An Interval Difference Method for Solving Hyperbolic Partial Differential Equations

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Abstract: The paper presents a proposition of construction the interval method of second order for solving hyperbolic PDE. The central-backward difference interval method for one dimensional wave equation is taken into consideration. The suitable Dirichlet and Cauchy conditions are satisfied for the string with fixed endpoints. The estimations of suitable truncation errors are proposed.


1 The Wave Equation

The string, as an example of the one dimensional wave equation, is taken into consideration [2] and [3], with following assumptions:

- the string is well flexible and homogeneous (mass of string per unit length $\varrho$ is a constant),
- the tension $T$ of the string is constant and larger than the force of gravity (no other external forces act on the string),
- damping effects are neglected,
- the amplitude is not too large,
- each inside point of the string can move only in the vertical direction.

If the string is stretched between two points (see Fig. 1) where $x = 0$ and $x = L$ and $u = u(x,t)$ denotes the amplitude of the string's displacement, then $u$ satisfies the wave equation

$$v^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x, t)}{\partial t^2} = 0,$$

in the region where $0 < x < L$ and time $t > 0$, where $v^2 = T/\varrho = \text{const.}$

Since the ends of string are secured to the $x$-axis, $u$ must also satisfy the Dirichlet boundary conditions given by

$$u(0, t) = 0$$
$$u(L, t) = 0$$
for $t > 0$. (2)

The initial position and velocity of the string are given by the Cauchy initial conditions at the time $t = 0$

$$u(x, 0) = \varphi(x)$$
$$\left. \frac{\partial u(x,t)}{\partial t} \right|_{t=0} = \psi(x)$$
for $0 < x < L$, (3)

where $\varphi(x)$ and $\psi(x)$ are given functions.

2 The Central-Backward Difference Method

To set up the central-backward difference method the positive numbers $n$ and $m$, which adequately describe number of points $x$ and $t$, are selected. The space-time grid with the mesh points $(x_i, t_j)$ is received, where

$$\Delta x = h = \frac{L}{n} > 0, \quad \Delta t = k > 0,$$

such that

$$x_i = i \cdot h,$$
$$t_j = j \cdot k$$
(4)
for $i = 0, 1, \ldots, n$, and $j = 0, 1, \ldots, m$.

The Taylor series for each interior mesh point $(x_i, t_j)$ are used to obtain the central-backward difference formulas

$$\frac{\partial^2 u(x_i, t_j)}{\partial x^2} = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{\Delta x^2} - \frac{k^2}{\Delta t^2} \frac{\partial^2 u(x_i, t_j)}{\partial t^2},$$

$$\frac{\partial^2 u(x_i, t_j)}{\partial t^2} = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{\Delta t^2} - \frac{k^2}{\Delta x^2} \frac{\partial^2 u(x_i, t_j)}{\partial x^2},$$

(5)

for $i = 1, 2, \ldots, n - 1$ and $j = 2, 3, \ldots, m$, where

$$\xi_i \in (x_{i-1}, x_{i+1}),$$

$$\eta_j \in (t_{j-2}, t_j).$$

Substituting the formulas (5) into the wave equation (1), gives

$$\frac{\partial^2}{\partial x^2} \left( u_{i-1,j} + u_{i+1,j} \right) - \left( \frac{2 \partial^2}{\partial t^2} + 1 \right) u_{i,j} + 2u_{i,j-1} - u_{i,j-2} = 0,$$

(6)

where $u_{i,j}$ denotes $u(x_i, t_j)$ for each $i$ and $j$.

Neglecting the truncation errors in the formula (6) the central-backward difference method is generated

$$\frac{\partial^2}{\partial x^2} \left( u_{i-1,j} + u_{i+1,j} \right) - \left( \frac{2 \partial^2}{\partial t^2} + 1 \right) u_{i,j} + 2u_{i,j-1} - u_{i,j-2} = 0,$$

(7)

for $i = 1, 2, \ldots, n - 1$ and $j = 2, 3, \ldots, m$.

