Journal of Uncertain Systems
Vol.3, No.1, pp.3-10, 2009
Online at: www.jus.org.uk

Some Research Problems in Uncertainty Theory

Baoding Liu*

Uncertainty Theory Laboratory, Department of Mathematical Sciences
Tsinghua University, Beijing 100084, China

Received 10 March 2008; Accepted 2 June 2008

Abstract

In addition to the four axioms of uncertainty theory, this paper presents the fifth axiom called product measure axiom. This paper also gives an operational law of independent uncertain variables and a concept of entropy of continuous uncertain variables. Based on the uncertainty theory, a new uncertain calculus is proposed and applied to uncertain differential equation, finance, control, filtering and dynamical systems. Finally, an uncertain inference will be presented.

Keywords: uncertain variable, uncertain measure, uncertain calculus, finance, uncertain inference

1 Introduction

Fuzziness is a basic type of subjective uncertainty. The concept of fuzzy set was initiated by Zadeh [15] via membership function in 1965. In order to measure a fuzzy event, Liu and Liu [7] introduced the concept of credibility measure in 2002. Credibility theory was founded by Liu [8] in 2004 and refined by Liu [9] in 2007 as a branch of mathematics for studying the behavior of fuzzy phenomena.

However, a lot of surveys showed that the subjective uncertainty cannot be modeled by fuzziness. This means that some real problems cannot be processed by credibility theory. In order to deal with this type of uncertainty, Liu [9] founded an *uncertainty theory* that is a branch of mathematics based on normality, monotonicity, self-duality, and countable subadditivity axioms.

This paper will review the uncertainty theory and propose some further research problems. Section 2 presents a product measure axiom. Section 3 gives a definition of independence and an operational law of independent uncertain variables. Section 4 proposes an identification function to describe uncertain variables and proves a measure inversion theorem. Section 5 presents a new definition of entropy for continuous uncertain variables. An uncertain calculus is given in Section 6, including canonical process, uncertain integral and chain rule. Section 7 gives a concept of stability of solution of uncertain differential equation. After that, we discuss the applications in finance, control, filtering and dynamical systems. An uncertain inference will be provided in Section 12.

2 Uncertain Measure

Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an *event*. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to each event Λ a number $\mathcal{M}\{\Lambda\}$ which indicates the level that Λ will occur. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu [9] proposed the following four axioms:

Axiom 1. *(Normality)* $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2. *(Monotonicity)* $\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}$ *whenever* $\Lambda_1 \subset \Lambda_2$.

Axiom 3. *(Self-Duality)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ.

Axiom 4. *(Countable Subadditivity) For every countable sequence of events* $\{\Lambda_i\}$, *we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{1}$$

*Corresponding author. Email: liu@tsinghua.edu.cn (B. Liu).

The law of contradiction tells us that a proposition cannot be both true and false at the same time, and the law of excluded middle tells us that a proposition is either true or false. Self-duality is in fact a generalization of the law of contradiction and law of excluded middle. In other words, a mathematical system without self-duality assumption will be inconsistent with the laws. This is the main reason why self-duality axiom is assumed.

Definition 1 *(Liu [9]) The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the normality, monotonicity, self-duality, and countable subadditivity axioms.*

Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots, n$. Write $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ and $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$. It is easy to verify that

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda} \min_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} + \sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c} \min_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} \le 1. \tag{2}$$

Based on this inequality, we may accept the following product measure axiom.

Axiom 5. *(Product Measure Axiom) Let Γ_k be nonempty sets on which $\mathcal{M}_k$ are uncertain measures, $k = 1, 2, \cdots, n$, respectively. Then the product uncertain measure on Γ is*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \displaystyle\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda} \min_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\}, & \textit{if} \displaystyle\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda} \min_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \displaystyle\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c} \min_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\}, & \textit{if} \displaystyle\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c} \min_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, & \textit{otherwise} \end{cases} \tag{3}$$

for each event $\Lambda \in \mathcal{L}$.

