A Two-Layer Approach for Tailings in Open Channels

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Abstract

In a number of locations around the world the tailings storage area is sufficiently lower than the mineral processing plant to allow disposal by gravity. Most past efforts to represent open channel flows have been empirical and have focused on representing the solid phase as a viscosity problem for which the Manning equation could be used. Empirical equations developed for stormwater flows are difficult to use for tailings as they were often developed for cases of solids that do not exceed few thousand parts per million.

As large tailings systems develop to transport solids with volumetric concentration in the range of 20% to 30% different forces interact. The coarsest particles move as a bed load in the bottom layer, while the fines move in a suspended mode in the upper layer. Bed forms develop at the interface between the two layers that include ripples, dunes, anti-dunes and flat planes. With large volumes of solids, a Coulombic force develops that is independent of speed but must be accounted for in friction losses while collisions between particles give rise to the Bagnold stress. The proposed two-layer approach provides a tool to examine the combination of these forces for design of long distance tailings transport in open channels.

Introduction

The size of metal concentrators and oilsand processing plants has grown tremendously over the last few decades. Copper concentrators may dispose 80,000 metric tonnes of dry solids per day as waste or tailings. Oilsand plants handle up to 150,000 tonnes per day. The high cost of water associated with these efforts pushes the concentration of slurry mixtures to 55% – 70% by weight. When the tailings storage facility is located at a much lower elevation than the mineral processing plant, or when a pit of an oilsand operation is converted to tailings storage, slurry transport by gravity in open channels becomes an economic option.

It is therefore a challenge for the design engineer to combine the different forces that influence the flow regime.The equilibrium of all these forces ultimately establishes the parameters for design of open channels for highly concentrated tailings. A new equation is proposed to account for shear stress, dune resistance, Coulombic friction, Bagnold stress, lift, dissipative and centrifugal forces in the final calculation of slope in steady state uniform motion.

1.0 PRINCIPLES OF THE APPROACH

In the conventional problem for transport of sediments in rivers and sewers, the bed of solids is considered to be fairly shallow compared to the depth of the liquid. The active bed thickness is often considered to be the thickness of two d_{90} diameters plus the height of the bedform.

In the approach we are proposing in this paper, the bottom layer is much thicker and transports the fraction of solids coarser than 74 µm, while fine solids are transported in suspension in the upper layer. The interface between the two layers may take on a wavy shape called bedforms such as ripples, dunes, antidunes (upstream migrating or downstream migrating types), or may be simply a flat plane.

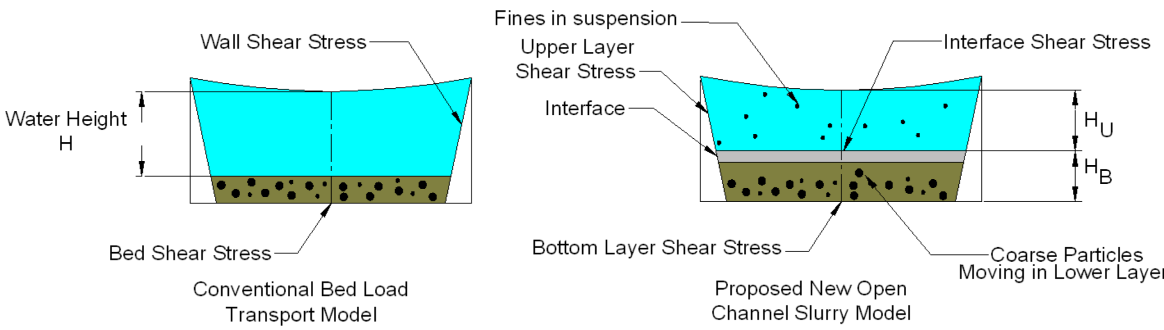


Figure 1: Comparison between conventional models of sediment transport in open channel and proposed approach for open channel flow of coarse tailings at high concentration

Figure (1) presents a comparison between the conventional model for sediment transport and the approach to be discussed in this paper. In each layer density takes a different value, the conservation of mass is expressed by equation (1) across a plane

$$\rho_m A_m U_m = \rho_U A_U U_U + \rho_B A_B U_B \quad (1)$$

$$A_m = A_U + A_B \quad (2)$$

with ρ the density, A sectional area of mixture, U average velocity in the channel, subscript m refers to total mixture, subscript B to bottom layer and U to upper layer.

Forces and stresses develop in each layer and are listed in table (1) with a graphical representation in Figure (2). In this paper, steady state flow is assumed.

Table (1) Characteristics of flow in bottom and upper layers

Bottom Layer of stratified coarse	**Upper Layer of Suspended Sediments**
❖ Component of Bed Weight in plane of inclination for bottom layer (F_{GB})	❖ Component of Weight in plane of inclination for upper layer (F_{GU})
❖ Bed Shear stress (τ_{vB})	❖ Interfacial Shear stress (τ_I)
❖ Drag force due to vegetation in bed (D_v)	❖ Viscous shear stress against the walls τ_{wU}
❖ Columbic friction force (F_c)	❖ Shear stress due to bedform (τ_f)
❖ Bagnold Dispersive Stress (τ_{BA})	❖ Turbulent Dispersive Stress (τ_{dispU})
❖ Turbulent Dispersive Stress (τ_{dispB})	❖ Centrifugal force (CF_U)
❖ Near Wall Lift Force (L_p)	❖ Pressure losses due to fittings ΣF_{fU}
❖ Centrifugal force (CF_B)	❖ Entrainment Force acting on lower layer

	F_E
❖ Pressure losses due to fittings ΣF_{fB}	
❖ Entrainment force from upper layer F_E	

2.0 DYNAMICS OF THE LOWER LAYER

Due to the difference in average velocity between the upper and bottom layer, an entrainment force is assumed to exist to transfer energy from the upper to the bottom layer. Since the interface between the bottom and the top layer is assumed to be made of bedforms, the dune length λ is assumed to be the characteristic length.

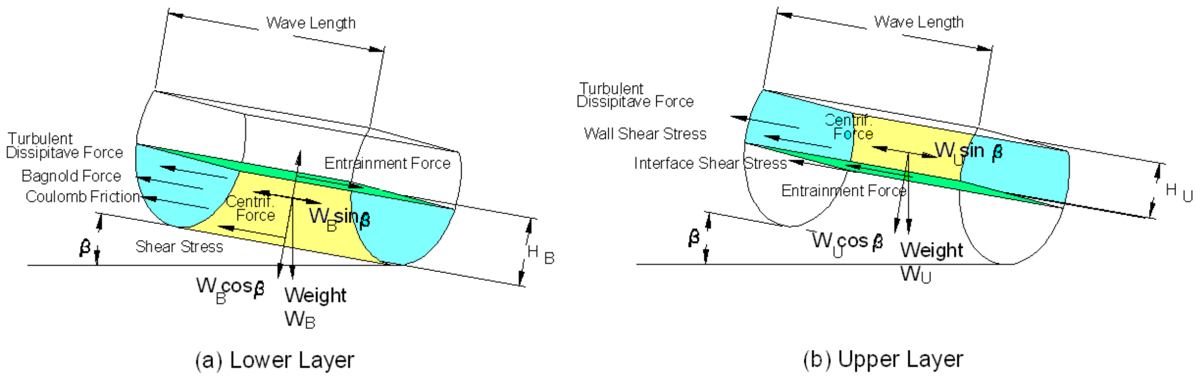


Figure 2: Concept of forces in steady motion for each layer – The upper layer exerts an entrainment force FE on the lower layer that is seen as resistance by the upper layer.