The boundary conditions (2), for $i = 0$ and $i = n$ as well as $j = 2, 3, \ldots, m$, give

$$u_{0,j} = 0,$$

$$u_{n,j} = 0,$$

(8)

The initial conditions (3), for $i = 1, 2, \ldots, n - 1$ and $j = 0$, we can write as follows

$$u_{i,0} = \varphi_i,$$

$$u_{i,1} = \varphi_i + k \psi_i + \frac{k^2}{2} \varphi_i'' + \frac{k^3}{6} \varphi_i'''.$$

(9)

To obtain the second equation in the formula (9) the derivative

$$\frac{\partial u(x_i, t_j)}{\partial t} \bigg|_{t=0}$$

by the forward difference approximation of third order Taylor series in the variable $t$ about $t_1$ for points $(x_i, t_1)$ is replaced

$$u_{i,1} = u_{i,0} + k \cdot \frac{\partial u_{i,0}}{\partial t} + \frac{k^2}{2} \cdot \frac{\partial^2 u_{i,0}}{\partial t^2} +$$

$$+ \frac{k^3}{6} \cdot \frac{\partial^3 u_{i,0}}{\partial t^3} + \frac{k^4}{24} \cdot \frac{\partial^4 u(x_i, \eta_j)}{\partial t^4},$$

(10)

where $\eta_j \in (t_0, t_1)$ for $i = 1, 2, \ldots, n - 1$.

If the first equation in the formula (9) is differentiated twice in respect to $x$ (see [1]) then we can write

$$\frac{\partial^2 u(x_i, t_0)}{\partial t^2} = v^2 \frac{\partial^2 u(x_i, t_0)}{\partial x^2} = v^2 \frac{\partial^2 u(x_i, t_0)}{\partial x^2} =$$

$$= v^2 \varphi(x_i)'' = v^2 \varphi_i''.$$  

(11)

Analogously, if $\frac{\partial^3 u(x_i, t_0)}{\partial x^3}$ exists, then we can write

$$\frac{\partial^3 u(x_i, t_0)}{\partial x^3} = v^3 \varphi_i'''.$$

(12)

Substituting (11), (12) and the first equation (9) into formula (10), gives

$$u_{i,1} = \varphi_i + k \psi_i + \frac{k^2}{2} \varphi_i'' + \frac{k^3}{6} \varphi_i'''' +$$

$$+ \frac{k^4}{24} \cdot \frac{\partial^4 u(x_i, \eta_j)}{\partial t^4},$$

(13)

where $\eta_j \in (t_0, t_1)$ for $i = 1, 2, \ldots, n - 1$.

Neglecting the truncation error, the second dependence of the formula (9) is obtained.

## 3 The Central-Backward Difference Interval Method

In conventional difference methods the truncation errors frequently are neglected. The idea of the interval methods is to involve these errors into interval solutions. All interval solutions must satisfied the theorem

**Theorem 1** For the mesh points $(x_i, t_j) \in (X_1, T_j)$ the exact solution is included in the interval solution

$$u(x_i, t_j) \in U(X_i, T_j) = U_{i,j},$$

for $i = 1, 2, \ldots, n - 1$, $j = 1, 2, \ldots, m - 1$.

We consider the central-backward difference method (7) and the second initial condition (9) together their truncations errors.

The truncation error $E_M$ (the right side of eq.(6)) is in the form

$$E_M = \frac{k^4}{12} \cdot \frac{\partial^4 u(x_i, \eta_j)}{\partial t^4} - \frac{k^2}{12} \cdot \frac{\partial^4 u(x_i, \eta_j)}{\partial x^4},$$

(14)

where

$$\xi_i \in (x_{i-1}, x_{i+1}), i = 1, 2, \ldots, n - 1,$$

$$\eta_j \in (t_{j-2}, t_j), j = 2, 3, \ldots, m.$$  

The approximation error $E_C$ of the second initial condition (9) is defined as

$$E_C = \frac{k^4}{24} \cdot \frac{\partial^4 u(x_i, \eta_j)}{\partial t^4},$$

(15)
where 
\[ \tilde{\eta}_i \in (t_0, t_1), i = 1, 2, \ldots, n - 1 \] .

To obtain the interval method the estimations of truncations errors \( E_M \) and \( E_C \) are needed. To obtain estimations of \( \frac{\partial^4 u(x, \eta_j)}{\partial x^4} \) and \( \frac{\partial^4 u(x, \eta_j)}{\partial t^4} \) in (14), the wave equation (1) twice with respect to \( x \) and twice with respect to \( t \) is differentiated:

\[
\frac{\partial^4 u(x, t_j)}{\partial x^4} = \frac{1}{4!} \frac{\partial^4 u(x, t_i)}{\partial x^4 \partial t^4}, \tag{16}
\]

\[
\frac{\partial^4 u(x, t_j)}{\partial t^4} = v^2 \frac{\partial^4 u(x, t_i)}{\partial x^4 \partial t^2},
\]

for \( i = 1, 2, \ldots, n - 1, j = 2, 3, \ldots, m \).

