Relativistic symmetries in Yukawa-type interactions with Coulomb-like tensor

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A B S T R A C T

This study presents the solutions of the Dirac equation with a new suggested Yukawa-type potential for any spin–orbit quantum number \( \kappa \) interacting with a Coulomb-like tensor interaction. In the presence of spin and pseudospin (p-spin) symmetries, the approximate energy eigenvalues and wave functions are obtained by means of the parametric Nikiforov–Uvarov (pNU) method and the asymptotic iteration method (AIM). The numerical results show that the Coulomb-like tensor interaction removes degeneracies between spin and p-spin state doublets. The bound state solutions of the Schrödinger and Klein–Gordon equations for this new potential have also been presented. Our analytical results are in exact agreements with previous works.

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1. Introduction

Relativistic symmetries of the Dirac Hamiltonian had been discovered many years ago. However, these symmetries have been recently recognized empirically in nuclear and hadronic spectroscopies [1]. Within the framework of Dirac equation, the p-spin symmetry used to feature deformed nuclei and the superdeformation to establish an effective shell-model [2–4]. The spin symmetry is relevant to mesons [5] and occurs when the difference of the scalar \( S(r) \) and vector \( V(r) \) potentials are constant, i.e., \( \Delta(r) = C_s \), and the p-spin symmetry occurs when the sum of the scalar and vector potentials are constant, i.e., \( \Sigma(r) = C_{ps} \) [6,7]. The p-spin symmetry refers to a quasi-degeneracy of single nucleon doublets with non-relativistic quantum number \( (n, l, j = l + 1/2) \) and \( (n - 1, l + 2, j = l + 3/2) \), where \( n, l \) and \( j \) denote the single nucleon radial, orbital and total angular quantum numbers, respectively [8,9]. Furthermore, the total angular momentum is \( j = \tilde{l} + \tilde{s} \), where \( \tilde{l} = l + 1 \) pseudo-angular momentum and \( \tilde{s} \) is p-spin angular momentum [10]. Recently, the tensor potential has been introduced into the Dirac equation with the substitution \( \tilde{p} \rightarrow \tilde{p} - im\tilde{s} \alpha U(r) \) and a spin–orbit coupling is added to the Dirac Hamiltonian [11,12]. Lisboa et al. [13] have studied a generalized relativistic harmonic oscillator for spin-1/2 particles by considering a Dirac Hamiltonian that contains quadratic vector and scalar potentials together with a linear tensor potential under the conditions of p-spin and spin symmetry. Alberto et al. [14] studied the contribution of the isoscalar tensor coupling to the realization of p-spin symmetry in nuclei. Akçay [15] solved exactly the Dirac equation with scalar and vector quadratic potentials including a Coulomb-like tensor potential. He also solved exactly the Dirac equation for a linear and Coulomb-like term containing the tensor potential too [16]. Aydoğdu and Sever [17] obtained exact solution of the Dirac equation for the

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pseudoharmonic potential in the presence of linear tensor potential under the p-spin symmetry and showed that tensor interactions remove all degeneracies between members of p-spin doublets. Ikhdair and Sever [10] solved approximately Dirac–Hulthén problem under spin and p-spin symmetric limits including Coulomb-like tensor potential with for arbitrary spin–orbit coupling number \( \kappa \). Very recently, Hamzavi et al. [18,19] presented exact solutions of the Dirac equation for Mie-type potential and approximate solutions of the Dirac–Morse problem with a Coulomb-like tensor potential. Very recently, Hamzavi et al. [20] introduced a novel potential and named it as inversely quadratic Yukawa (IQY) potential. The authors solved Dirac–IQY problem in the presence of spin and p-spin symmetric limits with Coulomb-like tensor interaction. Over the past few years, the Schrödinger, Klein–Gordon (KG) and the Dirac equations have been solved for various types of quantum potential models by different authors [21–40].

In this work, we introduce a novel potential in the form of Yukawa potential very similar to the combinations of the IQY potential [20] and Yukawa potential [40] on the entire positive line range, \( r \in (0, \infty) \) as shown in Fig. 1. In the short screening regime, the \( V_\text{Y-type}(r) \) is an intermediate between \( V_{\text{IQY}}(r) \) and \( V_{\text{Y}}(r) \) as seen shown in Fig. 1. The new Yukawa-type potential can be written as:

\[
V(r) = -V_0 \left(1 + \frac{1}{r} e^{-ar}\right)^2 = -\frac{A}{r^2} e^{-2ar} - \frac{B}{r} e^{-ar} - C, \quad A = C = V_0, \quad B = 2V_0,
\]

where \( a \) is the screening parameter and \( V_0 \) is the coupling strength of the potential. The above potential is simply a combination of the IQY-plus-Yukawa potential. It should come asymptotically to a finite value as \( r \to \infty \) and should become infinite at \( r = 0 \). It is worth mentioning that a form of Yukawa potential has been earlier studied by Taseli [41] in obtaining modified Laguerre basis for hydrogen-like systems. Furthermore, different forms of the Yukawa potential have also been used to obtain the effective range functions by Kermode and co researchers [42].

The motivation of the present work is to test a new suggested potential being more general from IQY potential proposed and studied in our previous work [20] and having a very similar behavior as the Yukawa potential. Therefore, we shall investigate the bound states of a spin-1/2 particle exposed to the field of Yukawa-type potential (1) in view of spin and p-spin symmetries and a Coulomb field interaction. We calculate the energy eigenvalues and the corresponding wave functions are expressed in terms of Jacobi polynomials.

