STRONG CONVERGENCE FOR SEMIGROUP OF ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

Balwant Singh Thakur¹ and Mohammad Saeed Khan²

¹School of Studies in Mathematics
Pt. Ravishankar Shukla University
Raipur (C.G.) 492010, India
e-mail: balwantst@gmail.com

²Department of Mathematics and Statistics
Sultan Qaboos University
P.O. Box 36, PCode 123 Al-Khod, Muscat
Sultanate of Oman, Oman
e-mail: mohammad@squ.edu.om

Abstract. In this paper, we propose a viscosity iteration process for semigroup of asymptotically pseudocontractive mappings, and prove a strong convergence theorem in uniformly convex Banach space for the proposed iteration process.

1. Introduction

Let $E$ be a real Banach space, $E^*$ be its dual space, $K$ a nonempty closed convex subset of $E$ and $J : E \to 2^{E^*}$ the normalized duality mapping defined by

$$Jx = \{ f \in E^* : \langle x, f \rangle = \| x \|^2 = \| f \|^2, \| f \| = \| x \| \}, \text{ for all } x \in E,$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing between $E$ and $E^*$. The single-valued normalized duality mapping is denoted by $j$.

A mapping $T : K \to K$ is said to be

---

¹2000 Mathematics Subject Classification: 47H20, 47H06, 47H10.
²Keywords: Semigroup of asymptotically pseudocontractive mappings, strongly pseudocontractive mapping, viscosity iteration process, iterative approximation, variational inequality.
- nonexpansive, if
  \[ \|Tx - Ty\| \leq \|x - y\| \]
  for all \( x, y \in K \),
- pseudocontractive, if there exists some \( j(x - y) \in J(x - y) \) such that
  \[ \langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 \]
  for all \( x, y \in K \),
- strongly pseudocontractive, if there exists a constant \( \alpha \in (0, 1) \) such that
  \[ \langle Tx - Ty, j(x - y) \rangle \leq \alpha \|x - y\|^2 \]
  for all \( x, y \in K \),
- asymptotically nonexpansive [9], if there exists a sequence \( \{k_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} k_n = 1 \) such that
  \[ \|T^n x - T^n y\| \leq k_n \|x - y\| \]
  for all \( x, y \in K \) and \( n \in \mathbb{N} \),
- asymptotically pseudocontractive [13], if there exists a sequence \( \{k_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} k_n = 1 \) such that
  \[ \langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 \]
  for all \( x, y \in K \), and \( n \in \mathbb{N} \).

It can be seen from the above definitions that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is asymptotically pseudocontractive. A mapping \( T \) is called uniformly \( L \)-Lipschitzian, if there exists \( L > 0 \) such that \( \|T^n x - T^n y\| \leq L \|x - y\| \), for all \( x, y \in K \) and for each integer \( n \geq 1 \). Uniformly asymptotically regular if \( \|T^{n+1} x - T^n x\| \to 0 \) as \( n \to \infty \) for all \( x \in K \).

Let \( K \) be a closed convex subset of a Banach space \( E \) and \( \mathbb{R}^+ \) the set of nonnegative real numbers. \( T := \{T(t) : t \in \mathbb{R}^+\} \) is said to be strongly continuous semigroup of asymptotically pseudocontractive mappings from \( K \) into \( K \) if the following conditions are satisfied [5]:

1. \( T(0)x = x \) for all \( x \in K \);
2. \( T(s + t) = T(s) \circ T(t) \) for all \( s, t \in \mathbb{R}^+ \);
3. there exist \( \{k_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} k_n = 1 \) and \( j(x - y) \in J(x - y) \) such that
   \[ \langle (T(t_n))^n x - (T(t_n))^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 , \forall t_n > 0, x, y \in K \]
4. for each \( x \in K \), the mapping \( T(\cdot)x \) from \( \mathbb{R}^+ \) into \( K \) is continuous.

If in the above definition, condition (3) is replaced by the following condition:
there exist \( \{k_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} k_n = 1 \) such that

\[
\| (T(t_n))^n x - (T(t_n))^n y \| \leq k_n \| x - y \|, \quad \forall t_n > 0, \ x, y \in K
\]

then \( \mathcal{T} \) is called strongly continuous semi-group of asymptotically nonexpansive mappings on \( K \).