Let

\[
\left| \frac{\partial^4 u(x, t_j)}{\partial x^4 \partial t^2} \right| \leq M
\]

for each mesh point \( (x, t_j), \ i = 0, 1, \ldots, n, \ j = 0, 1, \ldots, m \).

Then

\[
\frac{\partial^4 u(x, t_j)}{\partial x^4 \partial t^2} \in [-M, M]. \tag{17}
\]

The relation (17) into formulas (16) is put as follows

\[
\frac{\partial^4 u(x, t_j)}{\partial x^4 \partial t} = \frac{1}{4!} \left[ -M, M \right],
\]

\[
\frac{\partial^4 u(x, t_j)}{\partial t^4} = v^2 \left[ -M, M \right].
\]

Thus, these relations for \( \xi_i \in (x_{i-1}, x_{i+1}), \ i = 0, 1, \ldots, n \) and \( \eta_j \in (t_{j-1}, t_j), \ j = 2, 3, \ldots, m \) are in the forms

\[
\frac{\partial^4 u(\xi_i, \eta_j)}{\partial x^4} \in \frac{1}{4!} \left[ -M, M \right], \tag{18}
\]

\[
\frac{\partial^4 u(x, \eta_j-1)}{\partial t^4} \in v^2 \left[ -M, M \right].
\]

Putting the relations (18) into (14) gives

\[
E_M \in \frac{k^2}{12} \left( \frac{\partial^2 u(x, t_j)}{\partial x^2} \right)^2 \left[ -M, M \right]. \tag{19}
\]

The derivative \( \frac{\partial^4 u(x, \tilde{\eta}_i)}{\partial x^4} \) in (15) for \( \tilde{\eta}_i \in (t_0, t_1), \ i = 1, 2, \ldots, n - 1, \) is a special case of derivative \( \frac{\partial^4 u(x, \eta_j-1)}{\partial x^4} \) in (14) for \( \eta_{j-1} \in (t_{j-2}, t_j) \) and \( j = 2 \). Then the estimation of \( \frac{\partial^4 u(x, \eta_j-1)}{\partial x^4} \) is true for \( \frac{\partial^4 u(x, \tilde{\eta}_i)}{\partial x^4} \). Putting the second relation (18) into (15) gives

\[
E_C \in \frac{k^2 v^2}{24} \left[ -M, M \right]. \tag{20}
\]

In order to estimate value \( M \) the Taylor series in the variable \( t \) about \( t_j \) and in the variable \( x \) about \( x_i \) for the formulas (5) are used as follows

\[
\frac{\partial^2 u(x, t_j)}{\partial x^2} = \frac{\partial^2 u(x, t_j)}{\partial x^2 \partial t^2} = \frac{\partial^2 u(x, t_j)}{\partial t^2 \partial x^2} \tag{21}
\]

\[
\frac{\partial^2 u(x, t_j)}{\partial x^2} \left( \frac{\partial^2 u(x, t_j)}{\partial x^2} \right) = \frac{u(x_{i+1, j+1}) - 2u(x_{i+1, j}) + u(x_{i+1, j-1})}{h^2 k^2} -
\]

\[
\frac{k^2}{12h^2} \left( \frac{\partial^4 u(x, \eta_j)}{\partial x^4} \right) -
\]

\[
-2 \frac{u(x_{i+1, j+1}) - 2u(x_{i+1, j}) + u(x_{i+1, j-1})}{h^2 k^2} -
\]

\[
- \frac{k^2}{12h^2} \left( \frac{\partial^4 u(\xi_i, \eta_j)}{\partial x^4} + \frac{\partial^4 u(x_{i-1, \eta_j})}{\partial x^4} \right) -
\]

\[
- \frac{\partial^4 u(x_{i-1, \eta_j})}{\partial x^4} \tag{22}
\]

where

\[
\eta_j, \tilde{\eta}_j, \tilde{\eta}_j \in (t_{j-1}, t_{j+1}) \text{ for } j = 1, 2, \ldots, m - 1, \]