This paper is organized as follows. In Section 2, we briefly introduce the Dirac equation with scalar and vector potential with any spin–orbit quantum number \( \kappa \) including tensor interaction under spin and p-spin symmetric limits. The parametric Nikiforov–Uvarov (pNU) method is presented in Section 3. The energy eigenvalue formulas and corresponding eigenfunctions are obtained in Section 4. Special cases such as the Schrödinger and Klein–Gordon solutions for Yukawa-type, IQY and Yukawa potentials are obtained in Section 5. Our concluding remarks are given in Section 6.

2. Dirac equation with tensor coupling potential

The Dirac equation for fermionic massive spin-1/2 particles moving in the field of an attractive scalar potential \( S(r) \), a repulsive vector potential \( V(r) \) and a tensor potential \( U(r) \) (in units \( \hbar = c = 1 \)) is

\[
[i\vec{\gamma} \cdot \vec{\partial} + \beta(M + S(r)) - i\beta \vec{\gamma} \cdot \vec{U}(r)]\psi(\vec{r}) = (\mathcal{E} - V(r))\psi(\vec{r}),
\]

with

\[
\begin{align*}
\text{(a)} & \quad V_{\text{Y}}(r) = -V_0 e^{-ar} \\
\text{(b)} & \quad V_{\text{IQY}}(r) = -V_0 (1 + e^{-ar})^2 \\
\end{align*}
\]

Fig. 1. The behavior of the Yukawa-type potential compared with Yukawa potential [40] and IQY potential [20] for screening parameter values (a) \( a = 0.10 \text{ fm}^{-1} \) and (b) \( a = 0.05 \text{ fm}^{-1} \). The coupling strength of the potential is taken as 1.0 fm\(^{-1}\).
\[ \psi_{n\lambda}(\vec{r}) = \left( \begin{array}{c} f_{n\lambda}(\vec{r}) \\ g_{n\lambda}(\vec{r}) \end{array} \right) = \frac{1}{\tilde{T}} \left( \begin{array}{c} F_{n\lambda}(r) Y_{jm}^l(\theta, \varphi) \\ iG_{n\lambda}(r) Y_{jm}^{l'}(\theta, \varphi) \end{array} \right), \]

where \( E \) is the relativistic binding energy of the system, \( \tilde{p} = -i \tilde{\nabla} \) is the three-dimensional momentum operator and \( M \) is the mass of the fermionic particle. \( \tilde{x} \) and \( \tilde{t} \) are the 4 \times 4 usual Dirac matrices [20]. Further, \( f_{n\lambda}(\vec{r}) \) is the upper (large) component and \( g_{n\lambda}(\vec{r}) \) is the lower (small) component of the Dirac spinors. \( Y_{jm}^l(\theta, \varphi) \) and \( Y_{jm}^{l'}(\theta, \varphi) \) are spin and p-spin spherical harmonics, respectively, and \( m \) is the projection of the angular momentum on the z-axis.

Following our previous work [20], we obtain two Schrödinger-like differential equations for the upper and lower radial spinor components

\[ \frac{d^2}{dr^2} - \frac{\kappa(k+1)}{r^2} + 2\frac{\kappa}{r} U(r) + \frac{dU(r)}{dr} - U^2(r) F_{n\lambda}(r) + \frac{1}{(M+E_{n\lambda}-\Delta(r))} \left( \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n\lambda}(r) = \left[(M+E_{n\lambda}-\Delta(r))(M-E_{n\lambda}+\Sigma(r))\right] F_{n\lambda}(r), \]

and

\[ \frac{d^2}{dr^2} - \frac{\kappa(k-1)}{r^2} + 2\frac{\kappa}{r} U(r) + \frac{dU(r)}{dr} - U^2(r) G_{n\lambda}(r) + \frac{1}{(M-E_{n\lambda}+\Sigma(r))} \left( \frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n\lambda}(r) = \left[(M+E_{n\lambda}-\Delta(r))(M-E_{n\lambda}+\Sigma(r))\right] G_{n\lambda}(r), \]

respectively, where \( \kappa(k-1) = \tilde{l}(\tilde{l}+1) \) and \( \kappa(k+1) = \tilde{l}(\tilde{l}+1) \). The quantum number \( \kappa \) is related to the quantum numbers for spin symmetry \( l \) and p-spin symmetry \( \tilde{l} \) as

\[ \kappa = \begin{cases} -(l+1) = -(j+1/2)(s_{1/2}, p_{3/2}, \text{etc.}) & j = l + \frac{1}{2}, \quad \text{aligned spin}(\kappa < 0) \\ +l = +(j+1/2)(p_{1/2}, d_{3/2}, \text{etc.}) & j = l - \frac{1}{2}, \quad \text{unaligned spin}(\kappa > 0), \end{cases} \]

and the quasidegenerate doublet structure can be expressed in terms of a p-spin angular momentum \( \tilde{s} = 1/2 \) and pseudoorbital angular momentum \( \tilde{l} \), which is defined as

\[ \kappa = \begin{cases} -\tilde{l} = -(j+1/2)(s_{1/2}, p_{3/2}, \text{etc.}) & j = \tilde{l} + \frac{1}{2}, \quad \text{aligned p-spin}(\kappa < 0) \\ (\tilde{l}+1) = +(l+1/2)(d_{1/2}, f_{3/2}, \text{etc.}) & j = \tilde{l} - \frac{1}{2}, \quad \text{unaligned p-spin}(\kappa > 0), \end{cases} \]

where \( \kappa = \pm 1, \pm 2, \ldots \).

The lower radial wave functions \( F_{n\lambda}(r) \) and \( G_{n\lambda}(r) \) are

\[ G_{n\lambda}(r) = \frac{1}{(M+E_{n\lambda}-\Delta(r))} \left( \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n\lambda}(r), \]

\[ F_{n\lambda}(r) = \frac{1}{(M-E_{n\lambda}+\Sigma(r))} \left( \frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n\lambda}(r), \]

with \( \Delta(r) = V(r) - S(r) \) and \( \Sigma(r) = V(r) + S(r) \).