\( \mathcal{T} \) is said to have a fixed point if there exists \( x_0 \in K \) such that \( T(t)x_0 = x_0 \) for all \( t \geq 0 \). We denote by \( F \) the set of fixed point of \( \mathcal{T} \), i.e. \( F := \bigcap_{t \in \mathbb{R}^+} F(T(t)) \).

Numerous problems in mathematics and physical sciences can be recast in terms of a fixed point problem for nonexpansive mappings. Due to practical importance of these problems, algorithms for finding fixed points of nonexpansive mappings continue to be a flourishing topic of interest in fixed point theory.

The most straightforward attempt to solve the fixed point problem for nonexpansive mappings is by Picard iteration :

\[
x_{n+1} = Tx_n, \quad \forall n \geq 0 \ (x_0 \in K)
\] (1.1)

Unfortunately, algorithm (1.1) may fail to produce a norm convergence sequence \( \{x_n\} \).

In view of celebrated Banach contraction principle, the attempt to approximate fixed point of nonexpansive self mappings seems very promising :

For given \( u \in K \) and each \( t \in (0,1) \) define a contraction \( T_t : K \to K \) by

\[
T_t x = tu + (1 - t)Tx \quad \forall x \in K.
\]

Clearly \( T_t \) is \((1 - t)\) contraction, so by Banach contraction principle, it has a unique fixed point \( z_t \in K \), i.e. \( z_t \) is the unique solution of equation

\[
z_t = tu + (1 - t)Tz_t,
\] (1.2)

here \( z_t \) is defined implicitly.

In 1967, Browder [2] proved that \( z_t \) defined by (1.2) converges strongly to a fixed point of \( T \) as \( t \to 0 \). In the same year, Halpern [10] devised an explicit iteration method which converges in norm to a fixed point of \( T \), the iteration process is known as Halpern iterative method and defined as below :

For a sequence \( \{\alpha_n\} \) in \((0,1)\), obtain the modified version of (1.1)

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0
\] (1.3)
Further, it is proved that the sequence \( \{x_n\} \) defined by (1.3) converges strongly to a fixed point of \( T \) if \( \{\alpha_n\} \) satisfies certain conditions.

It is an interesting problem to extend results related to nonexpansive, asymptotically nonexpansive, pseudocontractive, asymptotically pseudocontractive mappings to semigroup of respective mappings.

Suzuki [14] proved the following result for strongly continuous semigroup of nonexpansive mappings:

**Theorem S.** Let \( K \) be a closed convex subset of a Hilbert space \( H \). Let \( \{T(t) : t \in \mathbb{R}^+\} \) be a strongly continuous semigroup of nonexpansive mappings on \( K \) such that \( F = \bigcap_{t \in \mathbb{R}^+} F(T(t)) \neq \emptyset \). Let \( \{\alpha_n\} \) and \( \{t_n\} \) be sequences of real numbers satisfying \( 0 < \alpha_n < 1, \ t_n > 0 \) and \( \lim n t_n = \lim n \alpha_n/t_n = 0 \). Fix \( u \in K \) and define a sequence \( \{u_n\} \) in \( K \) by
\[
u_n = \alpha_n u + (1 - \alpha_n) T(t_n) \nu_n
\]
for \( n \in \mathbb{N} \). Then \( \{u_n\} \) converges strongly to the element of \( F \) nearest to \( u \).

Chidume [5] proved following result for strongly continuous semigroup of asymptotically pseudocontractive mappings in the setting of Banach space:

**Theorem C.** Let \( K \) be a closed convex and bounded subset of a real uniformly convex Banach space \( E \) having uniformly Gâteaux differential norm, \( L \leq N(E)^{1/2} \). Let \( T := \{T(t) : t \in \mathbb{R}^+\} \) be a strongly continuous uniformly asymptotically regular and uniformly \( L \)-Lipschitzian semigroup of asymptotically pseudocontractive mappings from \( K \) into \( K \) with a sequence \( \{k_n\} \subset [1, \infty) \). Then for \( u \in K, \ t_n > 0 \) and \( s_n \in (0,1) \), there exists a sequence \( \{x_n\} \in K \) satisfying the following condition:
\[
x_n = \alpha_n u + (1 - \alpha_n) (T(t))^n x_n,
\]
where \( \alpha_n := (1 - s_n/k_n) \). Moreover, if \( \lim t_n = \lim (\alpha_n/t_n) = 0, \ (k_n - 1) \to 0 \) as \( n \to \infty \), and
\[
\|x_n - (T(t))^m x\|^2 \leq \langle x_n - (T(t))^m x, j(x_n - x) \rangle,
\]
\( \forall m, n \geq 1, \ \forall x \in C, \ t \in \mathbb{R}^+, \) where \( C := \{x \in K : \Phi(y) = \min_{z \in K} \Phi(z)\} \)
where \( \Phi(z) := \lim \|x_n - z\|^2 \) for each \( z \in K \). Then \( \{x_n\} \) converges strongly to a fixed point of \( T \).