Remark 1: In fact, the product measure axiom may also be defined in other ways. For example,

$$\mathcal{M}\{\Lambda\} = \begin{cases} \displaystyle\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda} \prod_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\}, & \text{if} \displaystyle\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda} \prod_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \displaystyle\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c} \prod_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\}, & \text{if} \displaystyle\sup_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c} \prod_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, & \text{otherwise.} \end{cases} \tag{4}$$

3 Uncertain Variables

Definition 2 *(Liu [9]) An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event.*

Definition 3 *The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^m \{\xi_i \in B_i\}\right\} = \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\} \tag{5}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

Theorem 1 *The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^m \{\xi_i \in B_i\}\right\} = \max_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\} \tag{6}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

Proof: It follows from the self-duality of uncertain measure that $\xi_1, \xi_2, \cdots, \xi_m$ are independent if and only if

$$\mathcal{M}\left\{\bigcup_{i=1}^{m}\{\xi_i \in B_i\}\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{m}\{\xi_i \in B_i^c\}\right\} = 1 - \min_{1\le i\le m}\mathcal{M}\{\xi_i \in B_i^c\} = \max_{1\le i\le m}\mathcal{M}\left\{\xi_i \in B_i\right\}.$$

Thus the proof is complete.

Theorem 2 *(Operational Law) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables, and $f : \Re^n \to \Re$ a measurable function. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable such that*

$$\mathcal{M}\{\xi \in B\} = \begin{cases} \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B}\ \min\limits_{1\le k\le n}\mathcal{M}_k\{\xi_k \in B_k\}, & \text{if } \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B}\ \min\limits_{1\le k\le n}\mathcal{M}_k\{\xi_k \in B_k\} > 0.5 \\ 1 - \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B^c}\ \min\limits_{1\le k\le n}\mathcal{M}_k\{\xi_k \in B_k\}, & \text{if } \sup\limits_{f(B_1,B_2,\cdots,B_n)\subset B^c}\ \min\limits_{1\le k\le n}\mathcal{M}_k\{\xi_k \in B_k\} > 0.5 \\ 0.5, & \text{otherwise,} \end{cases}$$

where $B, B_1, B_2, \cdots, B_n$ are Borel sets of real numbers.

Proof: It follows from the product measure axiom immediately.

4 Identification Function

A random variable may be characterized by a probability density function, and a fuzzy variable may be described by a membership function. This section will introduce an *identification function* to characterize an uncertain variable.

Definition 4 *An uncertain variable ξ is said to have an identification function (λ, ρ) if (i) $\lambda(x)$ is a nonnegative function and $\rho(x)$ is a nonnegative and integrable function such that*

$$\sup_{x\in B}\lambda(x) + \int_B \rho(x)\mathrm{d}x \ge 0.5 \quad \textit{and/or} \quad \sup_{x\in B^c}\lambda(x) + \int_{B^c}\rho(x)\mathrm{d}x \ge 0.5 \tag{7}$$

for any Borel set B of real numbers; (ii) we have

$$\mathcal{M}\{\xi \in B\} = \begin{cases} \sup\limits_{x\in B}\lambda(x) + \int_B \rho(x)\mathrm{d}x, & \textit{if } \sup\limits_{x\in B}\lambda(x) + \int_B \rho(x)\mathrm{d}x < 0.5 \\ 1 - \sup\limits_{x\in B^c}\lambda(x) - \int_{B^c}\rho(x)\mathrm{d}x, & \textit{if } \sup\limits_{x\in B^c}\lambda(x) + \int_{B^c}\rho(x)\mathrm{d}x < 0.5 \\ 0.5, & \textit{otherwise.} \end{cases} \tag{8}$$

Remark 2: Some uncertain variables do not have their own identification functions. In other words, it is not true that every uncertain variable may be represented by an appropriate identification function.