2.1. P-spin symmetry limit

The p-spin symmetry occurs when \( \Sigma(r) = C_{ps} = \text{constant} \) [7]. This means that \( \frac{d\Sigma(r)}{dr} = 0 \). Then p-spin symmetry is exact in the Dirac equation [43–48]. In this part, we are taking \( \Delta(r) \) as the novel Yukawa-type potential (1) and tensor potential as Coulomb-like interaction, that is,

\[ \Delta(r) = -4z^2A \frac{e^{-4zR_c}}{(1-e^{-4zR_c})^2} - 2zB \frac{e^{-2zR_c}}{1-e^{-2zR_c}} - C, \]

\[ U(r) = -\frac{\tilde{T}}{2}, T = \frac{2\tilde{T}^2}{4z^2}, r \geq R_c, \]

where \( R_c = 7.78 \text{ fm} \) is the Coulomb radius, \( Z_a \) and \( Z_b \) denote the charges of the projectile \( a \) and the target nuclei \( b \), respectively [10]. In the presence of this symmetry, Eq. (5) can be rewritten in the simple form:

\[ \left( \frac{d^2}{dr^2} - \frac{\kappa(k-1)}{r^2} + \gamma \left( 4z^2A \frac{e^{-4zR_c}}{(1-e^{-4zR_c})^2} + 2zB \frac{e^{-2zR_c}}{1-e^{-2zR_c}} + C \right) - \tilde{p}^2 \right) G_{n\lambda}(r) = 0, \]

where \( \kappa = \tilde{l} \) and \( \kappa = \tilde{l} + 1 \) for \( \kappa < 0 \) and \( \kappa > 0 \), respectively. Also, we identified \( \Delta_{ps} = \kappa + T, \gamma = E_{n\lambda} - M - C_{ps} \) and \( \tilde{p}^2 = (M+E_{n\lambda})(M-E_{n\lambda}+C_{ps}) \).
2.2. Spin symmetry limit

In the spin symmetry limit $\frac{d^2\psi}{dr^2} = 0$ or $\Delta(r) = C_r = \text{constant}$ [43–48], with $\Sigma(r)$ is taken as Yukawa-type potential (1) and $U(r)$ as the Coulomb-like tensor interaction. Thus, Eq. (4) recasts in the form:

$$\left[ \frac{d^2}{dr^2} - \frac{\eta_j}{r^2} + \gamma \left( 4x^2A \frac{e^{-4x}}{(1-e^{-2x})^2} + 2xB \frac{e^{-2x}}{1-e^{-2x}} + C \right) - \beta^2 \right] F_{inc}(r) = 0,$$

where $\kappa = l$ and $\kappa = l - 1$ for $\kappa < 0$ and $\kappa > 0$, respectively. Also, $\eta_k = \kappa + T + 1$, $\gamma = M + E_{inc} - C_s$ and $\beta^2 = (M - E_{inc})(M + E_{inc} - C_s)$.

Since the Dirac equation with the Yukawa-type potential has no exact solution, we use an approximation for the centrifugal term as [20,40,49–54],

$$\frac{1}{r^2} \approx \lim_{\tau \to 0} \left[ 4d^2 \frac{e^{-2x}}{(1-e^{-2x})^2} \right].$$

Finally, for the solutions of Eqs. (10) and (11) with the above approximation, we will employ the parametric NU method which is briefly introduced in the following section.

3. Parametric NU method

This powerful mathematical tool solves second order differential equations. Let us consider the following differential equation [55–59],

$$\psi''_n(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \psi'_n(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi_n(s) = 0$$

(13)

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without need to check the validity of solution. We present a shortcut for the method. So, at first we write the general form of the Schrödinger-like Eq. (13) in a more general form as

$$\psi''_n(s) + \left( \frac{c_1 - c_2 s}{s(1-c_3s)} \right) \psi'_n(s) + \left( -\frac{p_2 s^2 + p_1 s - p_0}{s^2(1-c_3s)^2} \right) \psi_n(s) = 0$$

(14)

satisfying the wave functions

$$\psi_n(s) = \phi(s) y_n(s).$$

Comparing (14) with its counterpart (13), we obtain the following identifications:

$$\tilde{\tau}(s) = c_1 - c_2 s, \quad \sigma(s) = s(1-c_3s), \quad \tilde{\sigma}(s) = -p_2 s^2 + p_1 s - p_0,$$

(16)

Following the NU method [55], we obtain the energy equation [56,57]

$$c_2 n - (2n+1) c_5 + (2n+1) (\sqrt{c_6} + c_3 \sqrt{c_8}) + n(n-1) c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_6} c_9 = 0.$$

(17)

and the wave functions

$$\rho(s) = s^{c_1} (1-c_3s)^{c_{11}}, \quad \phi(s) = s^{c_1} (1-c_3s)^{c_{11}}, \quad c_{12}, > 0, \quad c_{13}, > 0, \quad c_{11} > -1, \quad c_{12} > -1, \quad \psi_{m}(s) = N_m s^{c_1} (1-c_3s)^{c_{11}},$$

(18)

where $P_n^{\mu\nu} (x), \mu > -1, \nu > -1$, and $x \in [-1, 1]$ are Jacobi polynomials with the constants are

$$c_4 = \frac{1}{2} (1-c_1), \quad c_5 = \frac{1}{2} (c_2 - 2c_1), \quad c_6 = c_3 + p_2, \quad c_7 = 2c_4c_5 - p_1,$$

$$c_8 = c_3 + p_0, \quad c_9 = c_3 (c_7 + c_3c_8) + c_6, \quad c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} - c_1 - 1, \quad c_{11} = 1 - c_1 + 2c_4 + 2\sqrt{c_8} - 1, \quad c_{12} = c_4 + \sqrt{c_8} > 0, \quad c_{13} = -c_4 + \frac{1}{c_1} (\sqrt{c_6} - c_5) > 0, \quad c_{13} \neq 0, \quad c_{12} > 0, \quad c_{13} > 0 and s \in [0, 1/c_3], \quad c_3 \neq 0.$$