On the other hand viscosity method provide an efficient approach to a large number of ill-posed problems (lack of existence, or uniqueness, or stability of a solution) coming from different branches of mathematics. A major feature of these methods is to provide as a limit of the solution of the approximate problems, a particular (possibly relaxed or generalized) solution of the original
First abstract formulation of the properties of the viscosity approximation have been given by A.N. Tykhonov [15] in 1963 when studying ill-posed problems (see [8] for details).

Let us now make precise the mathematical abstract setting. Let $X$ be an abstract space, given $f : X \to \mathbb{R}^+ \cup \{+\infty\}$ an extended real valued function, let us consider the minimization problem

$$\min \{ f(x) : x \in X \}$$

which is assumed to be ill-posed.

For any $\varepsilon > 0$, let us consider the approximate minimization problem

$$\min \{ f(x) + \varepsilon g(x) : x \in X \}$$

which is well posed due to nice properties of a nonnegative real valued function $g : X \to \mathbb{R}^+ \cup \{+\infty\}$. So, it is assumed that, for all $\varepsilon > 0$, there exists a solution $u_\varepsilon$ of $(P_\varepsilon)$. The central question is to study the convergence of the sequence $\{u_\varepsilon; \varepsilon \to 0\}$ and the characterization of its limit. The function $g$ is called viscosity function.

Using contraction mapping as a viscosity function, Moudafi [12] introduced viscosity approximation method of selecting a particular fixed point of a nonexpansive mapping. Given a real number $t \in (0, 1)$ and a contraction mapping $f : K \to K$ with contraction constant $\alpha \in [0, 1)$. Define a mapping $T_t = T_t^f : K \to K$ by

$$T_t x = tf(x) + (1 - t)Tx, \quad x \in K.$$  \hfill (1.4)

Clearly $T_t$ is a $(1 - t(1 - \alpha))$ contraction, and so has a unique fixed point $x_t = x_t^f \in K$. Thus $x_t$ is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1 - t)Tx_t.$$  \hfill (1.5)

Xu [16] studied the strong convergence of $x_t$ defined by (1.5) as $t \to 0$. He also introduced the following iterative algorithm to approximate fixed points of nonexpansive mappings: For arbitrary chosen $x_o \in K$, construct a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T x_n, \quad \forall n \geq 0.$$  \hfill (1.6)

More recently Cho and Kang [6] proved following theorem for strongly continuous semigroup of nonexpansive mappings:

**Theorem CK.** Let $K$ be a closed convex subset of a real uniformly convex Banach space $E$ having uniformly Gâteaux differential norm. Let $\{T(t) : t \in \mathbb{R}^+\}$
be a strongly continuous $L-$ Lipschitz semigroup of pseudocontractive mappings on $K$ such that $F \neq \emptyset$. Let $f: K \to K$ be a fixed bounded, continuous and strong pseudocontraction with the coefficient $\alpha \in (0,1)$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n \to \infty} t_n = \lim_{n \to \infty} \frac{t_n}{\alpha_n} = 0$. Let $\{x_n\}$ be a sequence generated in the following manner:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n) (T(t)) x_n, \quad \forall n \geq 1.$$ 

Assume that $\lim \|T(t)x_n - T(t)x^*\| \leq \lim \|x_n - x^*\|$, $\forall x^* \in C$, $t \geq 0$, where $C := \{x^* \in K : \Phi(x^*) = \min_{x \in K} \Phi(x)\}$ where $\Phi(x) := \lim \|x_n - x\|^2$ for each $x \in K$. Then $\{x_n\}$ converges strongly to a fixed point of $T$, which solves the following variational inequality

$$\langle (I - f)x^*, j(x^* - x) \rangle \leq 0, \quad \forall x \in F.$$ 

Motivated by the above results and a viscosity iteration defined by Ceng, Xu and Yao [3], in this paper we propose a viscosity iteration method (VIM) for strongly continuous semigroup of asymptotically pseudocontractive mappings and prove a strong convergence theorem for proposed VIM.