Theorem 3 *Suppose λ is a nonnegative function and ρ is a nonnegative and integrable function satisfying (7). Then there is an uncertain variable ξ such that (8) holds.*

Proof: Let $\Re$ be the universal set. For each Borel set B of real numbers, we define a set function

$$\mathcal{M}\{B\} = \begin{cases} \sup\limits_{x\in B}\lambda(x) + \int_B \rho(x)\mathrm{d}x, & \text{if } \sup\limits_{x\in B}\lambda(x) + \int_B \rho(x)\mathrm{d}x < 0.5 \\ 1 - \sup\limits_{x\in B^c}\lambda(x) - \int_{B^c}\rho(x)\mathrm{d}x, & \text{if } \sup\limits_{x\in B^c}\lambda(x) + \int_{B^c}\rho(x)\mathrm{d}x < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

It is clear that $\mathcal{M}$ is normal, increasing, self-dual, and countably subadditive. That is, the set function $\mathcal{M}$ is indeed an uncertain measure. Now we define an uncertain variable ξ as an identity function from the uncertainty space $(\Re, \mathcal{A}, \mathcal{M})$ to $\Re$. We may verify that ξ meets (8). The theorem is proved.

5 Entropy

This section provides a definition of entropy to characterize the uncertainty of uncertain variables resulting from information deficiency. Note that the discrete case has been defined by Liu [9].

Definition 5 *Suppose that ξ is a continuous uncertain variable. Then its entropy is defined by*

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mathcal{M}\{\xi \le x\})\mathrm{d}x, \tag{9}$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$.

It is easy to verify that when a continuous uncertain variable tends to a crisp number, its entropy tends to the minimum 0. In addition to the development of entropy itself, we also need a maximum entropy principle for uncertain variables.

6 Uncertain Calculus

Uncertain calculus, initialized by Liu [10] in 2008, is composed of canonical process, uncertain integral and chain rule. This paper would like to revise those concepts as follows.

Definition 6 *An uncertain process C_t is said to be a canonical process if*
(i) $C_0 = 0$ and C_t is sample-continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re. \tag{10}$$

Let C_t be a canonical process, and $\mathrm{d}t$ an infinitesimal time interval. Then $\mathrm{d}C_t = C_{t+\mathrm{d}t} - C_t$ is an uncertain process with $E[\mathrm{d}C_t] = 0$ and $\mathrm{d}t^2/2 \le E[\mathrm{d}C_t^2] \le \mathrm{d}t^2$.

Definition 7 *Let X_t be an uncertain process and let C_t be a canonical process. For any partition of closed interval $[a,b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as $\Delta = \max_{1\le i\le k}|t_{i+1} - t_i|$. Then the uncertain integral of uncertain process X_t with respect to C_t is*

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\to 0}\sum_{i=1}^{k} X_{t_i}\cdot(C_{t_{i+1}} - C_{t_i}) \tag{11}$$

provided that the limit exists almost surely and is an uncertain variable.

Theorem 4 *Let C_t be a canonical process, and let $h(t,c)$ be a continuously differentiable function. Define $X_t = h(t, C_t)$. Then we have the following chain rule*

$$\mathrm{d}X_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t. \tag{12}$$

Proof: Since the function h is continuously differentiable, by using Taylor series expansion, the infinitesimal increment of X_t has a first-order approximation

$$\Delta X_t = \frac{\partial h}{\partial t}(t, C_t)\Delta t + \frac{\partial h}{\partial c}(t, C_t)\Delta C_t.$$

Hence we obtain the chain rule because it makes

$$X_s = X_0 + \int_0^s \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \int_0^s \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t$$

for any $s \ge 0$.

7 Uncertain Differential Equation

Suppose C_t is a canonical process, and f and g are some given functions. Liu [10] presented the following uncertain differential equation,

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t. \tag{13}$$

An existence and uniqueness theorem is needed for the solution of uncertain differential equations.

Definition 8 *(Stability) An uncertain differential equation is said to be stable if for any given $\epsilon > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for any solutions X_t and Y_t, we have*

$$\mathcal{M}\{|X_t - Y_t| > \epsilon\} < \varepsilon, \quad \forall t > 0 \tag{14}$$

whenever $|X_0 - Y_0| < \delta$.

We should develop different types of stability, and discuss their relationship as well as their applications.

8 Uncertain Finance

If we assume that the stock price follows some uncertain differential equation, then we may produce a new topic of uncertain finance. As an example, let stock price follow geometric canonical process. Then we have a *stock model* in which the bond price X_t and the stock price Y_t are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eY_t\mathrm{d}t + \sigma Y_t\mathrm{d}C_t \end{cases} \tag{15}$$

where r is the riskless interest rate, e is the stock drift, σ is the stock diffusion, and C_t is a canonical process.