(19)

In the rather more special case of $c_3 = 0$, the wave function (15) becomes
\[ \lim_{c_3 \to 0} P_n^{(c_1, c_2)}(1 - 2c_3 s) = L_m^{(c_1)}(c_1 s), \quad \lim_{c_1 \to 0} (1 - c_3 s)^{c_3} = e^{c_3 s}, \quad \psi(s) = \mathcal{N} s^{c_1} e^{c_3 s} L_n^{(c_1)}(c_1 s). \] (20)

4. Bound state solutions of the Dirac equation

In this section, we shall solve the Dirac equation with our novel potential including tensor interaction by using the pNU and AIM.

4.1. P-spin symmetric case

4.1.1. Solution via pNU

Firstly, we obtain solution of Eq. (10) via pNU, by using transformation of the form \( s = e^{-2\beta t} \). Thus we rewrite the equation as follows

\[ \left[ \frac{d^2}{ds^2} + \frac{1 - s}{s(1 - s)} \frac{d}{ds} + \frac{1}{s^2 (1 - s)^2} \left( -\Lambda^r_k (\Lambda^r_k - 1)s + \gamma' s^2 - \frac{\beta^2}{4\alpha} (1 - s)^3 \right) \right] G_{m n}(s) = 0, \] (21)

with

\[ \Lambda^r_k (\Lambda^r_k - 1) = \Lambda_k (\Lambda_k - 1) - \frac{\gamma B}{2\alpha}, \quad \gamma' = \gamma \left( \frac{2 \alpha A - B}{2\alpha} \right), \quad \beta^2 = \beta^2 - \gamma C. \] (22)

Comparing Eq. (21) with Eq. (14), we obtain

\[ c_1 = 1, \quad p_2 = \frac{p_2^1}{4\pi^2} - \gamma', \] \[ c_2 = 1, \quad p_1 = -\Lambda^r_k (\Lambda^r_k - 1) + \frac{p_2^2}{4\pi^2}, \] \[ c_3 = 1, \quad p_0 = \frac{p_2^2}{4\pi^2}, \] (23)

and by the use of Eq. (19), we can obtain

\[ c_4 = 0, \quad c_5 = -\frac{1}{2}, \] \[ c_6 = \frac{1}{2} + \frac{p_2^2}{4\pi^2} - \gamma', \quad c_7 = \Lambda^r_k (\Lambda^r_k - 1) - \frac{p_2^2}{2\pi}, \] \[ c_8 = \frac{p_2^2}{4\pi^2}, \quad c_9 = (\Lambda^r_k - 1/2)^2 - \gamma', \] \[ c_{10} = \frac{\gamma'}{2}, \quad c_{11} = 2 \sqrt{(\Lambda^r_k - 1/2)^2 - \gamma'}, \] \[ c_{12} = \frac{\gamma'}{2\pi}, \quad c_{13} = \frac{1}{2} + \sqrt{(\Lambda^r_k - 1/2)^2 - \gamma'}. \] (24)

The energy equation can be obtained by using Eqs. (17), (23), and (24) to get

\[ \left( \frac{1}{2} + \sqrt{(\Lambda^r_k - 1/2)^2 - \gamma'} + \frac{\beta^2}{2\alpha} \right)^2 = \frac{\beta^2}{4\alpha^2} - \gamma', \] (25)

with

\[ \Lambda^r_k = \frac{1}{2} + \sqrt{(\kappa + T - 1/2)^2 - \frac{\beta^2}{2\pi} (E_{\text{nc}} - M - C_{\beta})}, \]
\[ \gamma' = (E_{\text{nc}} - M - C_{\beta}) \left( \frac{2\alpha A - B}{2\alpha} \right), \]
\[ \beta^2 = \sqrt{(M - E_{\text{nc}} + C_{\beta})(M + E_{\text{nc}} + C)}. \] (26)

4.1.2. Solutions via AIM

Again, we obtain approximate relativistic solution to the differential equation (10) via the AIM.1 In a similar fashion to pNU, we make a variable transformation of the form \( s = e^{-2\beta t} \) to obtain Eq. (21) and then proceed further by proposing a wave function of the form:

\[ G_{m n}(s) = s^{\gamma'} (1 - s)^{\frac{3}{2}} + \sqrt{1 - \Lambda^r_k} s^{1/2 - \gamma'} R(s) \] (27)

---

1 The details about this method can be found in Refs. [60,61].
Substituting this wave function into Eq. (21) we can find the following equation

\[
R''(s) + \left[ \frac{1}{2} + s \left( \frac{\dot{r} + 2\ddot{r} + 1}{s(1-s)} \right) \right] R'(s) - \left[ \frac{\left( \dot{r} + \ddot{r} \right)^2 + \ddot{r} - \left( \frac{\ddot{r}}{s} \right)^2}{s(1-s)} \right] R(s) = 0,
\]

(28)

where \( \ddot{r} = \frac{1}{2} + \sqrt{\left( \frac{1}{2} - \Lambda_k' \right)^2 - \ddot{\gamma}} \). Eq. (28) is a more convenient second-order homogeneous linear differential equation and the solution can be found by using asymptotic iteration method [60–78]. Since we have succeeded in writing Eq. (21) in the form [60,61], i.e.,

\[
R''(s) - \lambda_0(s)R'(s) - S_0(s)R(s) = 0.
\]

(29)

