International Publications (USA)

PanAmerican Mathematical Journal
Volume 17(2007), Number 1, 53–60

Convergence of Three-step Iteration Process for Asymptotically Quasi-Nonexpansive Type Mappings in Convex Metric Spaces

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Communicated by Jong K. Kim
(Received September 2006; Accepted October 2006)

Abstract

In complete convex metric space, the concept of asymptotically quasi-nonexpansive type mapping is defined. Moreover, three-step iteration process with errors is defined for this class of mappings. Some necessary and sufficient condition for convergence of this process to fixed point of asymptotically quasi-nonexpansive type mappings is proved. These results generalize and unify many important result in the literature.

2000 MSC: 47H09, 47H10.

Keywords: Three-step iteration process, convex metric space, asymptotically quasi-nonexpansive type mapping.

1 Introduction

Let $X$ be a complete convex metric space and $K$ be a nonempty closed convex subset of $X$. $T$ be a self mapping of $K$. $T$ is said to be nonexpansive provided $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$. Denote by $F(T)$ the set of fixed points of

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† This paper was supported by Dong-A University Research Fund in 2006.
that is, \( F(T) = \{ x \in K : Tx = x \} \). Throughout this paper we assume that \( F(T) \neq \emptyset \). \( T \) is said to be quasi-nonexpansive if \( d(Tx, p) \leq d(x, p) \) for all \( x \in K \) and \( p \in F(T) \). There has been a number of results on fixed points of nonexpansive and quasi-nonexpansive mappings in Banach and metric spaces. Petryshyn and Williamson [10], in 1973, proved sufficient and necessary condition for convergence of Picard and Mann iterates to fixed point of quasi-nonexpansive mappings. Ghosh and Debnath [4] extended results of [10] and gave the necessary and sufficient condition for convergence of Ishikawa iterative sequence to fixed point of asymptotically quasi-nonexpansive mappings. In 1972, the concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [3]. \( T \) is said to be asymptotically nonexpansive if there exists a sequence \( \{k_n\} \) in \([1, \infty)\) with \( \lim_{n \to \infty} k_n = 1 \) such that \( d(T^n x, T^n y) \leq k_n d(x, y) \) for all \( x, y \in K \) and \( n \in \mathbb{N} \).

The concept of asymptotically quasi-nonexpansive mappings is given by [12]. \( T \) is said to be asymptotically quasi-nonexpansive if there exists a sequence \( \{k_n\} \) in \([1, \infty)\) with \( \lim_{n \to \infty} k_n = 1 \) such that \( d(T^n x, p) \leq k_n d(x, p) \) for all \( x \in K \), \( p \in F(T) \) and \( n \in \mathbb{N} \).

Liu Qihou [6, 7, 8] proved some necessary and sufficient condition for convergence of Ishikawa iterative sequence and Ishikawa iterative sequence with errors to fixed points of asymptotically quasi-nonexpansive mappings in Banach space and in uniformly convex Banach spaces.

In 1974, Kirk [5] introduced asymptotically nonexpansive type mapping, where \( T \) satisfies : \( \limsup_{n \to \infty} \left( \sup_{x \in K} \{ d(T^n x, T^n y) - d(x, y) \} \right) \leq 0 \) for each \( y \in K \) which may hold even if none of the iterates of \( T \) is Lipschitzian.

In this paper, we first introduce concept of asymptotically quasi-nonexpansive type mapping in complete convex metric space. We also define three-step iteration process with errors and give some necessary and sufficient condition for convergence of iterative scheme to fixed point of the asymptotically quasi-nonexpansive type mappings in convex metric spaces.

### 2 Preliminaries

The concept of a convex metric space was introduced by Takahashi [14].

**Definition 2.1.** Let \((X, d)\) be a metric space, \( I = [0, 1] \), and \( \alpha, \beta, \gamma \) be real sequences in \( I \) with \( \alpha + \beta + \gamma = 1 \). A mapping \( W : X^3 \times I^3 \to X \) is said to be a convex structure on \( X \) if for all \( (x, y, z, \alpha, \beta, \gamma) \in X^3 \times I^3 \) the following condition is satisfied

\[
d(u, W(x, y, z, \alpha, \beta, \gamma)) \leq \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z),
\]

for all \( u \in X \). The metric space \( X \) with a convex structure \( W \) is called a convex metric space and denote it \((X, d, W)\).