European Option Prices

A European call option gives the holder the right to buy a stock at a specified time for specified price. Assume that the option has strike price K and expiration time s. Then the payoff from such an option is $(Y_s - K)^+$. Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rs)(Y_s - K)^+$. Hence the European call option price should be

$$f_c = \exp(-rs)E[(Y_s - K)^+]. \tag{16}$$

It is easy to verify the European call option price formula

$$f_c = \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} \left(1 + \exp\left(\frac{\pi(\ln y - es)}{\sqrt{3}\sigma s}\right)\right)^{-1} \mathrm{d}y. \tag{17}$$

A European put option gives the holder the right to sell a stock at a specified time for specified price. Assume that the option has strike price K and expiration time s. Then the payoff from such an option is $(K - Y_s)^+$. Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rs)(K - Y_s)^+$. Hence the European put option price should be

$$f_p = \exp(-rs)E[(K - Y_s)^+]. \tag{18}$$

It is easy to verify the European put option price formula

$$f_p = \exp(-rs)Y_0 \int_0^{K/Y_0} \left(1 + \exp\left(\frac{\pi(es - \ln y)}{\sqrt{3}\sigma s}\right)\right)^{-1} \mathrm{d}y. \tag{19}$$

Multi-factor Stock Model

Now we assume that there are multiple stocks whose prices are determined by multiple canonical processes. For this case, we have a multi-factor stock model in which the bond price X_t and the stock prices Y_{it} are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_{it} = e_iY_{it}\mathrm{d}t + \displaystyle\sum_{j=1}^{n}\sigma_{ij}Y_{it}\mathrm{d}C_{jt},\ i=1,2,\cdots,m \end{cases} \tag{20}$$

where r is the riskless interest rate, e_i are the stock drift coefficients, σ_{ij} are the stock diffusion coefficients, C_{it} are independent canonical processes, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$.

Portfolio Selection

For the stock model (20), we have the choice of $m+1$ different investments. At each instant t we may choose a portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ (i.e., the investment fractions meeting $\beta_t+\beta_{1t}+\cdots+\beta_{mt}=1$). Then the wealth Z_t at time t should follow uncertain differential equation

$$\mathrm{d}Z_t = r\beta_tZ_t\mathrm{d}t + \sum_{i=1}^{m}e_i\beta_{it}Z_t\mathrm{d}t + \sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}\beta_{it}Z_t\mathrm{d}C_{jt}. \tag{21}$$

Portfolio selection problem is to find an optimal portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ such that the expected wealth $E[Z_s]$ is maximized and variance $V[Z_s]$ is minimized at terminal time s. In order to balance the two objectives, we may, for example, maximize the expected wealth subject to a variance constraint, i.e.,

$$\begin{cases} \max E[Z_s] \\ \text{subject to:} \\ \qquad V[Z_s] \le \overline{V} \end{cases} \tag{22}$$

where $\overline{V}$ is a given level. This is just the so-called *mean-variance model* in finance.

No-Arbitrage

The stock model (20) is said to be *no-arbitrage* if there is no portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ such that for some time $s>0$, we have

$$\mathcal{M}\{\exp(-rs)Z_s \ge Z_0\} = 1, \quad \mathcal{M}\{\exp(-rs)Z_s > Z_0\} > 0 \tag{23}$$

where Z_t is determined by (21) and represents the wealth at time t. We may prove that the stock model (20) is no-arbitrage if and only if its diffusion matrix $(\sigma_{ij})_{m\times n}$ has rank m, i.e., the row vectors are linearly independent.