2.1 Incipient Motion

The flow of cohesionless particles can be divided into three forms of transport

- bed load transport
- suspended load
- wash load

In the bed load transport form, the coarser particles are jumping, rolling, moving in a regime called "saltation" often forming dunes and anti-dunes. Bedforms form at the interface of bottom and top layers with a wave length λ and an amplitude Λ.The wave length λ is assumed to be the characteristic length.

The transition from a stationary state to the formation of a bed load is often called "incipient motion". A number of equations have been developed to account for the effects of viscosity, particle size, density of solids, density of liquid, hydraulic radius of flow, hydraulic radius of bed, and bed shear stress. Tailings systems operate well above the regime of incipient motion, but we will discuss this regime briefly. One method widely accepted is based on the Shields Curve which defines the limit between possible and impossible motion.

2.1.1 Non-dimensional Particle Diameter

The Van Rijn approach (1984) as discussed by Bertrand-Krajewski (2006) calculates the critical mobility parameter θ_{crit} as a function of the non-dimensional particle diameter D*.

$$D^* = d_{50}\left(\frac{g(\rho_s - \rho_l)}{\nu^2 \rho_l}\right)^{1/3} \tag{3}$$

Where d_{50} is the particle size diameter passing 50% of the solids, ν is the kinematic viscosity, ρ_s is the particle density and ρ_l is the liquid density.

2.1.2 Critical Shear (or friction) Velocity

The critical mobility parameter θ_{crit} (also sometimes called the critical shear parameter) is defined as

$$\theta_{crit} = \frac{u^{*2}_{crit}}{d_{50B}\, g(\rho_s / \rho_l - 1)} \tag{4}$$

From the Van Rijn approach (1984a)' the curve θ_{crit} is formulated by five equations as per table (2).

Table (2) Equations for Critical Mobility parameter

Value of Non-Dimensional Diameter D*	critical mobility parameter θ_c
D*<4	$\theta_{crit} = 0.24/D^*$
4<D*<10	$\theta_{crit} = 0.14/(D^*)^{0.64}$
10< D*<20	$\theta_{crit} = 0.04/(D^*)^{0.10}$
20< D*<150	$\theta_{crit} = 0.013\,(D^*)^{0.29}$
150< D*	$\theta_{crit} = 0.055$

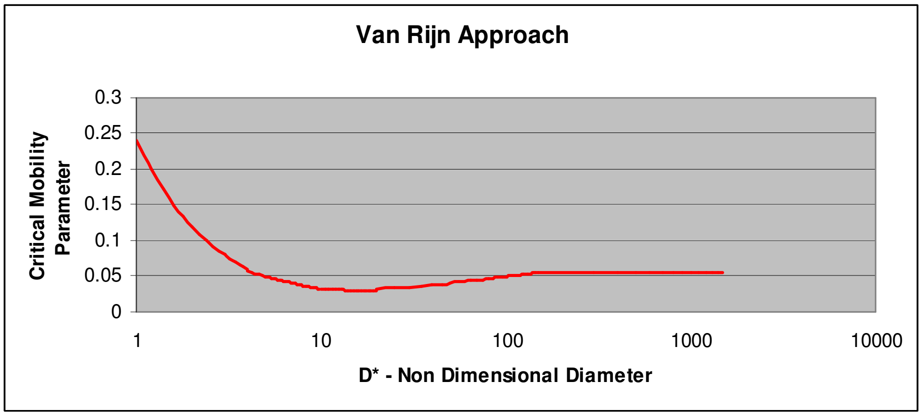


Figure (3) The Critical Mobility Parameter after the Van Rijn Equations

The critical shear (or friction) velocity u^*_{crit} can then be calculated from the critical mobility parameter.

2.2 Concepts of Critical Speed

A number of empirical equations are available for critical speed in slurry launders.

Dominguez (1996) self-cleaning speed developed on copper tailings

$$V_D = 1.833\sqrt{\left(8gR_H\frac{\rho_s-\rho_m}{\rho_m}\right)}\left[\frac{d_{85}}{R_H}\right]^{0.158} 1.2^{(3100/Re^*)} \quad (5)$$

With Reynolds Number Re*= $\frac{\rho_m R_H(\sqrt{gR_H})}{\mu}$

R_H= hydraulic radius,d_{85}= 85% passing diameter for particles,ρ_m=density of slurry mixture, ,ρ_s=density of solid particles,g=acceleration due to gravity (9.81 m/s^2),μ=viscosity of carrier fluid.

Novak & Nalluri (1975) proposed the following equation for critical Speed

$$V_c = 0.61\sqrt{\left[gd_p\frac{\rho_s-\rho_l}{\rho_l}\right]}\left[\frac{d_p}{R_H}\right]^{-0.27} \quad (6)$$

Particle size in mm- suitable for rectangular channels $0.01<d_p/R_H<1$,Circular channels $0.01<d_p/R_H<0.8$

Novak & Nalluri (1984) developed a Bed Critical Speed

$$V_{bc} = 0.54\sqrt{\left[gd_{PB}\frac{\rho_s-\rho_l}{\rho_l}\right]}\left[\frac{d_{PB}}{R_B}\right]^{-0.38} \quad (7)$$

R_B=Hydraulic Radius of Bed is defined as in the Einstein approximation.

2.4 Viscous Force

As the solids are transported in the bottom layer, viscous friction develops. The Swamee-Jaime friction factor (Lindeburg,1997) is modified by using the hydraulic radius to

$$f_{DB} = \left(\frac{0.25}{\{\log_{10}[0.0675k_B/R_{HB} + 5.74/(4R_{HB}\rho_l U_B/\mu_l)^{0.9}]\}^2}\right) \quad (8)$$

which is valid from the Reynolds Number $5000< (4R_{HB}\rho_l U_B/\mu_l) <100{,}000$, where U_B is the average speed in the bottom layer. The roughness k_B is traditionally taken as anywhere from $2d_{50}$ to $3d_{90}$.The viscous shear stress for the bottom layer is calculated as

$$\tau_B = \rho_B U_{fB}^2 = \frac{1}{2}\rho_B f_{NB} U_B^2 = \frac{1}{8}\rho_B f_{DB} U_B^2 \qquad (9)$$

Where U_{fB} is the friction velocity of the bottom layer, f_{NB} is the Fanning coefficient and f_{DB} is the Darcy-Weisbach or Moody coefficient. The resultant force within a narrow tailings conduit is based on an area covering the wet perimeter WP_B by the unit length λ. In section 3.4, a sidewall correction will be reviewed.