Then, for sufficiently large \( k \) the following recurrence relation can be used to determine the \( \lambda_k \) and \( S_k \) values

\[
\lambda_k(s) = \lambda_{k-1}(s) + \lambda_0(s)\lambda_{k-1}(s), \quad S_k(s) = S_{k-1}(s) + S_0(s)\lambda_{k-1}(s), \quad k = 1, 2, 3, \ldots
\]

(30)

Thus, the energy eigenvalue equation is obtained from the roots of the following quantization condition:

\[
\delta_k = \left| \begin{array}{cc}
\lambda_k(s) & S_k(s) \\
\lambda_{k+1}(s) & S_{k+1}(s)
\end{array} \right| = 0, \quad k = 1, 2, 3, \ldots
\]

(31)

By using Eq. (31) and recurrence relation given in Eq. (30), we have

\[
\delta_0 = \left| \begin{array}{cc}
\lambda_0(s) & S_0(s) \\
\lambda_1(s) & S_1(s)
\end{array} \right| = 0 \Rightarrow \frac{\ddot{r}}{2s} + \frac{1}{2} + \sqrt{\left( \frac{1}{2} - \Lambda_0' \right)^2 - \ddot{\gamma}} = 0 + \sqrt{\frac{\ddot{r}^2}{4s^2} - \ddot{\gamma}},
\]

\[
\delta_1 = \left| \begin{array}{cc}
\lambda_1(s) & S_1(s) \\
\lambda_2(s) & S_2(s)
\end{array} \right| = 0 \Rightarrow \frac{\ddot{r}}{2s} + \frac{1}{2} + \sqrt{\left( \frac{1}{2} - \Lambda_1' \right)^2 - \ddot{\gamma}} = -1 + \sqrt{\frac{\ddot{r}^2}{4s^2} - \ddot{\gamma}},
\]

\[
\delta_2 = \left| \begin{array}{cc}
\lambda_2(s) & S_2(s) \\
\lambda_3(s) & S_3(s)
\end{array} \right| = 0 \Rightarrow \frac{\ddot{r}}{2s} + \frac{1}{2} + \sqrt{\left( \frac{1}{2} - \Lambda_2' \right)^2 - \ddot{\gamma}} = -2 + \sqrt{\frac{\ddot{r}^2}{4s^2} - \ddot{\gamma}},
\]

\[
\vdots
\]

\[
\delta_n = \left| \begin{array}{cc}
\lambda_n(s) & S_n(s) \\
\lambda_{n+1}(s) & S_{n+1}(s)
\end{array} \right| = 0 \Rightarrow \frac{\ddot{r}}{2s} + \frac{1}{2} + \sqrt{\left( \frac{1}{2} - \Lambda_n' \right)^2 - \ddot{\gamma}} = -n + \sqrt{\frac{\ddot{r}^2}{4s^2} - \ddot{\gamma}}.
\]

(32)

From the generalization of the above sequence, i.e., the \( n \)th term of the series, we can find

\[
\left( n + \frac{1}{2} + \sqrt{\left( \frac{1}{2} - \Lambda_n' \right)^2 - \ddot{\gamma}} \right)^2 = \frac{\ddot{r}^2}{4s^2} - \ddot{\gamma}.
\]

(33)

which is similar to pNU result of Eq. (25) and the notations \( \ddot{\gamma}, \Lambda_n' \), and \( \ddot{r} \) have been defined previously by Eq. (26). It is worthy to note that once \( \beta = \omega = 0 \), Eq. (25) or Eq. (33) becomes identical to Eq. (30) of Ref. [20] with same energy spectra already found in Table 1 and Table 2 of [20].

Furthermore, substituting the explicit forms of \( \Lambda_n', \ddot{\gamma} \), and \( \ddot{r}^2 \) (26) into Eq. (25) or Eq. (33), one can readily obtain the required energy formula in closed form. In the limiting case when the screening parameter \( \alpha \to 0 \) (low screening regime), the potential approximates as \( V_{QY}(r) = -V_0 \lim_{\alpha \to 0} (1 + \frac{3}{4} e^{-\alpha r})^2 \approx -\frac{a}{r} - \frac{b}{r^2} + c \), where the potential parameters are defined as \( a = -V_0, b = -2(1 - x)V_0, c = -(1 - x)^2 V_0 \). This potential is well known as Mie-type potential [18,27]. The energy eigenvalue equation for this potential has recently been found in Ref. [27] as
Table 2
The spin symmetric bound state energy levels (in unit of fm$^{-1}$) of the GIQY potential taking several values of $n$ and $\kappa$.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$n$, $\kappa &lt; 0$</th>
<th>$E_{\text{nl}, aT = 0.5}$</th>
<th>$E_{\text{nl}, aT = 0}$</th>
<th>$n$, $\kappa &gt; 0$</th>
<th>$E_{\text{nl}, aT = 0.5}$</th>
<th>$E_{\text{nl}, aT = 0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0, -2</td>
<td>0.52063601</td>
<td>0.54846403</td>
<td>0, 1</td>
<td>0.56241728</td>
<td>0.54846403</td>
</tr>
<tr>
<td>2</td>
<td>0, -3</td>
<td>0.62641728</td>
<td>0.69619834</td>
<td>0, 2</td>
<td>0.75313824</td>
<td>0.69619834</td>
</tr>
<tr>
<td>3</td>
<td>0, -4</td>
<td>0.75313824</td>
<td>0.79872127</td>
<td>0, 3</td>
<td>0.83505474</td>
<td>0.79872127</td>
</tr>
<tr>
<td>4</td>
<td>0, -5</td>
<td>0.83505474</td>
<td>0.86407329</td>
<td>0, 4</td>
<td>0.88739206</td>
<td>0.86407329</td>
</tr>
<tr>
<td>1</td>
<td>1, -2</td>
<td>1.02063601</td>
<td>1.04846403</td>
<td>1, 1</td>
<td>1.07821833</td>
<td>1.04846403</td>
</tr>
<tr>
<td>2</td>
<td>1, -3</td>
<td>1.12641728</td>
<td>1.19619834</td>
<td>1, 2</td>
<td>1.28640391</td>
<td>1.19619834</td>
</tr>
<tr>
<td>3</td>
<td>1, -4</td>
<td>1.25313824</td>
<td>1.30872127</td>
<td>1, 3</td>
<td>1.35820409</td>
<td>1.30872127</td>
</tr>
<tr>
<td>4</td>
<td>1, -5</td>
<td>1.32063601</td>
<td>1.34846403</td>
<td>1, 4</td>
<td>1.39821833</td>
<td>1.34846403</td>
</tr>
</tbody>
</table>