A nonempty subset \( K \) of \( X \) is said to be convex if \( W(x, y, z, \alpha, \beta, \gamma) \) belongs to \( K \) for all \( x, y, z \in X \) and \( \alpha, \beta, \gamma \in [0, 1] \).
It is important to note that each linear normed space is a special example of convex metric space, but there exists some convex metric spaces which cannot be embedded into any normed spaces [14].

Now, we define the concept of asymptotically quasi-nonexpansive type mapping in metric spaces:

**Definition 2.2.** Let a map $T : K \rightarrow K$, $F(T) \neq \emptyset$ be asymptotically quasi-nonexpansive type if for each $p \in F(T)$,

$$\limsup_{n \to \infty} \left[ \sup_{x \in K} \{ d(T^n x, p) - d(x, p) \} \right] \leq 0. \tag{2.1}$$

**Remark 2.1.** It is easy to see that quasi-nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive type mappings are special cases of asymptotically quasi-nonexpansive type mappings.

In 2000, Noor [9] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces by using the techniques of updating the solution and auxiliary principle. Glowinski and Le Tallec [2] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown [2] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. Thus we conclude that three-step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences.

Recently, Xu and Noor [16] introduced and studies a three-step iterative scheme to approximate fixed points of asymptotically nonexpansive self mappings in Banach space.

Now, we define three-step iteration process with errors for asymptotically quasi-nonexpansive type mappings in convex metric spaces as follows:

**Definition 2.3.** Let $(X, d, W)$ be a convex metric space, $K$ be nonempty closed convex subset of $X$, and $T : K \rightarrow K$ be an asymptotically quasi-nonexpansive type mapping. Let $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ be sequences in $[0, 1]$ with $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ and $\sum_{i=1}^{\infty} \gamma_n^{(i)} < +\infty$, $1 \leq i \leq 3$, $n = 0, 1, 2, \ldots$. For any given $x_0 \in K$, we define three-step iteration process with errors $\{x_n\}$ by

$$\begin{cases}
x_{n+1} = W(Ty_n, x_n, u_n^{(1)}, \alpha_n^{(1)}, \beta_n^{(1)}, \gamma_n^{(1)}), \\
y_n = W(Tz_n, x_n, u_n^{(2)}, \alpha_n^{(2)}, \beta_n^{(2)}, \gamma_n^{(2)}), \\
z_n = W(Tx_n, x_n, u_n^{(3)}, \alpha_n^{(3)}, \beta_n^{(3)}, \gamma_n^{(3)}),
\end{cases} \tag{2.2}$$

where $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$, $\{u_n^{(3)}\}$ are bounded sequences in $K$. 

...
Since \( \{u_n^{(1)}\}, \{u_n^{(2)}\}, \{u_n^{(3)}\} \) are bounded sequences in \( K \), we can put
\[
M = \sup_{n \geq 1} d(u_n^{(1)}, p) \lor \sup_{n \geq 1} d(u_n^{(2)}, p) \lor \sup_{n \geq 1} d(u_n^{(3)}, p),
\]
where \( a \lor b = \max\{a, b\} \). Then \( M \) is a finite number.

**Lemma 2.1.** [13] Suppose that \( \{a_n\} \) and \( \{b_n\} \) are two sequences of nonnegative numbers such that \( a_{n+1} \leq a_n + b_n \), for all \( n \geq 1 \). If \( \sum_{n=1}^\infty b_n \) converges, then \( \lim_{n \to \infty} a_n \) exists.

## 3 Main Results

In this section, we give some necessary and sufficient conditions for convergence of three-step iteration processes with errors to fixed point of asymptotically quasi-nonexpansive type mappings in convex metric spaces.

**Lemma 3.1.** Let \((X, d, W)\) be a complete convex metric space, \( K \) be a closed convex subset of \( X \), and \( T : K \to K \) be a mapping of asymptotically quasi-nonexpansive type with \( F(T) \neq \emptyset \). Let the iteration process \( \{x_n\} \) be defined by (2.2). If \( p \in F(T) \), then \( \lim_{n \to \infty} d(x_n, p) \) exists.