9 Uncertain Control

A *control system* is assumed to follow the uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t, Z_t)\mathrm{d}t + g(t, X_t, Z_t)\mathrm{d}C_t \tag{24}$$

where X_t is the state and Z_t is a control. Assume that R is the return function and T is the function of terminal reward. If we want to maximize the expected return on $[0, s]$ by using an optimal control, then we have the following control model,

$$\begin{cases} \displaystyle\max_{Z_t} E\left[\int_0^s R(t, X_t, Z_t)\mathrm{d}t + T(s, X_s)\right] \\ \text{subject to:} \\ \mathrm{d}X_t = f(t, X_t, Z_t)\mathrm{d}t + g(t, X_t, Z_t)\mathrm{d}C_t. \end{cases} \tag{25}$$

Hamilton-Jacobi-Bellman equation provides a necessary condition for extremum of stochastic control model, and Zhu's equation [16] provides a necessary condition for extremum of fuzzy control model. What is the necessary condition for extremum of general uncertain control model? How do we find the optimal control?

10 Uncertain Filtering

Suppose an uncertain system X_t is described by an uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{26}$$

where C_t is a canonical process. We also have an observation $\widehat{X}_t$ with

$$\mathrm{d}\widehat{X}_t = \widehat{f}(t, X_t)\mathrm{d}t + \widehat{g}(t, X_t)\mathrm{d}\widehat{C}_t \tag{27}$$

where $\widehat{C}_t$ is another canonical process that is independent of C_t. The *uncertain filtering problem* is to find the best estimate of X_t based on the observation $\widehat{X}_t$.

One outstanding contribution to stochastic filtering problem is Kalman-Bucy filter. How do we filter the noise away from the observation for general uncertain process?

11 Uncertain Dynamical System

Usually a stochastic dynamical system is described by a stochastic differential equation, and a fuzzy dynamical system is described by a fuzzy differential equation. Here we define an *uncertain dynamical system* as an uncertain differential equation. Especially, a first-order uncertain system is a first-order uncertain differential equation

$$\dot{X}_t = f(t, X_t) + g(t, X_t)\dot{C}_t, \tag{28}$$

and a second-order uncertain system is a second-order uncertain differential equation

$$\ddot{X}_t = f(t, X_t, \dot{X}_t) + g(t, X_t, \dot{X}_t)\dot{C}_t, \tag{29}$$

where

$$\dot{X}_t = \frac{\mathrm{d}X_t}{\mathrm{d}t}, \quad \ddot{X}_t = \frac{\mathrm{d}\dot{X}_t}{\mathrm{d}t}, \quad \dot{C}_t = \frac{\mathrm{d}C_t}{\mathrm{d}t}. \tag{30}$$

We should develop a new theory for uncertain dynamical systems.

12 Uncertain Inference

Uncertain logic was designed by Li and Liu [5] in 2009 as a generalization of logic for dealing with uncertain knowledge. A key point in uncertain logic is that the truth value of an uncertain proposition is defined as the uncertain measure that the proposition is true. One advantage of uncertain logic is the well consistency with classical logic. This paper proposes a framework of uncertain inference for uncertain propositions. Assume that we have two universal sets $\mathbb{X}$ and $\mathbb{Y}$. An uncertain relation between $\mathbb{X}$ and $\mathbb{Y}$ is a function that takes“uncertain variable” values and is written as $y = g(x)$ which is called an *uncertain relation equation* between $\mathbb{X}$ to $\mathbb{Y}$. First we have the following *inference rule*:

$$\begin{array}{l} \text{Relation: } y = g(x) \\ \text{Fact: } \mathbb{X} \text{ is } \xi \\ \hline \text{Infer: } \mathbb{Y} \text{ is } g(\xi). \end{array} \tag{31}$$

Sometimes, instead of knowing the perfect relation between $\mathbb{X}$ and $\mathbb{Y}$, we only have a rule “if $\mathbb{X}$ is ξ then $\mathbb{Y}$ is η”. For this case, we have the following inference rule:

$$\begin{array}{l} \text{Rule: If } \mathbb{X} \text{ is } \xi \text{ then } \mathbb{Y} \text{ is } \eta \\ \text{Fact: } \mathbb{X} \text{ is } \xi^* \\ \hline \text{Infer: } \mathbb{Y} \text{ is } \eta^*. \end{array} \tag{32}$$