2.5 Coulombic Friction Force

At high solids concentrations in tailings, a certain percentage of the particles in the lower layer will not be fully suspended. A mechanical friction then develops between these particles and the walls of the open channel, called Coulombic Friction. Considering the buoyancy force, the submerged weight is used. Defining the mechanical friction factor as f_c, the Coulombic force for a horizontal flow can be expressed as

$$F_c = f_c\,(\rho_s - \rho_L)A_B\ \lambda.g.c_{BS} \qquad (10)$$

Where C_{BS} is the volumetric concentration of coarse solids in contact with the wall, also called contact load. In the Mohr-Coulomb model, the dry Coulombic friction factor is usually expressed as

$$f_c = \tan\phi \qquad (11)$$

Where ϕ is called the angle of repose or internal failure of a static granular body. For dry sand $f_c \approx 0.5$. Pudasaini et al (2005) discussed the importance of Coulombic friction in the flow of debris in open channels under steady and unsteady flows. They proposed modifications to the dry friction factor based on pore pressure distribution for mud. As centrifugal forces compress the solids, the pore pressure change and hence the friction factor for the Coulombic force computation will also change.

To evaluate the concentration of particles in an open-channel O'Brien (1933) proposed that the rate of transfer of solids upward must be in equilibrium with the downward exchange of momentum due to gravitational forces

$$V_t\, c_y = -\beta V' l_{mb} \frac{dc}{dy} = \varepsilon_s \frac{dc}{dy} \qquad (12)$$

V_t=terminal velocity of particle, l_{Bm} is the mixing length, V'=average of absolute values of fluctuations of velocity normal to the flow, β= correlation coefficient ≅1.0, c_y=concentration at height y from wall, c=concentration of particles, ε_s= mass transfer coefficient.

In pipe flow, the contact load is expressed by the Shook-Roco equation (SRC 2000) as

$$\frac{c_{BS}}{c_v} = e^{-\Gamma} \quad \text{where } \Gamma = 0.124 Ar^{-0.061}\left(\frac{U^2}{g d_p}\right)^{0.28}\left(\frac{d_p}{D_i}\right)^{-0.431}\left(\frac{\rho_s - \rho_l}{\rho_l}\right)^{-0.272} \qquad (13)$$

Where Ar is the Archimedean number defined in the following equation

$$Ar = \frac{4}{3\mu_l^2} d_p^3 g(\rho_s - \rho_l)\rho_l$$

Equation (13) applies for Ar<300,000. Modifying equation (13) for open channels, but it is proposed that

$$c_{bs} = c_v e^{-\Gamma} \text{ with } \Gamma = 0.124 Ar^{-0.061} \left(\frac{U^2}{gd_p}\right)^{0.28} \left(\frac{d_p}{4R_H}\right)^{-0.431} \left(\frac{\rho_s - \rho_l}{\rho_l}\right)^{-0.272} \quad (14)$$

Where the average velocity *U,* and hydraulic radius R_H are taken over the entire flow. Equation (14) needs confirmation by empirical data.

2.6 Lift Force

Wilson et al (2008) conducted an analysis on spherical particles and correlated the shear (friction) velocity with the lift force. The lift force occurs at walls with strong curvature such as pipes . Lift may be due to spinning of particles by the Magnus effect. Wilson et al proposed that the ratio of lift force to submerged weight of a sphere be expressed as

$$\frac{L}{((\rho_s - \rho_l)g} = C_L \left[\frac{(3/32) f_L U}{g(\rho_s ; \rho_l)/\rho_l}\right] \quad (15)$$

where C_L is the average lift coefficient.

The effect of the lift force is to reduce the resistance due to the Coulombic force by lifting off the solids. Data on lift coefficient for non-spherical particles such as those likely to be encountered in tailings is not readily available. Defining the stratification ratio in terms of energy gradient and volumetric concentration as

$\psi = \frac{\rho_l (i_m - i_l)}{c_v(\rho_s - \rho_l)}$, and the particle Reynolds Number based on friction velocity Re*= $\frac{\rho_j U_f d_{50}}{\mu_l}$

The experimental work of Wilson et al (2008) on spheres between 0.3 mm and 0.7 mm in pipe diameters between 0.1 m and 1m led to the following relationship: $\psi = 0.70(\text{Re}^*)^{0.33}$. Within the scope of the open channel problem we can state that the net lift force is obtained through reference to the wet perimeter WP_B and wave length λ

.

$$L_p = \xi C_{Lav} \rho_l f_{DB} U_B^2 WP_B \lambda \quad (16)$$

Where ξ is a function the particles size and their concentration at the wall and C_{Lav} is their average lift coefficient.

2.7 Bagnold Dispersive Force

The interactions of coarse solids in a moving fluid result in collisions. While the normal inter-granual stress is essentially due to the (weight-buoyancy) of the solids, at high speed a phenomenon develops resulting in a dispersive shear stress and a force called the Bagnold's dispersive force. Wang et al (2000) expressed the Bagnold stress as

$$\tau_{BA} = \alpha\rho_s \left(\frac{1}{\sqrt[3]{c_{v\,max} / c_v} - 1} \right)^2 d_{50c}^2 \left[\frac{du}{dy} \right]^2 \tag{17}$$

Where α is the Bagnold coefficient c_v is the mixture volumetric coefficient of particles undergoing collisions, with diameter d_{50c} for particles undergoing collisions, and c_{vmax} is the maximum volumetric concentration that can be achieved and (du/dy) is the shear rate.. Figure (4) shows a representation for Bagnold stress. For very fine tailings this stress is negligible. Johnson (1996) reported a value of 0.025 for sand while Mih 1993,Yadav 2008 reported values of 0.013 – 0.056 for different materials.

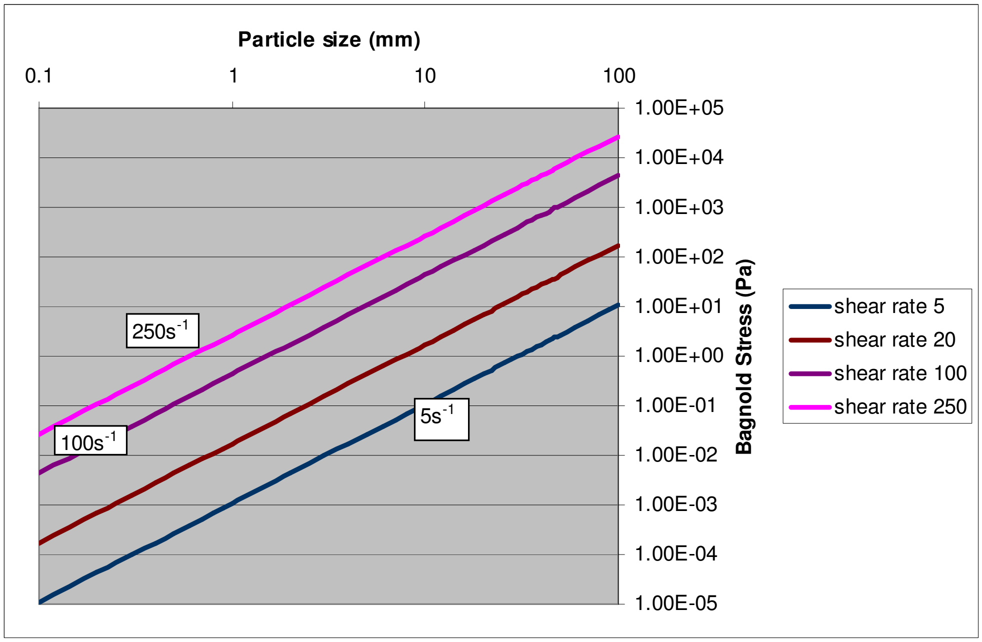


Figure (4) Bagnold Stress ,c_v=30%,c_{vmax}=70%,ρ_s=2700 kg/m^3,α=0.0011

In turbulent regime, the shear rate is established from the law of the wall for very dilute mixtures

$$\frac{du}{dy} = \frac{U_{fB}}{\kappa y} \tag{18}$$

Where κ is the von Karman constant, and y is the height above wall boundary, U_{fB} = friction velocity in the bottom layer. For water κ ≈ 0.405, but its value drops with concentrated slurry mixtures in open channels. The law of the wall is usually limited to 20% of the depth of the flow H for open channel. It is not known whether it applies well for hyper-concentrated mixtures.