**Proof.** Put,
\[
A_n = \sup_{x \in K} [d(T^n x, T^n y) - d(x, y)] \lor 0, \quad \text{for all } n \geq 1,
\]
so that \( \sum_{n=1}^\infty A_n < +\infty \). Now,
\[
d(x_{n+1}, p) = d(W(T^n y_n, x_n, u_n^{(1)}, \alpha_n^{(1)}, \beta_n^{(1)}, \gamma_n^{(1)}), p)
\]
\[
= \alpha_n^{(1)} d(T^n y_n, p) + \beta_n^{(1)} d(x_n, p) + \gamma_n^{(1)} d(u_n^{(1)}, p)
\]
\[
\leq \alpha_n^{(1)} [d(T^n y_n, p) - (y_n, p)] + \alpha_n^{(1)} d(y_n, p) + \beta_n^{(1)} d(x_n, p) + \gamma_n^{(1)} d(u_n^{(1)}, p)
\]
\[
\leq \alpha_n^{(1)} \sup_{x \in K} [d(T^n x, p) - (x, p)] + \alpha_n^{(1)} d(y_n, p) + \beta_n^{(1)} d(x_n, p) + \gamma_n^{(1)} d(u_n^{(1)}, p)
\]
\[
\leq \alpha_n^{(1)} A_n + \alpha_n^{(1)} d(y_n, p) + \beta_n^{(1)} d(x_n, p) + \gamma_n^{(1)} M, \tag{3.1}
\]
\[
n d(y_n, p) = d(W(T^n z_n, x_n, u_n^{(2)}, \alpha_n^{(2)}, \beta_n^{(2)}, \gamma_n^{(2)}), p)
\]
\[
\leq \alpha_n^{(2)} d(T^n z_n, p) + \beta_n^{(2)} d(x_n, p) + \gamma_n^{(2)} d(u_n^{(2)}, p)
\]
\[
\leq \alpha_n^{(2)} A_n + \alpha_n^{(2)} d(z_n, p) + \beta_n^{(2)} d(x_n, p) + \gamma_n^{(2)} M, \tag{3.2}
\]
and
\[
n d(z_n, p) \leq \alpha_n^{(3)} A_n + (\alpha_n^{(3)} + \beta_n^{(3)}) d(x_n, p) + \gamma_n^{(3)} d(u_n^{(3)}, p)
\]
\[
\leq \alpha_n^{(3)} A_n + d(x_n, p) + \gamma_n^{(3)} M. \tag{3.3}
\]
Substituting (3.3) into (3.3), we have
\[
d(y_n, p) \leq \alpha_n^{(2)} A_n + \alpha_n^{(2)} \left[ \alpha_n^{(3)} A_n + d(x_n, p) + \gamma_n^{(3)} M \right] \\
+ \beta_n^{(2)} d(x_n, p) + \gamma_n^{(2)} M \\
\leq \alpha_n^{(2)} (1 + \alpha_n^{(3)}) A_n + \left( \alpha_n^{(2)} + \beta_n^{(2)} \right) d(x_n, p) \\
+ \alpha_n^{(2)} \gamma_n^{(3)} M + \gamma_n^{(2)} M \\
\leq \alpha_n^{(2)} (1 + \alpha_n^{(3)}) A_n + d(x_n, p) + \left( \alpha_n^{(2)} \gamma_n^{(3)} + \gamma_n^{(2)} \right) M.
\] (3.4)

Substituting (3.4) into (3.1), we have
\[
d(x_{n+1}, p) \leq \alpha_n^{(1)} A_n + \alpha_n^{(1)} \left[ \alpha_n^{(2)} (1 + \alpha_n^{(3)}) A_n + d(x_n, p) + \left( \alpha_n^{(2)} \gamma_n^{(3)} + \gamma_n^{(2)} \right) M \right] \\
+ \beta_n^{(1)} d(x_n, p) + \gamma_n^{(1)} M \\
= \alpha_n^{(1)} (1 + \alpha_n^{(2)} (1 + \alpha_n^{(3)})) A_n + \left( \alpha_n^{(1)} + \beta_n^{(1)} \right) d(x_n, p) \\
+ \left[ \alpha_n^{(1)} \alpha_n^{(2)} \gamma_n^{(3)} + \alpha_n^{(1)} \gamma_n^{(2)} + \gamma_n^{(1)} \right] M \\
\leq \alpha_n^{(1)} (1 + \alpha_n^{(2)} (1 + \alpha_n^{(3)})) A_n + d(x_n, p) \\
+ \left[ \alpha_n^{(1)} \alpha_n^{(2)} \gamma_n^{(3)} + \alpha_n^{(1)} \gamma_n^{(2)} + \gamma_n^{(1)} \right] M \\
= d(x_n, p) + B_n,
\] (3.5)