2. Preliminaries

Let $E$ be a real normed space of dimension $\geq 2$. The norm of $E$ is said to be uniformly Gâteaux differentiable if for each $y \in S_1(0) := \{x \in E : \|x\| = 1\}$ the limit $\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$ exist uniformly for $x \in S_1(0)$.

Let $l^\infty$ be the Banach space of all bounded real-valued sequences. A Banach limit $\lim$ is a bounded linear functional on $l^\infty$ such that

$$\|\lim\| = 1, \quad \lim_{n \to \infty} t_n \leq \lim_{n \to \infty} t_n \leq \lim_{n \to \infty} t_n,$$

and $\lim t_n = \lim t_{n+1}$ for all $t_n \in l^\infty$.

We need following results to prove our main result:

**Lemma 2.1.** ([1]) Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $E$, $\{x_n\}$ a bounded sequence in $K$, $\lim$ a Banach limit, and $\Phi$ a real valued function on $K$ defined by $\Phi(z) = \lim \|x_n - z\|^2$, $z \in K$. Then the set $M$ defined by

$$M = \left\{ u \in K : \lim \|x_n - u\|^2 = \inf \lim \|x_n - z\|^2 \right\}$$

is a nonempty closed convex bounded set and has exactly one point.
Lemma 2.2. ([7]) Let $E$ be a Banach space, $K$ be a nonempty closed convex subset of $E$ and $T : K \to K$ be a continuous and strong pseudocontraction. Then $T$ has a unique fixed point.

Lemma 2.3. ([4]) For any $x, y \in E$ the following holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).$$

3. Main Results

Theorem 3.1. Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Let $T := \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous uniformly asymptotically regular and uniformly $L-$ Lipschitzian semigroup of asymptotically pseudocontractive mappings form $K$ into $K$ such that $F := \bigcap_{t \in \mathbb{R}^+} F(T(t)) \neq \emptyset$. Let $f : K \to K$ be a fixed bounded, continuous strong pseudocontraction with constant $\alpha \in (0, 1)$. Let $\{x_n\}$ be a sequence generated by

$$x_n = \left(1 - \frac{1}{k_n}\right)x_n + \frac{1 - \alpha_n}{k_n}fx_n + \frac{\alpha_n}{k_n}(T(t_n))^nx_n \tag{3.1}$$

where $\{\alpha_n\}$ and $\{t_n\}$ are sequences of real numbers satisfying

$$\left\{\begin{array}{l}
0 < \alpha_n < \frac{1 - \alpha}{k_n - \alpha} \quad \text{and} \quad \lim_{n \to \infty} \frac{k_n - 1}{1 - \alpha_n} = 0, \\
 t_n > 0 \ (\forall n) \quad \text{and} \quad \lim_{n \to \infty} t_n = 0 = \lim_{n \to \infty} \frac{1 - \alpha_n}{t_n}.
\end{array}\right. \tag{3.2}$$

Assume that $\lim\|x_n - (T(t))^mx^*\| \leq \lim\|x_n - x^*\|$, for all $x^* \in M$, $m \geq 1$, $t \in \mathbb{R}^+$, where $M := \{x^* \in K : \Phi(x^*) = \inf_{x \in K} \Phi(x)\}$ with $\Phi(x) := \lim\|x_n - x\|^2$, for all $x \in K$. Then $\{x_n\}$ converges to $x^* \in F$ which solves the variational inequality:

$$\langle (f - I)x^*, j(x - x^*) \rangle \leq 0 \quad \forall \ x \in F. \tag{3.3}$$