2.8 Turbulent Dispersive Force

As particles try to stay in suspension they develop a turbulent dispersive force. Wang (2000) proposed an equation that can be applied to the lower layer The turbulent dispersive stress is expressed as

$$\tau_{dispB} = \rho_B l_{mB}^2 \left[\frac{du}{dy} \right]_{wB}^2 \tag{19}$$

With l_{Bm} is the mixing length, and the shear rate *(du/dy)* is taken at the wall of the bottom layer, as per equation (18). Takahashi (2007) explained that for debris flows with large particles, the mixing length can reach the size of particle diameter.

2.9 Dominating mode of friction

In certain flows collisions predominate, in others the particles are sufficiently spaced to avoid collisions.

Iverson and Lahusen (1993) argued that the different mechanism of Bagnold, viscous friction or collisions may dominate depending on the flow characteristics. They proposed to examine three important numbers

The Bagnold Number defined as

$$N_{Ba} = \frac{\rho_s d_p^2}{\mu} \frac{du}{dy} \sqrt{\frac{c_v^{1/3}}{(c_{v\,max}^{1/3} - c_v^{1/3})}} \tag{20}$$

The Savage Number

$$N_{Sa} = \frac{d_p^2}{c_v g H_B \tan\phi} \left(\frac{du}{dy}\right)^2 \tag{21}$$

The friction Number

$$N_{Fr} = \frac{\rho_s c_v g \; H_B \tan\varphi}{\left(\frac{du}{dy}\right)\mu} \tag{22}$$

Where φ is the internal angle of repose

Iverson and Lahusen (1993) proposed that

- the Bagnold stresses due to particles collision dominate over the viscous friction when the Bagnold Number N_{Ba}>450, but the viscous mode of friction dominates when N_{Ba}<40.
- Grain collisions dominate grain friction when the Savage Number N_{Sa}>0.1.
- viscous friction dominates when N_{Fr}>1400.

2.10 Centrifugal Force

The Centrifugal force is usually treated as problem of unsteady flow, as sharp turns cause changes of velocity. For a large radius turn, (typical radius> 50 pipe diameters), uniform flow is assumed and the centrifugal force for the control volume is expressed as

$$CF_B = \rho_B A_B \lambda \frac{U_B^2}{R_c} \tag{23}$$

2.11 Interlayer Entrainment Force

Since the upper layer is assumed to move at a higher average speed U_U than the lower layer at the speed U_B, it exerts an entrainment force F_E at the interface between the two layers. This force will be examined from the equilibrium of forces in the upper layer.

3.0 DYNAMICS OF THE UPPER LAYER

Because of the high speeds associated with the transport of tailings bedforms will most likely be encountered near the deposition speed and when the mixture is highly stratified.

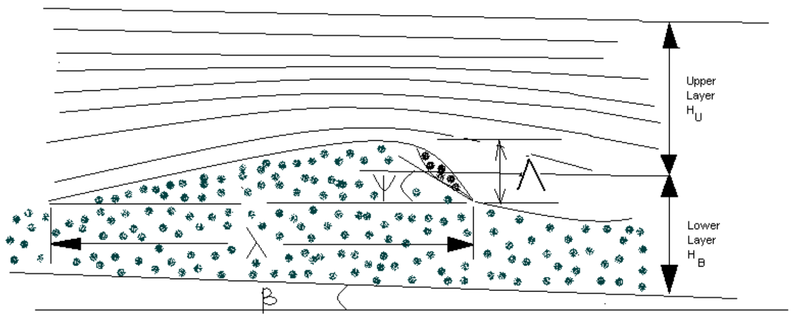


Figure (5) Parameters of Bedform showing length λ and amplitude Λ

3.1 Friction Factor for a Flat Interface

The Chezy Coefficient is an important parameter in open channel flows to determine friction losses. It is related to the Darcy-Weisbach friction factor or Fanning friction factor used in conventional fluid dynamics as

$$Ch = \left(\sqrt{8g / f_D}\right) = \left(\sqrt{2g / f_N}\right) \tag{24}$$

f_D = Darcy-Weisbach friction factor, f_N = Fanning friction factor

The Chezy number is dimensional. A non-dimensional equivalent often used in the literature is expressed as

$$c = \frac{U}{U_f} = \frac{Ch}{\sqrt{g}} \tag{25}$$

In conventional open-channel theory the velocity distribution above a flat bed follows a logarithmic profile. Considering the top of the bottom layer as a false bed on which the upper layer slides at an average speed U_U, the laws of sediment transport can be adapted

$$\frac{u(z)}{U_f} = \frac{1}{\kappa}\ln\left(\frac{z}{z_o}\right) \tag{26}$$

Where κ is the von Karman coefficient (typically 0.405), U_f = shear or friction velocity at H_B, u(z)= local velocity at an elevation z above the bed. z_o is the thickness of the bed and is often assumed to relate to the bed roughness k_s by $k_s=30z_0$

At the interface z = H_B, the velocity u(z) = U_i, (interface velocity). The profile of velocity in the upper layer is therefore expressed as

$$\frac{u_u(z)}{U_{fi}} = \frac{u_u(z)}{U_i} * \frac{U_i}{U_{fi}*} = \frac{1}{\kappa}\ln\left(\frac{z}{H_B}\right) + \frac{1}{\kappa}\ln\left(\frac{H_B}{z_o}\right) \tag{27}$$

Defining a value for z_0 at the height H_B, and as the roughness k_s= $30z_0$, equation (27) is rewritten as

$$\frac{u_u(z)}{U_{fi}} = \frac{1}{\kappa}\ln\left(\frac{z}{H_B}\right) + \frac{1}{\kappa}\ln\left(30\frac{H_{\mathbf{B}}}{k_s}\right) \tag{28}$$

The last term on the right of equation (28) is defined as the non-dimensional Chezy number at the interface of the two layers

$$c_i = \frac{1}{\kappa}\ln\left(\frac{30H_B}{k_s}\right) \tag{29}$$

The Chezy number at the interface is obtained from substituting equation (29) into equation (25)

$$Ch_i = \frac{\sqrt{g}}{\kappa}\ln\left(\frac{30H_B}{k_s}\right) = 7.735\ln\left(\frac{30H_B}{k_s}\right) \tag{30}$$

And substituting into equation (24), the Darcy and Fanning friction factors at the interface are derived to be

$$f_{DI} = \frac{1.32}{\left(\ln(30H_B/k_s)\right)^2} \tag{31a}$$

And

$$f_{NI} = \frac{0.328}{\left(\ln(30H_B/k_s)\right)^2} \tag{31b}$$

The calculated friction factor is used to compute the shear stress at the interface due to a flat interface.

$$\tau_{BI} = 0.125 f_{DI}\rho_U U_U^2 \tag{32}$$

The roughness k_s is traditionally taken as anywhere from $2d_{50}$ to $3d_{90}$ (e.g.van Rijn) but some authors have developed equations where the roughness increases with the bed concentration.