where \( B_n = \alpha_n^{(1)} (1 + \alpha_n^{(2)} (1 + \alpha_n^{(3)})) A_n + \left[ \alpha_n^{(1)} \alpha_n^{(2)} \gamma_n^{(3)} + \alpha_n^{(1)} \gamma_n^{(2)} + \gamma_n^{(1)} \right] M \). Since \( \sum_{n=1}^{\infty} A_n < +\infty \), we see that \( \sum_{n=1}^{\infty} B_n < +\infty \). By Lemma 2.1, \( \lim_{n \to \infty} d(x_n, p) \) exists.

**Lemma 3.2.** Let \((X, d, W)\) be a complete convex metric space, \(K\) be a closed convex subset of \(X\), and \(T : K \to K\) be a mapping of asymptotically quasi-nonexpansive type with \(F(T) \neq \emptyset\). Let the iteration process \(\{x_n\}\) be defined by (2.2). If \(\lim_{n \to \infty} d(x_n, F(T)) = 0\), then \(\{x_n\}\) is Cauchy, where \(d(x_n, F(T))\) denotes the distance from the point \(x_n\) to the set \(F(T)\).

**Proof.** Since \(\lim_{n \to \infty} d(x_n, F(T)) = 0\), for all \(\varepsilon > 0\) there exists \(k(\varepsilon) \in \mathbb{N}\) such that for all \(n \geq k(\varepsilon)\)
\[
d(x_n, F(T)) < \frac{\varepsilon}{2}
\] (3.6)

This implies that there exist \(p \in F(T)\) such that for all \(n \geq k(\varepsilon)\)
\[
d(x_n, p) < \frac{\varepsilon}{2}
\] (3.7)
Since the sequence \( \{d(x_n, p)\} \) is nonincreasing, we have for \( m, n \geq k(\varepsilon) \) that
\[
d(x_n, x_m) \leq d(x_n, p) + d(x_m, p) = 2d(x_k(\varepsilon), p) < \varepsilon
\]
which shows that \( \{x_n\} \) is Cauchy.

**Theorem 3.3.** Let \((X, d, W)\) be a complete convex metric space, \(K\) be a closed convex subset of \(X\), and \(T : K \to K\) be a mapping of asymptotically quasi-nonexpansive type with \(F(T)\) being a nonempty closed set. Let the iteration process \(\{x_n\}\) be defined by (2.2). Then

1. \(\lim_{n \to \infty} d(x_n, F(T)) = 0\) if \(\{x_n\}\) converges to a fixed point of \(T\).
2. \(\{x_n\}\) converges to a point in \(F(T)\) if \(\lim_{n \to \infty} d(x_n, F(T)) = 0\).

**Proof.** (1) Since \(F(T)\) is closed and the map \(x \mapsto d(x, F(T))\) is continuous,
\[
\lim_{n \to \infty} d(x_n, F(T)) = d(\lim_{n \to \infty} x_n, F(T)) = 0.
\]
(2) From Lemma 3.2, \(\{x_n\}\) is a Cauchy sequence, so \(\{x_n\}\) converges to a point, say \(p\) in \(K\). Since \(F(T)\) is closed,
\[
\lim_{n \to \infty} d(x_n, F(T)) = d(\lim_{n \to \infty} x_n, F(T)) = 0
\]
implies that \(p \in F(T)\).

**Theorem 3.4.** Let \((X, d, W)\) be a complete convex metric space, \(K\) be a closed convex subset of \(X\), and \(T : K \to K\) be a mapping of asymptotically quasi-nonexpansive type with \(F(T)\) being a nonempty closed set. Let the iterative process \(\{x_n\}\) be defined by (2.2). Then \(\{x_n\}\) converges to a point in \(F(T)\) if and only if \(\liminf_{n \to \infty} d(x_n, F(T)) = 0\).