Proof. First we show that the sequence $\{x_n\}$ generated by (3.1) is well defined. For each $n \in \mathbb{N}$, define a mapping $\widetilde{T}_n$ as follows

$$\widetilde{T}_n x := \left(1 - \frac{1}{k_n}\right)x + \frac{1 - \alpha_n}{k_n}fx + \frac{\alpha_n}{k_n}(T(t_n))^nx, \quad \forall x \in K,$$
then
\[
\langle \overline{T}_n x - \overline{T}_n y, j(x - y) \rangle \\
= \left( 1 - \frac{1}{k_n} \right) \langle x - y \rangle + \frac{1 - \alpha_n}{k_n} \langle fx_n - fy \rangle + \frac{\alpha_n}{k_n} \langle (T(t_n))^n x - (T(t_n))^n y \rangle, j(x - y) \rangle \\
\leq \left( 1 - \frac{1}{k_n} \right) \|x - y\|^2 + \frac{1 - \alpha_n}{k_n} \|x - y\|^2 + \alpha_n \|x - y\|^2 \\
= \left( 1 - \frac{1}{k_n} + \frac{\alpha(1 - \alpha_n) + \alpha_n}{k_n} \right) \|x - y\|^2
\]

and \( \left( 1 - \frac{1}{k_n} + \frac{\alpha(1 - \alpha_n) + \alpha_n}{k_n} \right) < 1 \) by choice of \( \alpha_n \). So \( \overline{T}_n \) is continuous and strongly pseudocontractive mapping, therefore from Lemma 2.2, the mapping \( \overline{T}_n \) has a unique fixed point say \( x_n \in K \), that is, the equation

\[
x_n = \left( 1 - \frac{1}{k_n} \right) x_n + \frac{1 - \alpha_n}{k_n} fx_n + \frac{\alpha_n}{k_n} (T(t_n))^n x_n
\]

has a unique solution for each \( n \in \mathbb{N} \).

Next we show that \( \{x_n\} \) is bounded. For any fixed \( p \in F \), from Lemma 2.3, we have

\[
\|x_n - p\|^2 \\
= \left( 1 - \frac{1}{k_n} \right) \langle x_n - p \rangle + \frac{1 - \alpha_n}{k_n} \langle fx_n - p \rangle + \frac{\alpha_n}{k_n} \langle (T(t_n))^n x_n - p \rangle, j(x_n - p) \rangle \\
= \left( 1 - \frac{1}{k_n} \right) \langle x_n - p, j(x_n - p) \rangle + \frac{1 - \alpha_n}{k_n} \langle fx_n - fp, j(x_n - p) \rangle \\
+ \frac{1 - \alpha_n}{k_n} \langle fp - p, j(x_n - p) \rangle + \frac{\alpha_n}{k_n} \langle (T(t_n))^n x_n - p, j(x_n - p) \rangle \\
\leq \left( 1 - \frac{1}{k_n} \right) \|x_n - p\|^2 + \frac{1 - \alpha_n}{k_n} \|x_n - p\|^2 \\
+ \frac{1 - \alpha_n}{k_n} \langle fp - p, j(x_n - p) \rangle + \alpha_n \|x_n - p\|^2 \\
= \left[ 1 - \frac{1}{k_n} + \frac{(1 - \alpha_n)\alpha}{k_n} + \alpha_n \right] \|x_n - p\|^2 + \frac{1 - \alpha_n}{k_n} \langle fp - p, j(x_n - p) \rangle \\
= \left[ 1 - \frac{1 - \alpha(1 - \alpha_n) - \alpha_n k_n}{k_n} \right] \|x_n - p\|^2 + \frac{1 - \alpha_n}{k_n} \langle fp - p, j(x_n - p) \rangle \\
= (1 - \eta_n) \|x_n - p\|^2 + \frac{1 - \alpha_n}{k_n} \langle fp - p, j(x_n - p) \rangle,
\]
where $\eta_n = \frac{1-\alpha(1-\alpha_n) - \alpha_n k_n}{k_n}$. Therefore,
\[
\|x_n - p\|^2 \leq \frac{1 - \alpha_n}{k_n \eta_n} \langle fp - p, j(x_n - p) \rangle,
\] (3.4)
since using (3.2), we have
\[
\frac{1 - \alpha_n}{k_n \eta_n} = \frac{1}{1 - k_n^{-1} \alpha_n - \alpha_n} \rightarrow \frac{1}{1 - \alpha}
\] (3.5)
thus $\{x_n\}$ is bounded and so $\{f(x_n)\}$ and $\{(T(t))_n x_n\}$ are bounded.