3.2 Friction Factor for Interface Dunes and Antidunes

A number of equations are available in the literature to account for presence of bedforms. Adapting the model developed by Yalin (1992) to the proposed two layer model yields the additional non-dimensional Chezy number for dunes:

$$\frac{1}{c^2} \approx \frac{1}{c_i^2} + \frac{1}{2}\frac{\Lambda}{H_U}\left(\frac{\lambda}{\Lambda}\right)^2 \tag{33}$$

This is in-line with Graf (1971) who proposed that the total friction factor be expressed as

$$f_{Dt} = f_{Di} + f_{DF} \tag{34}$$

Where f_{Dt} = total Darcy friction factor, f_{Di} = Darcy friction factor for bed without dunes, and f_{DF} is the Darcy friction factor due to the bedforms. Using Yalin's approach we define the bedform non-dimensional Chezy Number as:

$$\frac{1}{c_f} = \frac{\sqrt{g}}{Ch_f} = \sqrt{\frac{f_{Nf}}{2}} = \sqrt{\frac{f_{Df}}{8}} \approx \frac{\Lambda}{2H_U}\left(\frac{\lambda}{\Lambda}\right)^2 \tag{35}$$

Zhang,Y,(1999) reviewed in her thesis various methods to calculate the friction factor for large dunes. A number of equations were derived by Zhang, but the following equation, for maximum steepness is selected as it applies more to the range of tailings. It is modified to use the height of the upper layer as the forcing mechanism of dune formation with Z= H_U/d_{50}

$$\chi_{\max} = \left\{\frac{\Lambda}{\lambda}\right\}_{\max} = 0.04(1-e^{-0.0119Z})\left[\frac{0.5}{1+3(\log_{10}Z-2.8)^2}+1\right] \tag{36}$$

Vanoni and Hwang (1967) proposed that the increase in Darcy friction factor for bedforms be expressed as

$$\frac{1}{\sqrt{f_{DF}}} = 3.3\log_{10}\left(\frac{\lambda R_{HI}^2}{\Lambda^2}\right) - 2.3 \tag{37}$$

With R_{HI} as the bed hydraulic radius. The shear stress due to bedforms is then expressed as

$$\tau_{Bf} = 0.125 f_{DF}\,\rho_U U_U^2 \tag{38}$$

Recking et al (2009) offered a correlation between the shear stress and two-dimensional antidunes and derived the empirical equation below. They indicated that antidunes persisted at the following conditions

$\theta/\theta_{crit} < 100$ for gentle slope, this was also confirmed by Yelin (1992)

$\theta/\theta_{crit} < 20$ for steep slope

However Van Rijn put a limit at $\theta/\theta_{crit} = 25$ for bedforms based on his flume tests. The difference is due to the fact that Recking and Yalin focused on natural dunes, while van Rijn (1984) lab work would suggest that the narrow rigid boundaries limit bedforms.

3.3 Concepts of Froude Number

Kennedy(1960,1961,1963,1969) used potential flow theory to define the shape of the bedforms. The work of Kennedy assumed that the water layer moves above a rigid bed with sediment transport limited at the interface between bed and upper layer in the horizontal direction. He defined a Froude Number for the water layer above the sediment bed. Adapting Kennedy's Froude Number to the 2-layer model

$$Fr_U = \frac{U_i}{\sqrt{gH_U}} \tag{39}$$

For thin layers the interface celerity is often replaced by the average velocity U_U. When the bed is merely 1% - 4% of the flow height, average flow depth and velocity are used. This approach is incorrect when dealing with higher volumetric concentration of solids.

O.E. Sequeiros (2008) proposed that in the presence of density currents and accelerating flows in the upper layer to use a densimetric Froude Number should be used rather than the conventional Froude Number to characterize the division of bed forms.

$$Fr_d = \frac{U_U}{\sqrt{gc_v H_U(\rho_s/\rho_l - 1)}} \tag{40}$$

Table (3) presents a comparison between the conventional use of Froude Number in the layer of water above sediments and the use of the Sequerios densimetric Froude Number.

Table (3) Comparison between Kennedy water based Froude Number
and Sequerios Densimetric Froude Number

Traditional Froude Number based on liquid layer above bed	**Sequerios Densimetric Froude Number**
- Ripples in laminar flow with d_{50}<0.6 mm, wave length up to 1000 d_{50}	
$Fr_U < 0.88$, subcritical flow, dunes	For 0.46 < Frd < 0.97 , dunes form
- between $0.8<FR_U<0.95$ plane bed forms with substantial transport of material (Bennett 1997) – However with coarse material these plane bed forms may not develop and the dunes are followed straight by anti-dunes.	For 1.04 < Frd <Frdm , Upstream Migrating Dunes were observed
- between the Froude Number of 1.0 to 1.5 a regime of flow called "upper flow regime" develops (Bennett 1997)	For F_{rd}> Frdm and Fr_d> 1.04, Downstream Migrating Dunes were observed
- At higher Froude Numbers anti-dunes form until the bed is flattened out once again.	Antidunes persisted up to F_{rd}=3

Kennedy (1960) proposed an equation for three dimensional antidunes. It is adapted to the two-layer model to yield

$$U_i^2 \approx \frac{g\lambda}{2\pi}\left(\sqrt{1+\left(\frac{\lambda}{l_w}\right)^2}\right) \qquad (41)$$

Where l_w is the cross-sectional length of the antidune, and is often taken as the channel width *b*.Recently Nunez et al (2010) confirmed the validity Kennedy's model on streams of slurry up to a weight concentration of 45%.

3.4 Side Wall Friction

A correction factor is needed for viscous friction. Considering the basic three shapes, a wet perimeter for the side walls is derived in table (4).

As shown in Figure (6) an open channel formed of three sections in the upper layer. There are two side areas A_s under the influence of the wall friction and one central area A_c under the influence of the interface friction.

The hydraulic Radius is R_{HS} for the side section and R_{HC} for the central section. Chezy Law establishes a correlation between the flow and head loss per unit length S_f as

$$Q = Ac\sqrt{gR_H S_f} \qquad (42)$$

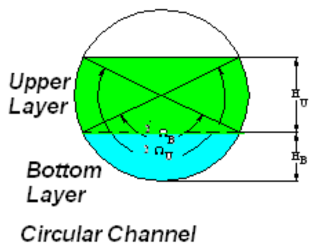
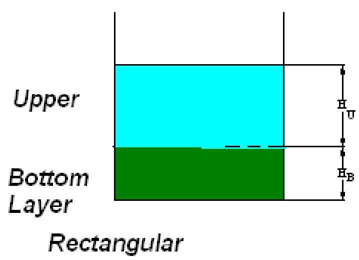
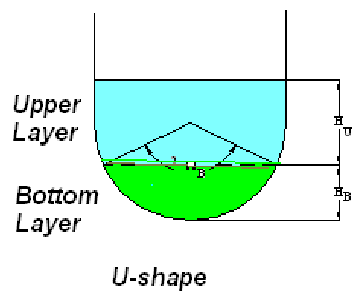
For each of the side section, a non-dimensional Chezy number c_s is defined. The flow into each side section is

$Q_s = c_s A_s \sqrt{gS_f R_{HS}}$ and for the central area $Q_I = c_I A_I \sqrt{gS_f R_{HI}}$

The total flow $Q = A_U U_U = Q_C + 2Q_s$ can be expressed as

$$c_T A_U \sqrt{R_{HU}} = c_i A_i \sqrt{R_{HI}} + 2c_s A_s \sqrt{R_{HS}} \qquad (43)$$

Table (4) Wet Perimeter for Sidewalls.