The special case when $a = c = 0$ and $C_{ps} = 0$ yields the energy formula for the Coulomb-like potential [27,79].

$$E_{\text{nl}} = -M \frac{4(n + \kappa)^2 - b^2}{4(n + \kappa)^2 + b^2}.$$  \hspace{1cm} (35)

For the case when $n \rightarrow \infty$, one obtains $E = -M$ (continuum states), that is, it shows that when $n$ goes to infinity the energy solution of Eq. (25) becomes finite (this is the exact $p$-spin symmetric case given by Eq. (38) of Ref. [79]).

On the other hand, to find the corresponding wave functions, referring to Eq. (18), we find the functions

$$\rho(s) = s^{\beta/2} (1 - s)^2 \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma}, \quad \phi(s) = s^{(\beta/2)x} (1 - s)^{1/2} \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma}.$$  \hspace{1cm} (36)

Hence, Eq. (18) with the help of the weight function $\rho(s)$ in Eq. (29) gives

$$y_n(s) = P_n \left( \frac{\beta/2}{2} \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma} (1 - 2s) \right).$$  \hspace{1cm} (37)

Furthermore, using $G_{nl}(s) = \phi(s)y_n(s)$, we get the lower component of the Dirac spinor as

$$G_{nl}(r) = A_{nl} (e^{-2\beta r}) \left(1 - e^{-2\beta r}\right)^{\frac{\beta}{2x} + 1} \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma} P_n \left( \frac{\beta/2}{2} \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma} (1 - 2e^{-2\beta r}) \right)$$

$$= A_{nl} (e^{-2\beta r}) \left(1 - e^{-2\beta r}\right)^{\frac{\beta}{2x} + 1} \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma} \times 2F_1 \left(-n, n + \beta/\alpha + 2 \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma} + 1, \beta/\alpha + 1; e^{-2\beta r} \right),$$  \hspace{1cm} (38)

where $A_{nl} = \frac{P_{nl}(n + \beta/\alpha + 1)}{\Gamma(\beta/2x + 1)} A_{nl}$ with $A_{nl}$ is the normalization constant. The upper component of the Dirac spinor can be calculated from Eq. (8b) as [80]

$$F_{nl}(r) = \frac{1}{(M - E_{nl} + C_{ps})} \left\{ \left[ (n + \beta/\alpha + 2 \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma} + 1) \left( e^{-2\beta r} \right)^{\frac{\beta}{2x} + 1} \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma} \right] e^{-2\beta r} \left(1 - e^{-2\beta r}\right) - \frac{\beta + H}{r} G_{nl}(r) \right\}$$

$$+ A_{nl} 2 \pi n \left( \frac{n + \beta/\alpha + 2 \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma} + 1}{\beta/\alpha + 1} \right) \left( e^{-2\beta r} \right)^{\frac{\beta}{2x} + 1} \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma} \times 2F_1 \left(-n + 1, n + \beta/\alpha + 2 \sqrt{\left(\Lambda_{n} - 1/2\right)^2 - \gamma} + 1, \beta/\alpha + 2 + e^{-2\beta r} \right),$$  \hspace{1cm} (39)

where $E_{nl} = M + C_{ps}$ and with the exact $p$-spin symmetry when $C_{ps} = 0$, only negative energy solution exists. The finiteness of our solution requires that the two-components of the wave function be defined over the whole positive range, $r \in [0, \infty)$. However, in the $p$-spin limit, if the positive energy is chosen, the upper-spinor component of the wave function will be no longer defined as obviously seen in Eq. (39). Furthermore, introducing the Coulomb-like tensor does not affect the negativity of the energy spectrum in the $p$-spin limit, but the main contribution is just the tendency to remove the degeneracy of the spectrum.

Of course, the energy eigenvalue Eq. (25), (33) admits two solutions (negative and positive), however, we are forced to choose the negative energy solution to make the wave function normalizable in the given range [80–82].
4.2. Spin symmetric case

To avoid repetition in our solution to Eq. (11),

$$\left[ \frac{d^2}{ds^2} + \frac{1 - s}{s(1 - s)} \frac{d}{ds} + \frac{1}{s(1 - s)^2} \left( -\eta_s^{nf}(\eta_n - 1)s + \gamma \frac{V_0s^2}{2} - \frac{\beta^2}{4\Delta^2} (1 - s)^2 \right) \right] F_{ns}(s) = 0, \quad (40)$$

with

$$\eta_s^{nf} = \frac{1}{2} + \sqrt{(\kappa + T + 1/2)^2 - \frac{B}{2\Delta}(M + E_{ns} - C)}, \quad \gamma' = \left( \frac{2\Delta A - B}{2\Delta} \right)(M + E_{ns} - C),$$

$$\beta' = \sqrt{(M + E_{ns} - C)(M - E_{ns} - C)}. \quad (41)$$

We follow the same procedures explained in the Section 4.1 to obtain the energy eigenvalue equation