The key problem is to construct an uncertain relation equation $y = g(x)$ between $\mathbb{X}$ and $\mathbb{Y}$ from the rule “if $\mathbb{X}$ is ξ then $\mathbb{Y}$ is η”. This paper suggests that the uncertain relation equation is defined as the conditional uncertain variable of η given $\xi = x$ (Liu [11]), i.e., $y = g(x) = \eta|_{\xi=x}$. Based on the uncertain relation equation

$y = g(x)$ and inference rule (31), from ξ^* we infer $\eta^* = g(\xi^*)$. Perhaps this method is too complex. This paper also suggests that η^* is the conditional uncertain variable of η given $\xi = \xi^*$, and denoted by

$$\eta^* = \eta|_{\xi=\xi^*}. \tag{33}$$

Remark 1: The uncertain relation equation may also be defined as uncertain variable such that $T(g(x) = t) = T(\xi \neq x) \vee T(\eta = t)$ or $T(g(x) = t) = T(\xi \neq x) \vee (T(\xi = x) \wedge T(\eta = t))$. Those two methods have been used in fuzzy inference by L.A. Zadeh.

Remark 2: The inference rule (33) is also applicable to fuzzy inference, random inference and hybrid inference provided that the conditional uncertain variable is replaced with conditional fuzzy variable, conditional random variable and conditional hybrid variable, respectively.

13 Conclusion

Uncertainty theory is a branch of mathematics for studying the behavior of subjective uncertainty. The essential differentia between fuzziness and uncertainty is that the former assumes

$$\mathcal{M}\{A \cup B\} = \mathcal{M}\{A\} \vee \mathcal{M}\{B\}$$

for any events A and B no matter if they are independent or not, and the latter assumes the relation only for independent events. However, a lot of surveys showed that

$$\mathcal{M}\{A \cup B\} \neq \mathcal{M}\{A\} \vee \mathcal{M}\{B\}$$

when A and B are not independent events. Perhaps this relation was abused in the fuzzy world. This is the main reason why we need the uncertainty theory.

Acknowledgments

This work was supported by National Natural Science Foundation of China Grant No.60874067.

References

[1] Black, F., and M. Scholes, The pricing of option and corporate liabilities, *Journal of Political Economy*, vol.81, pp.637-654, 1973.

[2] Gao, J., and X. Gao, A new stock model for credibilistic option pricing, *Journal of Uncertain Systems*, vol.2, no.4, pp.243-247, 2008.

[3] Li, P., and B. Liu, Entropy of credibility distributions for fuzzy variables, *IEEE Transactions on Fuzzy Systems*, vol.16, no.1, pp.123-129, 2008.

[4] Li, X., and B. Liu, Maximum entropy principle for fuzzy variables, *International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems*, vol.15, supp.2, pp.43-52, 2007.

[5] Li, X., and B. Liu, Hybrid logic and uncertain logic, *Journal of Uncertain Systems*, vol.3, no.2, pp.83-94, 2009.

[6] Liu, B., *Theory and Practice of Uncertain Programming*, Physica-Verlag, Heidelberg, 2002.

[7] Liu, B., and Y.K. Liu, Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy Systems*, vol.10, no.4, pp.445-450, 2002.

[8] Liu, B., *Uncertainty Theory*, Springer-Verlag, Berlin, 2004.

[9] Liu, B., *Uncertainty Theory*, 2nd ed., Springer-Verlag, Berlin, 2007.

[10] Liu, B, Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems*, vol.2, no.1, pp.3-16, 2008.

[11] Liu, B, *Uncertainty Theory*, 3rd ed., http://orsc. edu.cn/liu/ut.pdf.

[12] Peng, J., A general stock model for fuzzy markets, *Journal of Uncertain Systems*, vol.2, no.4, pp.248-254, 2008.

[13] Shafer, G., *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, NJ, 1976.

[14] Shannon, C.E., *The Mathematical Theory of Communication*, The University of Illinois Press, Urbana, 1949.

[15] Zadeh, L.A., Fuzzy sets, *Information and Control*, vol.8, pp.338-353, 1965.

[16] Zhu, Y., Fuzzy optimal control with application to portfolio selection, http://orsc.edu.cn/process/080117.pdf.