Now for any given $t > 0$, we have
\[
\|x_n - (T(t))^n x_n\| \leq \sum_{k=0}^{\left\lfloor \frac{t}{t_n} \right\rfloor - 1} \|T((k+1)t_n)^n x_n - (T(kt_n))^n x_n\|
\]
\[
+ \left\| T \left( \left[ \frac{t}{t_n} \right] t_n \right) x_n - (T(t))^n x_n \right\|
\]
\[
\leq \left[ \frac{t}{t_n} \right] L \|(T(t_n))^n x_n - x_n\|
\]
\[
+ L \left\| T \left( t - \left[ \frac{t}{t_n} \right] t_n \right) x_n - x_n \right\|
\]
\[
= \left[ \frac{t}{t_n} \right] (1 - \alpha_n) L \|(T(t_n))^n x_n - f(x_n)\|
\]
\[
+ L \max \{\|(T(s))^n x_n - x_n\| : 0 \leq s \leq t_n\}
\]
\[
\leq t \left( \frac{1 - \alpha_n}{t_n} \right) L \|(T(t_n))^n x_n - f(x_n)\|
\]
\[
+ L \max \{\|(T(s))^n x_n - x_n\| : 0 \leq s \leq t_n\}
\]
for $n \in \mathbb{N}$, which gives that
\[
\|x_n - (T(t))^n x_n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\] (3.6)

Thus,
\[
\|x_n - (T(t)) x_n\| \leq \|x_n - (T(t))^n x_n\| + \|(T(t))^n x_n - (T(t))^{n+1} x_n\|
\]
\[
+ \left\| (T(t))^{n+1} x_n - (T(t)) x_n \right\|
\]
\[
\leq (1 + L) \|x_n - (T(t))^n x_n\| + \|(T(t))^n x_n - (T(t))^{n+1} x_n\|
\]
therefore, from (3.6) and uniform asymptotic regularity of $T(t)$, we have
\[
\|x_n - (T(t)) x_n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\] (3.7)
On the other hand, since $K$ is closed, we see from Lemma 2.1 that $M$ is a nonempty closed convex bounded subset of $K$ and $M$ is singleton. For any $t \geq 0$, $x^* \in M$, we obtain by assumption that

$$\Phi ((T(t))^n x^*) = LIM \|x_n - (T(t))^n x^*\| \leq LIM \|x_n - x^*\| = \Phi(x^*).$$

That is $(T(t))^n M \subseteq M$, since $M$ is singleton, we have $(T(t))^n x^* = x^*$, by continuity of $T(t)$ we have $T(t)x^* = x^*$, i.e. there exists a unique $x^* \in M$ such that $x^* \in F$.

Now, for any $x \in F$, from (3.1), we have

$$\langle fx_n - x_n, j(x_n - x) \rangle$$

$$= \frac{1}{1 - \alpha_n} \langle x_n - (T(t_n))^n x_n, j(x_n - x) \rangle$$

$$= \frac{1}{1 - \alpha_n} [\langle x_n - x, j(x_n - x) \rangle - ((T(t))^n x_n - x, j(x_n - x))]$$

$$\geq \frac{1}{1 - \alpha_n} \left( \|x_n - x\|^2 - k_n \|x_n - x\|^2 \right)$$

$$= - \left( \frac{k_n - 1}{1 - \alpha_n} \right) \|x_n - x\|^2. \tag{3.8}$$

From (3.8), we have

$$LIM \langle x_n - fx_n, j(x_n - x) \rangle \leq LIM \left( \frac{k_n - 1}{1 - \alpha_n} \right) \|x_n - x\|^2 \to 0, \tag{3.9}$$

as $n \to \infty$. On the other hand, for any $s \in (0, 1)$, it follows from Lemma 2.3, that

$$\|x_n - x^* - s (fx^* - x^*)\|^2$$

$$\leq \|x_n - x^*\|^2 + 2 \langle -s (fx^* - x^*), j(x_n - x^* - s (fx^* - x^*)) \rangle$$

$$= \|x_n - x^*\|^2 - 2s \langle fx^* - x^*, j(x_n - x^*) \rangle$$

$$- 2s \langle fx^* - x^*, j(x_n - x^* - s (fx^* - x^*)) - j(x_n - x^*) \rangle.$$ 