$$\left( n + \frac{1}{2} + \sqrt{(\eta_s^{nf} - 1/2)^2 - \gamma'} + \frac{\beta'}{2\Delta} \right)^2 = \frac{\beta^2}{4\Delta^2} - \gamma', \quad (42)$$

and the corresponding wave functions for the upper Dirac spinor as

$$F_{ns}(r) = B_{ns} e^{\beta' r \left( 1 - e^{-2\Delta r} \right)^{1/2} \sqrt{(\eta_s^{nf} - 1/2)^2 - \gamma'}} P_n^{(\beta'/2\Delta)^{1/2}}(1 - 2e^{-2\Delta r}), \quad (43)$$

where $B_{ns}$ is the normalization constant. Finally, the lower-spinor component of the Dirac equation can also be obtained via Eq. (8a) as

$$G_{ns}(r) = \frac{1}{(M + E_{ns} - C_s)} \left\{ \left[ \frac{\beta'}{2\Delta} + \frac{\kappa + H}{r} \right] F_{ns}(r) \right. \left. + A_{ns} \left( n + \beta'/\Delta + 2\sqrt{(\eta_s^{nf} - 1/2)^2 - \gamma'} + 1 \right) \right. \\
\left. \times 2F_1 \left( -n + 1, n + \beta'/\Delta + 2\sqrt{(\eta_s^{nf} - 1/2)^2 - \gamma'} + 2, \beta'/\Delta + 2e^{-2\Delta r} \right) \right\}, \quad (44)$$

where $E_{ns} \neq -M + C_s$.

4.3. Numerical results

The tensor potential generates a new spin–orbit centrifugal term $\Lambda(\Lambda \pm 1)$ where $\Lambda = \Lambda_s'$ or $\eta_s'$. Some numerical results are given in Tables 1 and 2, where we have used the following parameters: $M = 5.0$ fm$^{-1}$, $V_0 = 1.0$, $C_\text{ps} = -5.5$ fm$^{-1}$ and $C_s = 6.0$ fm$^{-1}$. In Table 1, we consider the same set of p-spin symmetry doublets: $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(1f_{7/2}, 0h_{9/2})$, . . . . Also, in Table 2, we consider the same set of spin symmetry doublets: $(0p_{1/2}, 0d_{3/2})$, $(0d_{3/2}, 0f_{5/2})$, $(0f_{5/2}, 0g_{7/2})$, $(0g_{7/2}, 0h_{9/2})$, . . . . We have noticed that the tensor interaction removes the degeneracy between two states in spin doublets and p-spin levels. When $T = 0$, the energy levels of the spin (p-spin) aligned states and spin (p-spin) unaligned states move in the opposite directions. For example, in p-spin doublet $(1s_{1/2}, 0d_{3/2})$: when $T = 0$, $E_{1\pm 1} = E_{1\pm 2} = -5.729276633$ fm$^{-1}$, but when $T = 0.5$, $E_{1\pm 1} = -5.693455766$ fm$^{-1}$ with $\kappa < 0$ and $E_{1\pm 2} = -5.766798298$ fm$^{-1}$ with $\kappa > 0$.

5. Some special cases

We will consider two cases of much concern in the absence of tensor interaction.

5.1. Schrödinger solution

In the non-relativistic limit, the Schrödinger solution can be obtained from the exact spin symmetry case Eqs. (42) and (43) with $C_s = 0$. Applying the transformations $E_{ns} + M \approx 2\mu/h^2$, $E_{ns} - M \approx E_{nl}$ and $\kappa(\kappa + 1) \rightarrow l(l + 1)$, one obtains the energy formula

$$E_{nl} = -C - \frac{2\mu}{h^2} \left[ \frac{\alpha}{2\mu} \left( n + 1 + \sqrt{(l + 1/2)^2 - 2\mu A/h^2} \right)^2 + \frac{(2\Delta A - B)}{2\mu} \left( n + 1 + \sqrt{(l + 1/2)^2 - 2\mu A/h^2} \right) \right]^2, \quad (45)$$

and the radial wave functions
When $B = C = 0$, we obtain the non-relativistic energy levels of the IQY potential:

$$E_{nl} = \frac{2\mu}{\hbar^2} \left[ \frac{A}{2(n+l+1)} - \frac{\hbar^2(n+l+1)}{2\mu} \right]^2,$$

and wave functions

$$R_{nl}(r) = D_{nl} e^{-\sqrt{2\mu E_{nl}/\hbar^2} (1 - e^{-2\pi r})} P_n \left( \sqrt{-2\mu E_{nl}/\hbar^2}, 2\sqrt{(1+1/2)^2 - 2\mu A/\hbar^2} \right) \left( 1 - 2e^{-2\pi r} \right).$$

Further, when $A = C = 0$, and $B \to A$, then for Yukawa potential, the energy levels of Eq. (45) becomes

$$E_{nl} = \frac{2\mu}{\hbar^2} \left[ \frac{A}{2(n+l+1)} - \frac{\hbar^2(n+l+1)}{2\mu} \right]^2,$$

which is identical to Eq. (52) of Ref. [40] and wave functions are

$$R_{nl}(r) = D_{nl} e^{-\sqrt{2\mu E_{nl}/\hbar^2} (1 - e^{-2\pi r})} P_n \left( \sqrt{-2\mu E_{nl}/\hbar^2}, 2\sqrt{1+1/2} \right) \left( 1 - 2e^{-2\pi r} \right).$$

5.2. Spinless Klein–Gordon solution

As for the Klein–Gordon equation, for equally mixed scalar $S(r)$ and vector $V(r)$ potentials, the following Schrödinger-type equation can be found for the cases $S(r) = \pm V(r)$ [83]

$$\left[ \nabla^2 - (E_{nl} \pm M) V(r) + E_{nl}^2 - M^2 \right] \psi_{KG}(r) = 0.$$