Circle	Rectangular	U-shape
		
$2\Omega_B$ is the angle formed by the bottom layer $2\Omega_U$ by the upper layer in a pipe Wet Perimeter of walls $= 2R(\Omega_U - \Omega_B))$	Wet Perimeter for side wall shear stress $2H_U$	$2\Omega_B$ is the angle formed by the bottom layer Wet Perimeter $2(H_u + H_B - R) + R(\pi - 2\Omega_B)$

Equation (43) establishes the correlation between the interface non-dimensional Chezy number c_i, the sidewall non-dimensional Chezy number c_s and the overall upper layer non dimensional Chezy number c_T in the absence of bedforms. An upper friction factor is then developed for the walls.

$$f_{DU} = \left(\frac{0.25}{\{\log_{10}[0.0675 k_B / R_{HS} + 5.74/(4R_{Hs}\rho_l U_B / \mu_l)^{0.9}]\}^2} \right)$$

The viscous shear stress for the bottom layer is calculated as

$$\tau_{wU} = \rho_B U_{fUW}^2 = 0.5 f_{NU} \rho_U U_U^2 = 0.125 f_{DU} \rho_U U_U^2 \quad (44)$$

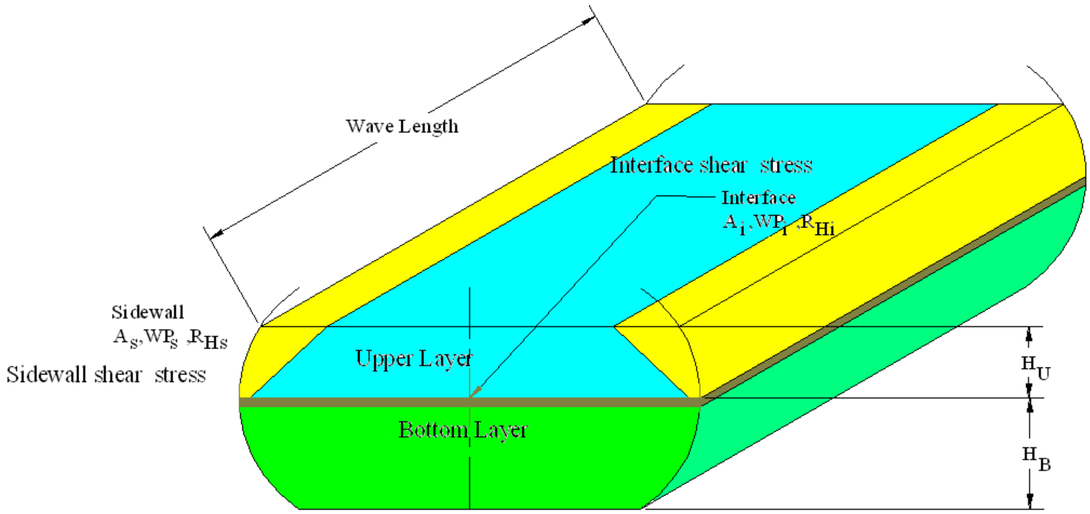


Figure (6) Concept of interface and wall shear stress

3.5 Turbulent Dispersive Force

An equation similar to (19) is developed for the upper layer

$$\tau_{dispU} = \rho_U l_{mU}^2 \left[\frac{du}{dy}\right]_{iU}^2 \quad (45)$$

Where l_{Bm} is the mixing length, and the shear rate *(du/dy)* is taken at the interface of the upper layer.

4.0 Equilibrium of Forces

The forces combine in the upper layer and lower layer to create an overall resistance. Ignoring second order interaction terms between these forces and assuming that they add up, the equilibrium of forces is established by the following equations.

4.1 Equilibrium of forces in Lower Layer

For the lower bed across a wavelength λ between point 1 and 2

$$F_{MB} + F_{GB} + F_{PB} + F_E = \tau_{vB} \lambda WP_B + f_c \,(W\cos\beta C_{BS} - \xi C_{Lav} f_{DB} U_B^2 WP_B \lambda\,) + CF_B + D_v + D_{ispB} + \Sigma F_{fB} \quad (46)$$

In steady flow, we ignore the force due to change of momentum F_{GB}*, and the force due to change of pressure* F_{BP}. If we assume that the centroids are at the same elevation at 1 and 2.If we ignore drag due to growth vegetation in natural channels, losses for fittings and if we substitute Equations (9),(10),(16),(17) (19),(23)equation (46) is simplified to

$$\rho_B A_B \lambda g \sin\beta + F_E = \frac{1}{8} f_{DB} \rho_B U_B^2 \lambda WP_B + f_c \left[(\rho_s - \rho_l) A_b \lambda g C_{BS} \cos\beta - \xi C_{Lav} \rho_l f_{DB} U_B^2 WP_B \lambda \right]$$

$$+ \left\{ \rho_B l_{mB}^2 + \alpha\rho_s \left(\frac{1}{\sqrt[3]{C_{vmax}/C_v} - 1} \right)^2 d_{50c}^2 \right\} \left[\frac{du}{dy} \right]^2 WP_B \lambda + \frac{\rho_B A_B \lambda U_B^2}{R_c} \qquad (47)$$

4.2 Equilibrium of forces in Upper Layer

At the upper layer we must account for the interface forces as well as wall stress.

$$F_{PU} + F_{GU} + F_{MU} = (\tau_{bI} + \tau_{bf}) \lambda b + 2\tau_{sw} H_U \lambda + C_U + D_{ispU} + \Sigma F_{fU} - F_E \qquad (48)$$

For steady uniform flow, momentum and pressure forces F_{MU} and F_{PU} are ignored, the centroids of the upper layer are assumed to be at the same elevation from the bottom of the channel.In the absence of fittings, and substituting equations (31),(37),(43) and (44) equation (47) can be written as:

$$\rho_U A_U \lambda g \sin\beta = \frac{1}{8}(f_{DI} + f_{DF}) \rho_u U_u^2 \lambda b + \frac{1}{4} f_{DU} \rho_U U_U^2 H_U \lambda + \rho_U A_U \lambda U_U^2 / R_c + \rho_U l_{mU}^2 \left[\frac{du}{dy} \right]^2 WP_U \lambda - F_E$$

The entrainment force is expressed as

$$F_E = \rho_U A_U \lambda g \sin\beta - \frac{1}{8}(f_{DI} + f_{DF}) \rho_u U_u^2 \lambda b - \frac{1}{4} f_{DU} \rho_U U_U^2 H_U \lambda - \rho_U A_U \lambda U_U^2 / R_c + \rho_U l_{mU}^2 \left[\frac{du}{dy} \right]^2 WP_U \lambda \qquad (49)$$