This implies that

$$\langle fx^* - x^*, j(x_n - x^*) \rangle$$

$$\leq \frac{1}{2s} \left[ \|x_n - x^*\|^2 - \|x_n - x^* - s (fx^* - x^*)\|^2 \right]$$

$$- \langle fx^* - x^*, j(x_n - x^* - s (fx^* - x^*)) - j(x_n - x^*) \rangle. \tag{3.10}$$

Since $E$ has uniform Gâteaux differential norm, so $j$ is norm-to-weak* uniformly continuous on bounded subsets of $E$. For any $\varepsilon > 0$, there exists $\delta > 0$
such that for all \( s \in (0, \delta) \), we have
\[
\langle fx^* - x^*, j(x_n - x^*) \rangle \leq \frac{1}{2s} \left[ \|x_n - x^*\|^2 - \|x_n - x^* - s(fx^* - x^*)\|^2 \right] + \varepsilon.
\]
Taking Banach limit \( \text{LIM} \) on the above inequality, we have
\[
\text{LIM} \langle fx^* - x^*, j(x_n - x^*) \rangle \leq \frac{1}{2s} \left[ \text{LIM} \|x_n - x^*\|^2 - \text{LIM} \|x_n - x^* - s(fx^* - x^*)\|^2 \right] + \varepsilon < \varepsilon.
\]
Now, since \( \varepsilon \) is arbitrary, this implies that
\[
\text{LIM} \langle fx^* - x^*, j(x_n - x^*) \rangle \leq 0. \tag{3.11}
\]
Again by inequality (3.4), we have
\[
\text{LIM} \|x_n - x^*\|^2 \leq \text{LIM} \frac{1 - \alpha_n}{k_n \eta_n} \langle fx^* - x^*, j(x_n - x^*) \rangle \leq 0,
\]
and hence
\[
\text{LIM} \|x_n - x^*\|^2 = 0 \tag{3.12}
\]
therefore, there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) which converges strongly to \( x^* \). Using (3.8), for any \( x \in F \), we have
\[
\langle x_{n_j} - fx_{n_j}, j(x_{n_j} - x) \rangle \leq \left( \frac{k_{n_j} - 1}{1 - \alpha_{n_j}} \right) \|x_{n_j} - x\|^2, \tag{3.13}
\]
taking limit in (3.13), we get
\[
\langle x^* - fx^*, j(x^* - x) \rangle \leq 0, \text{ for any } x \in F. \tag{3.14}
\]
Now, suppose there exists another subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) which converges strongly to \( z^* \) (say). Since \( \lim \|x_n - T(t)x_n\| = 0 \) for each \( t \in \mathbb{R}^+ \), we have that \( z^* \) is a fixed point of \( T \). Thus from (3.14), we have
\[
\langle x^* - fx^*, j(x^* - z^*) \rangle \leq 0. \tag{3.15}
\]
Now, since \( x^* \in F \), using (3.8) again, we get
\[
\langle x_{n_k} - fx_{n_k}, j(x_{n_k} - x^*) \rangle \leq \left( \frac{k_{n_k} - 1}{1 - \alpha_{n_k}} \right) \|x_{n_k} - x^*\|^2, \tag{3.16}
\]
taking limit in (3.16), we get
\[
\langle z^* - fz^*, j(z^* - x^*) \rangle \leq 0. \tag{3.17}
\]
Adding (3.15) and (3.17), we get
\[
\langle x^* - z^* + fz^* - fx^*, j(x^* - z^*) \rangle \leq 0.
\]
This gives
\[
\|x^* - z^*\|^2 \leq \langle fx^* - fz^*, j(x^* - z^*) \rangle \leq \alpha \|x^* - z^*\|^2.
\]
Since \( \alpha \in (0, 1) \), we have, \( x^* = z^* \). This proves that \( \{x_n\} \) converges strongly to \( x^* \in F \), which is the unique solution to the variational inequality (3.3). This completes the proof. \( \square \)

**Remark 3.2.** Theorem 3.1 includes as special case the corresponding results in [3, 5, 6, 11, 14, 16, 17].

**Remark 3.3.** In Theorem 3.1, viscosity iteration method involves strong pseudocontractive mapping, and therefore \( x^* \in F \) is the solution of larger class of variational inequality (3.3).

**References**