Under the restriction of equally mixed potentials $S(r) = V(r)$ (exact spin symmetry) and $\kappa = l$, the KG equation turns into a Schrödinger-like equation [84]. By setting $\kappa = l$ and $C = 0$, we obtain the energy equation from Eq. (42) as

$$\left( n + \frac{1}{2} + \sqrt{(l+1/2)^2 - A(M + E_{nl})} \right)^2 = (M + E_{nl}) \left[ \frac{(M - E_{nl} - C)}{4\lambda^2} + \frac{B}{2\lambda} - A \right],$$

and the wave function

$$R_{nl}(r) = B_{nl} e^{-\sqrt{(M + E_{nl})/(M - E_{nl} - C)} r} \left( 1 - e^{-2\pi r} \right)^{1/2} \sqrt{(l+1/2)^2 - A(M + E_{nl})} P_n \left( \sqrt{(M + E_{nl})/(M - E_{nl} - C)}, 2\sqrt{(l+1/2)^2 - A(M + E_{nl})} \right) \left( 1 - 2e^{-2\pi r} \right).$$

When $B = C = 0$, we obtain the energy equation and wave functions for the Klein–Gordon-IQY problem with exact spin symmetric case

$$\left( n + \frac{1}{2} + \sqrt{(l+1/2)^2 - A(M + E_{nl})} \right)^2 = (M + E_{nl}) \left[ \frac{(M - E_{nl} - 4\lambda^2 A)}{4\lambda^2} \right],$$

$$R_{nl}(r) = B_{nl} e^{-\sqrt{M^2 - E_{nl}^2} \left( 1 - e^{-2\pi r} \right)^{1/2} \sqrt{(l+1/2)^2 - A(M + E_{nl})} \times P_n \left( \sqrt{M^2 - E_{nl}^2}/2, 2\sqrt{(l+1/2)^2 - A(M + E_{nl})} \right) \left( 1 - 2e^{-2\pi r} \right).$$

On the other hand, when $S(r) = -V(r)$ (exact p-spin symmetry) and $\kappa = l + 1$, the energy equation and the wave functions are obtained via Eqs. (25), (33) and (38) as

$$\left( n + \frac{1}{2} + \sqrt{(l+1/2)^2 + A(M - E_{nl})} \right)^2 = (M - E_{nl}) \left[ \frac{(M + E_{nl} + C)}{4\lambda^2} - \frac{B}{2\lambda} + A \right],$$

and

$$R_{nl}(r) = A_{nl} e^{-\sqrt{(M - E_{nl})/(M + E_{nl} + C)} r} \left( 1 - e^{-2\pi r} \right)^{1/2} \sqrt{(l+1/2)^2 + A(M - E_{nl})} \times P_n \left( \sqrt{(M - E_{nl})/(M + E_{nl} + C)}, 2\sqrt{(l+1/2)^2 + A(M - E_{nl})} \right) \left( 1 - 2e^{-2\pi r} \right).$$
When $B = C = 0$, we obtain the energy equation and wave functions for the Klein–Gordon-IQY problem with exact p-spin symmetric case.

$$
\left(n + \frac{1}{2} + \sqrt{(1 + 2^n + \frac{2}{2})^2 + \frac{\sqrt{M^2 - E_{nl}^2}}{2\alpha}}\right)^2 = \frac{(M - E_{nl})(M + E_{nl} + 4\zeta^2\alpha)}{4\alpha^2},
$$

(58)

and

$$
R_{nl}(r) = A_{nl}e^{-\sqrt{M^2 - E_{nl}^2}[(1 - e^{-2\pi r})^{1/2} + \sqrt{(1 - 2\pi r)^{1/2} + \alpha(M - E_{nl})}] \times P_n^{\left(\sqrt{M^2 - E_{nl}^2} + 2\sqrt{(1 - 2\pi r)^{1/2} + \alpha(M - E_{nl})}\right)}(1 - 2e^{-2\pi r}).
$$

(59)

6. Conclusion

In this paper, we have introduced a novel potential which is an intermediate between the Yukawa potential [40] and the inversely quadratic Yukawa (IQY) potential [20], that mild the strong singularity of $1/r^2$ in IQY potential and the soft singularity $1/r$ in Yukawa potential. The behavior of this potential is very close to the other two potentials studied before in Refs. [20,40] as displayed in Fig. 1. Also, it can be easily reduced into the other potential forms [20,40]. Furthermore, we have obtained the approximate bound states of a Dirac particle confined to the field of GIQY interacting in the presence of spin and p-spin symmetries limits and a Coulomb-like tensor interaction in the form of $-C_0r$. We have used a parametric version of the powerful NU method [56] and the elegant AM approach [60,61]. Some numerical values of the energy levels are calculated in Tables 1 and 2 in view of p-spin and spin symmetries, respectively. Obviously, the degeneracy between the members of doublet states in p-spin and spin symmetries is removed by tensor interaction. The spin and p-spin spectrums of the present potential is identical to those ones obtained in [20] as the potential parameters $B = C = 0$. Furthermore, it has a continuum spectrum once $n \rightarrow \infty$. The relativistic spin symmetry in the absence of tensor interaction ($T = 0$) and when $C = 0$, reduces into the Schrödinger solution for the Yukawa potential [40] and the IQY potential [20] under appropriate transformations of parameters. Furthermore, the exact p-spin symmetry $C = 0$, provides the KG solution with mixed scalar and vector potential $S(r) = V(r)$ for particle, whereas the exact p-spin symmetric case corresponding to the KG solution with $S(r) = -V(r)$ case for anti-particle.

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References