\( \xi_i, \tilde{\xi}_i, \tilde{\xi}_i, \xi_i \in (x_{i-1}, x_{i+1}), \text{ for } i = 1, 2, \ldots, n - 1. \)

Let

\[
\frac{\partial^4 u(x, t_j)}{\partial x^2 \partial t^2} = \frac{\partial^4 u(x, t_j)}{\partial t^2 \partial x^2} \tag{23}
\]

together (17) for each mesh point \( u(x, t_j) \rightarrow u_{i,j} \), where \( i = 0, 1, \ldots, n \) and \( j = 0, 1, \ldots, m \). Neglecting truncations errors in formulas (21) and (22) we
can write them as
\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial^2 u_{i,j}}{\partial x^2} \right) \simeq \frac{u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}}{h^2} - 2\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + \frac{u_{i-1,j+1} - 2u_{i-1,j} + u_{i-1,j-1}}{h^2} - 2 \tag{23}
\]
and
\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial^2 u_{i,j}}{\partial x^2} \right) \simeq \frac{u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}}{h^2} - 2\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + \frac{u_{i-1,j+1} - 2u_{i-1,j} + u_{i-1,j-1}}{h^2} \tag{24}
\]
The right hand sides of the approximations (23) and (24) are the same, then constance \( M \) we can write as follows (see [5])
\[
M \simeq \frac{s}{h^2} \max_{i=1,\ldots,n-1, j=1,\ldots,m-1} \left| u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1} - 2u_{i,j+1} - 2u_{i,j-1} + u_{i-1,j+1} - 2u_{i-1,j} + u_{i-1,j-1} \right| \tag{25}
\]
where \( s = 1.5 \) (in this case is assumed that the approximation errors in (21) and (22) are not greater than 50\%). The terms \( u_{i,j} \to u(x_i, t_j) \) for \( i = 1, 2, \ldots, n - 1 \) and \( j = 1, 2, \ldots, m - 1 \) are calculated from the conventional central difference method.

Summarizing, the central-backward difference interval method is obtained as follows
\[
V^2 K^2 \frac{h^2}{12} (U(X_{i-1}, T_j) + U(X_{i+1}, T_j)) - 2V^2 K^2 \frac{h^2}{12} + 1) U(X_i, T_j) \tag{26}
\]
\[+ 2U(X_i, T_{j-1}) - U(X_i, T_{j-2}) = E_M, \tag{31}
\]
for \( i = 1, 2, \ldots, n - 1 \) and \( j = 2, 3, \ldots, m \), where \( E_M \) is given by the formula
\[
E_M \in \frac{K^2}{12} (H^2 + K^2 V^2) [-M, M]. \tag{27}
\]
The boundary conditions (2), for \( j = 1, 2, \ldots, m \), are as follows
\[
U(X_0, T_j) = [0, 0], \quad U(X_n, T_j) = [0, 0]. \tag{28}
\]
The initial conditions (3), for \( i = 1, 2, \ldots, n - 1 \), are given as
\[
U(X_i, T_0) = \Phi(X_i), \quad U(X_i, T_1) = \Phi(X_i) + K \Psi(X_i) + \frac{K^3 V^2}{2} \Phi''(X_i) + \frac{K^5 V^4}{6} \Phi'''(X_i) + E_C. \tag{29}
\]
The value \( M \) is calculated by the formula (25).

These relations are satisfied for each mesh point \( (x_i, t_j) \in (X_i, T_j) \), where \( (X_i, T_j) \) are interval representations (see [4] or [6]) of the suitable mesh points \( (x_i, t_j) \). The functions \( U, \Phi, \Psi \) and the values \( K, H, V \) are interval extension (see [7]) of functions \( u, \varphi, \psi \) and values \( k, h, v \) respectively.

To find the interval solution at the mesh points of the grid we have to solve the system of \((m-1) \times (m-1)\) interval linear equations for the \((m-1) \times (m-1)\) unknowns \( U_{i,j} \). This system we can write in the form
\[
\begin{bmatrix}
A & & & & & & & \\
B & & & & & & & \\
& & & & & & & \\
& & & & & & & \\
C & & & & & & &
\end{bmatrix}
\begin{bmatrix}
U_2 \\
U_3 \\
\vdots \\
\vdots \\
U_m
\end{bmatrix}
= \begin{bmatrix}
R_2 \\
R_3 \\
\vdots \\
\vdots \\
R_m
\end{bmatrix} \tag{31}
\]
where \( A_{(n-1) \times (n-1)} \) is tridiagonal matrix in the form
\[
A = \begin{bmatrix}
(2\Gamma^2 + 1) & \Gamma^2 & & & & & \\
\Gamma^2 & \ddots & \ddots & & & & \\
& \ddots & \ddots & \ddots & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \ddots & \ddots & \ddots & \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & & \ddots & \ddots \end{bmatrix} \tag{32}
\]
for \( \Gamma = V \cdot K \).