4.3 Equilibrium of forces for the complete flow

The forces in the upper and lower layers are obtained by substituting equation (49) into equation (46). Dividing by the characteristic length λ

$$\beta = \arcsin \left\{ \frac{1}{\rho_m A_m g} \left\{ \frac{1}{8}(f_{DI} + f_{DF}) \rho_U U_U^2 b + \frac{1}{4} f_{DU} \rho_U U_U^2 H_U + \frac{1}{8} f_{DB} \rho_B U_B^2 WP_B \right. \right.$$

$$+ f_c \left\{ (\rho_s - \rho_l) A_b g C_{BS} \cos\beta - \xi C_{Lav} \rho_l f_{DB} U_B^2 WP_B \right\} + \left\{ \rho_B l_{mB}^2 + \alpha\rho_s \left(\frac{1}{\sqrt[3]{C_{v\,max}/C_v} - 1} \right)^2 d_{50c}^2 \right\} \left[\frac{du}{dy} \right]^2 WP_B$$

$$\left. \left. + \rho_U l_{mU}^2 \left[\frac{du}{dy} \right]^2 WP_U + \frac{\rho_B A_B U_B^2}{R_c} + \frac{\rho_U A_u U_U^2}{R_c} \right\} \right\} \qquad (50)$$

At small angles of slope, sinβ≈tanβ and cosβ≈1. The bedform terms vanish at high shear rate.

Conclusion

Through the principle of conservation of mass and equilibrium of forces, it is possible to develop a set of equations for the slope of coarse cohesionless tailings that includes friction losses for the denser bottom layer, interface friction due to bedforms between upper and lower layers, Coulombic force, Bagnold dissipative friction and turbulent disspative friction, centrifugal force and to adjust for its influence by including terms of lift force. The final equation (50) indicates that there are areas needing field research such a measurement of lift for non-spherical particles, and dissipative force coefficients Through computational calculations, using an iterative approach, the proposed mathematical model provides a tool for the design of large long distance open channel tailings systems. The proposed approach will be extended in a separate paper to cover unsteady flow problems.

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Nomenclature

A	*Cross Sectional Area*	N_{SA}	Savage Number
b	*width of channel at the interface between layers.*	Q	Flowrate
c	*non-dimensional Chezy Coefficient*	q_B	solid load per unit of width
c_o	*non-dimensional Chezy Coefficient at the interlayer interface in the absence of bedforms*	R_c	Radius of curvature of channel
c_T	*non-dimensional Chezy Coefficient including sidewalls*	R_H	Hydraulic Radius
c_v	*Volumetric concentration of solids*	R_{HC}	Hydraulic Radius for central bed at the interface
C	*Chezy Number*	R_{HS}	Hydraulic Radius for side walls at the interface
C_{BS}	the volumetric concentration of coarse solids at wall	Re	Reynolds Number
C_D	*Drag Coefficient*	Re*	Reynolds Number based on or friction velocity
C_L	*Lift Coefficient*	Re	Reynolds Number
C_{Lav}	*Average Lift Coefficient*	S	Slope
Ch	*Dimensional Chezy Coefficient for total flow*	T	Transport Parameter
CF	*Centrifugal force*	U	Average Velocity
d_{50}	*Median particle size with 50% passing the diameter*	U_f	Friction velocity
d_{50c}	*Median particle size with 50% passing the diameter undergoing collisions*	u^*_{crit}	critical velocity for incipient motion
d_{85}	*particle size with 85% passing the diameter*	V_{CB}	Critical Speed for Asymmetric bed flow
d_{90}	*Particle size with 90% passing the diameter*	W	Weight
*D**	*Non-dimensional particle diameter*	WP	Wet Perimeter
D_{ispB}	*Bagnold Dispersive force*	z	depth measured from bottom of channel
D_v	*Drag due to vegetation in channel*	Z	Depth to particle diameter ratio
du/dy	*shear rate*		
f_c	*Coulomb Friction Coefficient*		***Greek Symbols***
f_D	*Darcy Weisbach friction factor*	α	Bagnold friction parameter
f_{disp}	*Bagnold Dispersive friction coefficient*	β	angle of inclination
F_L	*Durand Factor*	χ	steepness of bedform
f_N	*Fanning Friction Factor*		
F_{fB}	*force due to pressure loss across a fitting*	κ	Von Karman coefficient
F_{GU}	*force due to gravity in the direction of flow*	Λ	bedform amplitude
Fr	*Gilles Froude Number*	λ	bedform wave length
Fr_d	*Densimetric Froude Number*	μ	dynamic viscosity
Fr_U	*Kennedy Froude Number*	ν	kinematic viscosity
g	*acceleration due to gravity (9.8m/s²)*		
F_{GU}	*force due to gravity in the direction of flow*	θ	Shield Parameter
H	*Thickness of layer*	τ	shear stress
g	*acceleration due to gravity (9.8 m/s²)*	Ω	angle formed by wet perimeter in a pipe channel
H	*Height (Thickness) of layer*	ξ	Coefficient to calculate lift as a function of particle size and shape at the wall
k_b	bed roughness		***subcripts or superscripts***

k_s	Roughness coefficient at the interface due to flat bed	a	coefficient for Archimedean Number
L	lift	B	bottom layer
L_p	Lift due to particle at the wall	i	interface
l_m	mixing length for dispersive force	o	flat bed based
l_w	cross-length of 3D bedform	p	particle
N_{BA}	Bagnold Number	s	sidewall
N_{FR}	Friction Number	U	Upper Layer

Worked Example

In an oilsand application, open channel flow is considered to transport high grade tailings for in-pit disposal. Particle size distribution is presented in table (5). The open channel must transport 75,000 metric tonnes per day at a weight concentration of 60%. The solids density is 2.57 t/m^3 and the volume concentration is 36.87%. The total flow is 3250 m^3/h. The average slurry density is 1.58 t/m^3. Assume bends to be 50 times the channel width. Examine the effect if 5% of the solids consist of rejects at 20 mm diameter that may pass through the breakers.

The problem will be examined first by ignoring the presence of rejects.

To determine the volume of solids in the bed, it is assumed that particles larger than 74 μm will move in the bed, while the rest will be in the upper layer. Examining the particle size distribution of the cyclone underflow (fourth column in table 2), the -74 microns are screened off (figure 7) It appears that 7.7% of the particles will be in the upper layer.

The volume of solids in the bed is therefore 92.7% of the solids, or 1051.4 m^3/h. The maximum packing of solids for sand is 65%, but the bed in a first iteration is assumed to be 50%. The total volume of the bed is therefore 2103 m^3/h. The density of the bed is computed to be 1.85 t/m^3. The total mass flow of solids in the bed is 3890 kg/h.

In the upper layer, 239 t/h will be transported by 1000.7m^3/h of water. The total volume is 1089.27 m^3/h, mass 1239.91

t/h, density is 1.14 t/m^3.

Assuming a water viscosity of 1 cP, the viscosity is corrected for the presence of fines using the Landel equation to 1.4 cP due to a volumetric concentration of 7.7% of particles smaller than 74μm. The kinematic viscosity ν for the carrier is 1.39x10^{-6} m^2/s.

