval linear system of equations is used Gaussian elimination method.

### 4.2 Floating-point interval arithmetic

We define the interval number \( X = [x, \bar{x}] = \{ x \in R : x \leq \bar{x} \} \) (see e.g. [15, 17, 18]).

The conventional calculating on the computer gives solutions, which are loaded with the errors. There are initial-data errors, data representation errors, rounding errors and errors of methods. Our calculations have been made using IntervalArithmetic unit written in the Delphi Pascal language. This unit allowed to represent an input data in the form of interval and make all calculations in floating-point interval arithmetic (if a number or result has exact machine representation - two ends of interval number are equal, if not - then are two neighboring machine numbers) (see[10]). We can use some standard interval functions and give results in the form of intervals, in which the difference between ends is visible (e.g. if the ends of an interval number are not the same then number 0.1 in interval arithmetic is visible as
\[ [9.9999999999999999E-0002, 1.000000000000000001E-0002], \]
and the width of interval number is equal \( 6.77626357803440E-0021 \).

To sum up, the interval methods for solving Partial Differential Equations with using floating-point interval arithmetic give solutions, in form of intervals, which contain all possible numerical errors.

### 5 NUMERICAL EXPERIMENTS

We consider an example of string with assumptions written in section 2, which satisfied equation
\[ u_{xx} - \frac{1}{v^2} u_t = 0, \quad \text{for } 0 \leq x \leq \pi, \ t > 0, \ v = 0.01, \]
with the boundary conditions \( u(0,t) = u(\pi,t) = 0, \ t > 0, \) and initial conditions
\[ u(x,0) = 0, \quad \text{and } \frac{\partial u(x, t)}{\partial t} \bigg|_{t=0} = \sin(x), \ x \in [0, \pi]. \]

The solution of this problem is function \( u(x, t) = \frac{1}{v} \sin(vt) \sin(x). \)
We calculate estimations of truncation errors 

\[ \frac{\partial^4 u}{\partial x^4 \partial t^2} \leq M = \nu = 0.01 \]  

and  

\[ \frac{\partial^2 u(x,t)}{\partial t^2} \leq M_i = \frac{1}{\nu} = 100. \]

We compare the results, which are obtained with using conventional (CM) and interval (IM) methods. The solutions are presented on the Figure 3. and Figure 4.

6 CONCLUSIONS

- Interval methods for solving PDE problems in floating-point interval arithmetic give solutions, in form of intervals, which contain all possible numerical errors.

- The interval methods with floating point interval arithmetic guarantee correct digits in solutions. In the solutions of conventional methods we can obtain more correct digits than interval solutions. However, we do not know exactly, how many correct digits there are. That depends on a given problem.
The main point is to make good quality estimations of truncation errors. If these estimations are poor, the width of interval-solutions can not be satisfied.

Calculation time and computer memory (for large system of interval linear equations) are usually disadvantages of interval methods together with floating point interval arithmetic.

7 FURTHER STUDIES

First we will define boundary conditions for some functions. The most important things in the next studies will be to find better estimation of derivatives (constants $M$ and $M_1$) and to use other, faster exact method (without matrix) for solving the system of interval linear equations with the special form.

Next we will construct new interval methods for solving other physical problems involving the hyperbolic PDE (e.g. occur in the study of vibrating beams or in the transmission of electricity).

REFERENCES