**Proof.** The necessity of the condition is obvious. Thus, we will only prove the sufficiency.

For any \(n \in \mathbb{N}\) and \(p \in F(T)\), from (3.5) we have
\[
d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + B_n,
\]
\[
\sum_{n=1}^{\infty} B_n < +\infty.
\] By Lemma 2.1, we get that \(\lim_{n \to \infty} d(x_n, F(T))\) exists. It follows from
\[
\liminf_{n \to \infty} d(x_n, F(T)) = 0
\]
that
\[
\lim_{n \to \infty} d(x_n, F(T)) = 0.
\]
From Theorem 3.3, \(\{x_n\}\) converges to a point in \(F(T)\).
Remark 3.1. Theorem 3.3 and Theorem 3.4 partially extend and generalize Theorem 1.1 of Petryshyn and Williamson [10], Theorem 3.1 of Ghosh and Debnath [4], Theorem 2.1 of Chang, Cho and Zhou [1], Theorem 1 and 2 of Tian [15] to the case of the larger class of mapping, iteration process and the convex metric space considered here.

Let \((X, d, W)\) be a complete convex metric space and \(K\) be a closed convex subset of \(X\). A mapping \(T : K \to K\) is said to satisfy condition (A) if there is a nondecreasing function \(f : [0, \infty) \to [0, \infty)\) with \(f(0) = 0, f(r) > 0\) for all \(r \in (0, \infty)\) and \(d(x, Tx) \geq f(d(x, F(T)))\) for all \(x \in K\), where \(d(x, F(T)) = \inf\{d(x, z) : z \in F(T)\}\).

Condition (A) is originally introduced by Senter and Dotson [11].

Theorem 3.5. Let \((X, d, W)\) be a complete convex metric space, \(K\) be a closed convex subset of \(X\), and \(T : K \to K\) be a mapping of asymptotically nonexpansive type with \(F(T)\) being a nonempty closed set. Let the iterative process \(\{x_n\}\) be defined by (2.2). Let \(\{x_n\}\) be an approximate fixed point sequence for \(T\), that is, \(\lim_{n \to \infty} d(x_n, Tx_n) = 0\). Assume that \(T\) satisfies condition (A). Then \(\{x_n\}\) converges strongly to a fixed point of \(T\).

Proof. Let \(p \in F(T)\). Then the sequence \(\{d(x_n, p)\}\) is nonincreasing, so is \(\{d(x_n, F(T))\}\). Thus \(\lim_{n \to \infty} d(x_n, F(T)) = r\) for some \(r \geq 0\).

Now in view of Theorem 3.4, to complete the proof we must show that \(r = 0\). From condition (A), we have

\[
d(x_n, Tx_n) \geq f(d(x_n, F(T))) \geq f(r).
\]

Since \(\lim_{n \to \infty} d(x_n, Tx_n) = 0\), we get that \(f(r) = 0\) and so \(r = 0\). This completes the proof.

Remark 3.2. Theorem 3.5 generalizes theorem 3 of Liu [6], Theorem 3 of Tan and Xu [13] and Theorem 3 of Zeng [17] to the case of the iteration process (2.2), the asymptotically quasi-nonexpansive type mapping and the convex metric space considered here.

References


[5] W. A. Kirk, Fixed point theorems from non-Lipschitzian mappings of

[6] Q. Liu, Iterative sequences for asymptotically quasi-nonexpansive maps-


[8] Q. Liu, Iterative sequences for asymptotically quasi-nonexpansive mappings
Appl.* **266** (2002), 468-471.

[9] M. A. Noor, New approximation schemes for general variational inequali-

[10] W. V. Petryshyn and T. E. Williamson, Strong and weak convergence of the
sequence of successive approximations for quasi-nonexpansive mappings, *J.

[11] H. F. Senter and W. G. Dotson, Jr., Approximating fixed points of nonex-

families of mappings, in *Nonlinear analysis and applications (St. Johns,
(1982).

[13] K. K. Tan and H. K. Xu, Approximating fixed points of nonexpansive map-
301-308.

[14] W. Takahashi, A convexity in metric spaces and nonexpansive mappings-I,

[15] Y. X. Tian, Convergence of an Ishikawa type iterative scheme for asymptoti-
1905-1912.

[16] B. Xu, and M. A. Noor, Fixed-point iterations for asymptotically non-
444-453.

[17] L. C. Zeng, A note on approximating fixed points of nonexpansive mappings
250.