In the lower layer

- d_{50} which is found to be 260 μm, d_{85} which is found to be 570 μm , d_{90} which is found to be 1100 μm

Table (5) Particle Size Distribution from a coarse high grade tailings application

		Cumulative Passing			remove -74 microns	Γ	Contact load at the wall
mesh	microns	FEED	U/F	O/f			

8	2360	100	100	100		2.76	0
9	2000	100	100	100		3.20	0
10	1700	100	100	100		3.71	0
12	1400	100	100	100		4.41	0
14	1180	100	100	100		5.14	0
20	850	97.3	96.78	100	96.51	6.89	3.2868E-05
28	600	91.6	89.97	100	89.15	9.40	5.6153E-06
36	425	81.6	78.03	100	76.24	12.80	3.3007E-07
48	300	67.1	60.71	100	57.51	17.47	4.4625E-09
66	212	62.1	42.9	99.98	38.26	23.83	7.9361E-12
100	150	39.6	26.79	99.4	23.37	32.47	1.2716E-15
160	106	27.7	14.7	90.61	7.77	44.29	7.0281E-21
200	75	19.4	7.51	80.61	0.00	60.34	4.4598E-28
325	44	10.9	3.34	56.15		97.21	2.533E-44
	32	9	2.6	41.94		129.22	5.6014E-59
	20	7	1.95	33		196.71	2.4174E-88
	10	6.3	1.75	29.75		365.55	3.511E-162
	5	4.5	1.24	21.99		679.31	4.883E-298

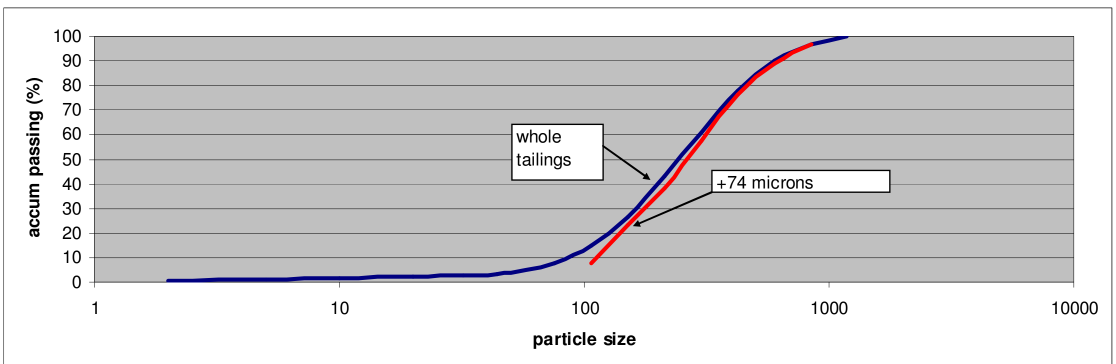

Figure (7) Particle distribution for worked example – the lower curve represents the bottom layer, and the upper curve the whole tailings

Applying equation (3) to the bottom layer D^* = 5.32. Using table (1), the non-dimensional critical mobility parameter is calculated from table (2), θ_c= 0.048. The critical shear velocity for incipient motion is calculated from equation (4) 0.0154 m/s. The corresponding critical shear stress for incipient motion is computed as 0.45 Pa.

A width of 750 mm is selected. The Dominguez equation is calculated to be 2.3 m/s. Calculations are done by iteration – the first will be illustrated here. In the first iteration the velocity for the bottom layer is assumed to be 2.2 m/s, average velocity of flow 2.33 m/s.

For the bottom layer, the height H_B=0.354 m, cross sectional area A_B=0.27 m^2, Hydraulic Radius R_{HB}=0.18 m. The wet perimeter WP_B=1.46 m

For the upper layer, the height H_U=0.153 m, average velocity 2.64 m/s, cross sectional area A_U=0.11 m^2, Hydraulic Radius R_{HU}=0.11 m. The wet perimeter WP_U=1.06 m

Total liquid height 0.507 m, total wet perimeter 1.76m, total Hydraulic radius 0.22 m, total flow area 0.38 m^2

Bottom Layer

Assuming a bed roughness of $2d_{90}$ or 1240 µm, the Darcy friction factor is calculated using equation (8) as 0.04. The resultant shear viscous shear stress is calculated using equation (9) as 45 Pa. The equivalent shear velocity is calculated to be 0.156 m/s.

The viscous force per unit length is obtained by multiplying the shear stress by the wet perimeter or $\tau_{vB} WP_B$ =66 N/m

Applying equation (14) for the concentration of solids at the wall, the contact load is tabulated in the last column of table (5).

The resultant Coulombic Force per unit length is calculated using equations (10),(11) and (14) to be 0.05 N/m.

Assuming a lift coefficient of 0.21, and that 10% of the particles are supported by the lift force, equation (16) yields -5.3 N/m.

The shear rate in the bottom layer is obtained by assuming a Von Karman coefficient of 0.35 and applying the law of the wall for 20% of the depth. It is calculated to be $4.4 s^{-1}$. Assuming a Bagnold coefficient of 0.013, the Bagnold stress is calculated to be 0.002 Pa and the resultant force per unit length 0.003 N/m. The Bagnold Number of 1.0 indicates minimum effect of collisions.

Assuming a mixing length of the order of d_{50}, the turbulent dispersive force per unit length is calculated from equation (19) to be 0.01N/m.

At a bend, the radius of curvature is 37.5 m. Applying equation (23) gives a force per unit length of 63.4 N/m.

Interface and Upper Layer

The ratio of shield number stress to critical shear number is calculated as 116. Hence applying the criteria that the ratio must be less than 100 for bedforms to persist, it is concluded that the interface will be flat.

From equation (31a), the Darcy friction factor at the interface is calculated to be 0.016.

The viscous force across the interface is obtained from equation (32) multiplied by 90% of the bed with to be 11 N/m.

The side walls are assumed to have a roughness of 70 µm. The wet perimeter = $2H_U$=0.31m – the resultant Darcy friction factor is 0.011. The resultant force per unit length is obtained by using equation (44) to be 3.4 N/m

The turbulent dispersive force in the upper layer is negligible.

If the launder goes through a bend with radius of 37.5 m, the centrifugal force per unit length of bend for the upper layer is computed to be 24.3 N/m.

Overall slope without rejects

The total force per unit length ignoring bends at the self cleaning speed is 77 N/m. The total density of the mixture is calculated to be 1607 kg/m^3. The total flow area is calculated to be 0.38 m^2.

Applying equation (50) $\beta = sin^{-1}\left(\frac{77}{1607 * 9.81 * .38}\right) = 1.28\%$

By adding the effects of centrifugal forces, the total force per unit length increases to 165N/m, or an equivalent slope of 2.75%.

Adding Rejects (no bends)

As rejects are added with 20 mm diameter, they develop on their own a non-dimensional particle size D=409 with critical Shield parameter of 0.055. The corresponding critical shear velocity is 0.134 m/s*

The volumetric concentration of the rejects is 1.34%. However it is assumed that they are not lifted and sliding with Coulombic resistance leading to a force per unit length calculated at 39.54 N/m.

The Bagnold force from the low volumetric concentration of the rejects is negligible so the main contribution of the rejects is to add 39.54 N/m as Coulombic force, thus raising the total force to 116.43 N/m leading to minimum slope of 1.94%

The critical shear of 0.134 m/s is however too close to the friction shear velocity of 0.156 m/s needed to maintain the flow of the bed. Further iteration leads to a need for more slope.