Diagonal matrices \( B_{(n-1) \times (n-1)} \) and \( C_{(n-1) \times (n-1)} \) are as follows
\[
B = \begin{bmatrix}
2 & & & & & & \\
& \ddots & \ddots & & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \ddots & \ddots & \ddots & \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots \end{bmatrix}, \\
C = \begin{bmatrix}
1 & & & & & & \\
& \ddots & \ddots & & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \ddots & \ddots & \ddots & \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots \end{bmatrix} \tag{33}
\]
Vectors \( U_j \) and \( R_j \), for \( j = 2, 3, \ldots, m \), are adequately vectors of unknowns and constants:
\[
U_j = \begin{bmatrix}
U_{1,j} \\
U_{2,j} \\
\vdots \\
U_{n-1,j}
\end{bmatrix}, \\
R_j = \begin{bmatrix}
R_{1,j} \\
R_{2,j} \\
\vdots \\
R_{n-1,j}
\end{bmatrix}. \tag{34}
\]
Elements of vector $R_j = [R_{1,j}, R_{2,j}, \ldots, R_{n-1,j}]^T$, for $i = 1, 2, \ldots, n-1$ and $j = 2, 3, \ldots, m$, (together with the Dirichlet (28) and Cauchy conditions (29)) are obtained by the formulas

$$R_{i,j} = E_M + U_i, 0 - 2U_{i,1} =$$

$$= E_M + \Phi(iH) - 2(\Phi(iH) + K\Psi(iH) + EC)$$

for $j = 2$.

$$R_{i,j} = E_M + U(X_i, T_1) =$$

$$= E_M + \Phi(iH) + K\Psi(iH) + EC$$

for $j = 3$ : \hspace{1cm} (35)

$$R_{i,j} = E_M$$

for $j = 4, 5, \ldots, m$.

To solve the interval linear systems of equations (31) the interval algorithm based on the floating point interval arithmetic is used.

## 4 The Floating Point Interval Arithmetic

The standard calculations on the computer give solutions with errors. There are initial-data errors, data representation errors, rounding errors and errors of methods. The central-backward difference interval method contains errors of this conventional method. All calculations have been made (in floating point interval arithmetic) using IntervalArithmetic unit written by professor Andrzej Marciniak in the Delphi Pascal language. This unit allows to represent all input data in the form of intervals and make all calculations of the floating point interval arithmetic. Some standard interval functions can be used to obtain the results in the form of intervals.

The interval number is defined as

$$X = [x, \overline{x}] = \{x \in \mathbb{R} : x \leq \overline{x} \leq \overline{x}\}.$$  

For example real number 0.1 is represented as following interval number

$$[0.099999999999999, 0.10000000000000001],$$

where the diameter of this interval number is equals

$$d = 6.77 \cdot 10^{-21}.$$  

Summarizing, calculations in floating point interval arithmetic together the interval methods for solving Partial Differential Equations give solutions, in form of intervals, which contain all possible numerical errors. In numerical experiments, the solutions obtained from conventional and presented interval method are compared.

## 5 Conclusion

- Interval methods for solving PDE problems in floating point interval arithmetic give solutions, in form of intervals, which contain all possible numerical errors.

- Presented interval method with floating point interval arithmetic guarantee correct digits in solutions. For example, in some interval solution

$$X = [1.570701610111265, 1.5707621013768904]$$

there are following warranted correct digits

$$1.5707.$$  

For adequately conventional real solution

$$x = 1.57073185574401,$$

we do not know, how many digits are corrected.

- The main point is to make good quality estimations for errors of methods. If these estimations are poor, the widths of interval-solutions can not be satisfied. (The diameters of interval solutions should be near zero.)

- Calculation time and computer memory (for large system of interval linear equations) are usually disadvantages of interval methods together with floating point interval arithmetic.

## References: