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Joint remote state preparation

Ba An Nguyen[1,2] **and Jaewan Kim**[2]

[1] Center for Theoretical Physics, Institute of Physics and Electronics, 10 Dao Tan, Thu Le, Ba Dinh, Hanoi, Vietnam

[2] School of Computational Sciences, Korea Institute for Advanced Study, 207-43 Cheongryangni 2-dong, Dondaemun-gu, Seoul 130-722, Korea

E-mail: nban@iop.vast.ac.vn and jaewan@kias.re.kr

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Abstract

Alice, Bob and Charlie are three remote parties. Alice and Bob share the classical knowledge of a secret qubit state. We consider the following question: 'how can Alice and Bob jointly prepare the qubit state for Charlie?' Two different protocols are proposed for such a joint remote state preparation. The first protocol uses a single GHZ state while the second one uses a pair of EPR states as the quantum channel whose entanglement is not necessarily maximal.

1. Introduction

Quantum networking has proven to be a promising solution for distributed quantum computing and distributed quantum information processing. A quantum network consists of many distant nodes, each of which contains just a moderate number of qubits. The different nodes are connected not only by conventional channels in terms of classical communication but also by quantum channels in terms of shared entanglement. It is believed that any global task can be done via local operations and classical communication (LOCC) provided that enough amount of classical communication and suitable entanglements are available (see, e.g., [1]). The problem is therefore to devise optimally possible protocol for a given task. One of the most striking global tasks is quantum teleportation [2] which is transmission of a qubit state from one to another remote location with dual usage of both classical and Einstein–Podolsky–Rosen (EPR) [3] channels without ever physically sending the qubit. The state to be teleported is unknown to both the sender and the receiver and should be given to the sender to perform a Bell-state measurement (BM) on it and a particle of the shared entangled channel. Since a BM has four possible results, two classical bits (cbits) are needed for broadcasting the obtained measurement outcome. A question one may ask is: 'what happens if the sender is provided with complete classical knowledge on the qubit state?' It turns out that then he/she can prepare the state for a remote receiver with a single-qubit von Neumann measurement (vNM), thus reducing the classical communication cost (CCC) to just one cbit. In this case the qubit state must not be given to the sender who is supposed to properly manipulate only the particle of the shared entangled state in such a way that the other particle at a remote location becomes exactly in the desired state or in a state up to an elementary correction operator. Such a process was termed remote state preparation (RSP) [4, 5] which in essence means remote preparation of a known quantum state. Experiments on RSP were demonstrated in [6] . Trade-off between shared entanglement and classical communication was investigated in [7]: in the high-entanglement limit the asymptotic CCC for RSP of a large number of general states is one cbit per qubit (i.e., half that of quantum teleportation). General RSP with the triple trade-off between three basic resources: 'classicality' (quantifiable in units of cbits), 'quantumness' (quantifiable in units of qubits) and 'nonlocality' (quantifiable in units of ebits), was also dealt with in [8]. So far various aspects of RSP have been addressed. These include oblivious RSP [9], continuous variable RSP [10], RSP of multipartite/higher-dimensional pure states [11], RSP of mixed states [12], RSP of many ensembles of state [13], etc. RSP at multiple locations [14] was studied as well.

Let Alice, Bob and Charlie be three remote parties. In this work we consider the so-called joint remote state preparation (JRSP) which is formulated as follows. Suppose Alice and Bob independently share the classical knowledge of a secret qubit state in such a way that no one alone is able to fully identify the state. The question is 'how can Alice and Bob jointly prepare the qubit state for Charlie?'. We propose two JRSP protocols: protocols 1 and 2. Protocol 1, which uses a single Greenberg–Horne–Zeilinger (GHZ) [15] state as the quantum channel, is presented in section 2. In this connection we note that a protocol similar to our protocol 1 has just recently appeared in [16] where, however, an incompatibility in normalization procedures is encountered. We shall elucidate this delicate issue as well in section 2. Section 3 is devoted to protocol 2 whose quantum channel is served by a pair of EPR states. We consider both maximally and nonmaximally entangled quantum channels in both protocols. Some discussions regarding the required amount and management of classical communication as well as the implement feasibility of the protocols are given in section 4, together with the conclusion.

2. Protocol 1

To be explicit we first consider the qubit state to be remotely prepared in the form

$$|q\rangle = \cos\theta|0\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle, \tag{1}$$

with θ, ϕ real parameters and $|0\rangle, |1\rangle$ the two eigenstates of the qubit in the computational basis. If the values of both θ and ϕ are known to Alice (Bob), then she (he) is able to remotely prepare the state $|q\rangle$ for Charlie following the original RSP protocol proposed in [5]. Of interest here is a situation when, say, for the sake of confidentiality, the complete classical knowledge of $|q\rangle$ is split between Alice and Bob so that neither Alice nor Bob alone can fully identify the state. Namely, let the complete classical knowledge of $|q\rangle$ be split in such a way that Alice knows only the value of θ (and not that of ϕ) while Bob knows only the value of ϕ (and not that of θ). As a consequence, both the teleportation [2] and the original RSP [5] protocols will not help. Of course, if Alice and Bob could get together to exchange their own data, then there would be no problems. But, remember, Alice and Bob are at spatially separated nodes. The question is: 'how can they, only by means of LOCC, jointly prepare the state $|q\rangle$ at Charlie's remote node?' Our first protocol for such a JRSP uses a single GHZ state in the form

$$|\mathrm{GHZ}\rangle_{ABC} = (\alpha|000\rangle + \beta|111\rangle)_{ABC} \tag{2}$$

of which qubit $A(B, C)$ belongs to Alice (Bob, Charlie). Here $|000\rangle_{ABC} \equiv |0\rangle_A\otimes|0\rangle_B\otimes|0\rangle_C$, $|111\rangle_{ABC} \equiv |1\rangle_A\otimes|1\rangle_B\otimes|1\rangle_C$ while the coefficients α, β are assumed real for simplicity and $\alpha^2 + \beta^2 = 1$ to satisfy the normalization condition. Without any loss of generality we set $\alpha^2 \geqslant \beta^2$. Note that $\alpha = \beta = 1/\sqrt{2}(\alpha^2 > \beta^2 > 0)$ corresponds to the case of maximal (nonmaximal) entanglement.

The protocol proceeds with each of Alice and Bob performing a vNM on her/his own qubit in an appropriate basis. Alice's measurement basis is $\{|\Phi\rangle_A, |\Phi_\perp\rangle_A\}$ which is related to the computational basis $\{|0\rangle_A, |1\rangle_A\}$ as

$$\begin{aligned}|\Phi\rangle_A &= \cos\theta|0\rangle_A + \sin\theta|1\rangle_A,\\ |\Phi_\perp\rangle_A &= \sin\theta|0\rangle_A - \cos\theta|1\rangle_A.\end{aligned} \tag{3}$$

Alice is able to use basis (3) since θ is known to her. As for Bob, he knows ϕ, so he is able to measure his qubit B in the basis $\{|\Psi\rangle_B, |\Psi_\perp\rangle_B\}$ which is determined through $\{|0\rangle_B, |1\rangle_B\}$ by

$$\begin{aligned}|\Psi\rangle_B &= \tfrac{1}{\sqrt{2}}(|0\rangle_B + \mathrm{e}^{\mathrm{i}\phi}|1\rangle_B),\\ |\Psi_\perp\rangle_B &= \tfrac{1}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\phi}|0\rangle_B - |1\rangle_B).\end{aligned} \tag{4}$$

In terms of $|\Phi\rangle_A$, $|\Phi_\perp\rangle_A$, $|\Psi\rangle_B$ and $|\Psi_\perp\rangle_B$ the quantum channel (2) reads

$$\begin{aligned}|\mathrm{GHZ}\rangle_{ABC} = \tfrac{1}{\sqrt{2}}[&\mathrm{i}|\Phi\rangle_A|\Psi\rangle_B\sigma_y(\beta\,\mathrm{e}^{-\mathrm{i}\phi}\sin\theta|0\rangle - \alpha\cos\theta|1\rangle)_C\\ &- \mathrm{i}|\Phi_\perp\rangle_A|\Psi\rangle_B\,\mathrm{e}^{-\mathrm{i}\phi}\sigma_y(\beta\cos\theta|0\rangle + \alpha\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_C\\ &- |\Phi\rangle_A|\Psi_\perp\rangle_B\,\mathrm{e}^{\mathrm{i}\phi}\sigma_x(\beta\,\mathrm{e}^{-\mathrm{i}\phi}\sin\theta|0\rangle - \alpha\cos\theta|1\rangle)_C\\ &+ |\Phi_\perp\rangle_A|\Psi_\perp\rangle_B\sigma_x(\beta\cos\theta|0\rangle + \alpha\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_C]\end{aligned} \tag{5}$$

with $\sigma_x = \{\{0, 1\}, \{1, 0\}\}$, $\sigma_y = \{\{0, -\mathrm{i}\}, \{\mathrm{i}, 0\}\}$ and $\sigma_z = \{\{1, 0\}, \{0, -1\}\}$ the well-known Pauli matrices. If the quantum channel is maximally entangled (i.e., $\alpha = \beta = 1/\sqrt{2}$), then equation (2) simplifies to

$$\begin{aligned}|\mathrm{GHZ}\rangle_{ABC} = \tfrac{1}{2}[&\mathrm{i}|\Phi\rangle_A|\Psi\rangle_B\sigma_y|q_\perp\rangle_C - \mathrm{i}|\Phi_\perp\rangle_A|\Psi\rangle_B\,\mathrm{e}^{-\mathrm{i}\phi}\sigma_y|q\rangle_C\\ &- |\Phi\rangle_A|\Psi_\perp\rangle_B\,\mathrm{e}^{\mathrm{i}\phi}\sigma_x|q_\perp\rangle_C + |\Phi_\perp\rangle_A|\Psi_\perp\rangle_B\sigma_x|q\rangle_C],\end{aligned} \tag{6}$$

where $|q_\perp\rangle = \mathrm{e}^{-\mathrm{i}\phi}\sin\theta|0\rangle - \cos\theta|1\rangle$ is the compliment state of $|q\rangle$. As is recognized from equation (6), the total success probability of JRSP for maximal entangled quantum channel is 1/2 : one-fourth comes from Alice–Bob finding $|\Phi_\perp\rangle_A$–$|\Psi\rangle_B$ (the second line of equation (6)) and another fourth comes from Alice–Bob finding $|\Phi_\perp\rangle_A$–$|\Psi_\perp\rangle_B$ (the last line of equation (6)). In the former (latter) case Charlie needs a correction by acting $\sigma_y(\sigma_x)$ on his qubit C to convert it to the desired state $|q\rangle_C$, modulo possible extra global phase factor.

On the other hand, as can be seen from equation (5), if entanglement of the quantum channel is nonmaximal (i.e., $\alpha^2 > \beta^2 > 0$), then, for the outcomes $|\Phi_\perp\rangle_A$–$|\Psi\rangle_B$ or $|\Phi_\perp\rangle_A$–$|\Psi_\perp\rangle_B$, the qubit C collapses into the (unnormalized) state

$$\tfrac{1}{\sqrt{2}}\sigma_y(\beta\cos\theta|0\rangle + \alpha\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_C \tag{7}$$

or

$$\tfrac{1}{\sqrt{2}}\sigma_x(\beta\cos\theta|0\rangle + \alpha\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_C. \tag{8}$$

After performing σ_y or σ_x on his qubit C, Charlie still needs additional actions. Namely, Charlie introduces an ancillary qubit C' in state $|0\rangle_{C'}$ then lets the qubits C and C' go through a two-qubit gate $V_{CC'}$ followed by measuring qubit C'. The gate $V_{CC'}$ has, in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}_{CC'}$, the following explicit form:

$$V_{CC'} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -\beta/\alpha & \sqrt{1-\beta^2/\alpha^2} & 0\\ 0 & \sqrt{1-\beta^2/\alpha^2} & \beta/\alpha & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{9}$$

which can be constructed by two controlled-NOT gates (CNOTs) and one controlled-ROTATION gate (CROT)

$$V_{CC'} = \mathrm{CNOT}_{CC'}\,\mathrm{CROT}_{C'C}\,\mathrm{CNOT}_{CC'}, \tag{10}$$

as shown in figure 1. The action of $\mathrm{CNOT}_{CC'}$ and $\mathrm{CROT}_{C'C}$ in equation (10) on a two-qubit state $|a\rangle_C|b\rangle_{C'}$ implies

$$\mathrm{CNOT}_{CC'}|a\rangle_C|b\rangle_{C'} = |a\rangle_C|b\oplus a\rangle_{C'}, \tag{11}$$

where $a, b \in \{0, 1\}$ and $\oplus$ stands for an addition mod 2, while

$$\mathrm{CROT}_{C'C}|a\rangle_C|b\rangle_{C'} = (R^b(\vartheta)|a\rangle_C)|b\rangle_{C'}, \tag{12}$$

where $R^0(\vartheta) \equiv I$ is the 2×2 identity operator and

$$R^1(\vartheta) \equiv R(\vartheta) = \begin{pmatrix}-\cos\vartheta & \sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix}, \tag{13}$$

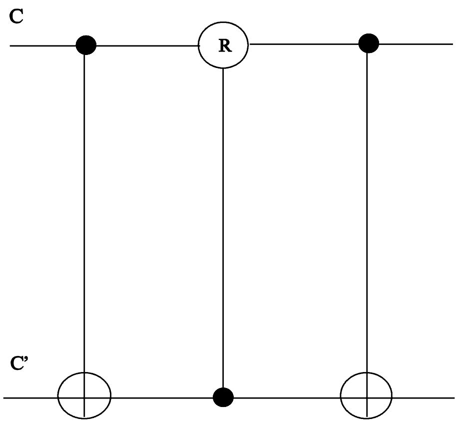

Figure 1. The gate $V_{CC'}$ defined by equation (9) in terms of two CNOTs and a CROT. The solid circle represents the control qubit.

with $\cos\vartheta = \beta/\alpha$, represents a formal rotation by an angle ϑ in the two-dimensional Hilbert space of a qubit.

Since

$$V_{CC'}\left[\frac{1}{\sqrt{2}}(\beta\cos\theta|0\rangle + \alpha\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_C|0\rangle_{C'}\right]$$

$$= \frac{\beta}{\sqrt{2}}|q\rangle_C|0\rangle_{C'} + \sqrt{\frac{\alpha^2-\beta^2}{2}}\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|0\rangle_C|1\rangle_{C'}, \quad (14)$$

the JRSP protocol succeeds when Charlie finds $|0\rangle_{C'}$, which takes place with a probability of $\beta^2/2$ (if, instead, Charlie finds $|1\rangle_{C'}$, the protocol fails because this projects qubit C onto $|0\rangle_C$). The total success probability in this nonmaximally entangled channel case is therefore $2\times\beta^2/2 = \beta^2$.

To be more general we now consider the qubit state to be prepared in the form

$$|Q\rangle = x|0\rangle + y|1\rangle, \quad (15)$$

where x is real and y is a complex nonzero parameter satisfying the normalization condition

$$x^2 + |y|^2 = 1. \quad (16)$$

In addition,

$$x = ac, \qquad y = bd \quad (17)$$

with a, c real and b, d complex nonzero coefficients. The complete classical information of the state $|Q\rangle$ is independently shared between Alice and Bob in the following way: while the coefficients a, b are known only to Alice (and by no means to Bob), the coefficients c, d are known only to Bob (and by no means to Alice). This situation was dealt with in [16], in which it is further assumed that

$$a^2 + |b|^2 = 1, \quad (18)$$

$$c^2 + |d|^2 = 1. \quad (19)$$

Unfortunately, condition (16) is not compatible with (18) and (19). In fact, from equations (16)–(19) it follows that

$$a^2 = \frac{c^2}{2c^2-1}, \quad (20)$$

$$|b|^2 = \frac{c^2-1}{2c^2-1} \quad (21)$$

or

$$c^2 = \frac{a^2}{2a^2-1}, \quad (22)$$

$$|d|^2 = \frac{a^2-1}{2a^2-1}. \quad (23)$$

Since both a^2 and $|b|^2$ are nonzero positive, from equations (20) and (21) it requires that c^2 must be greater than 1. But for such c there are no d to meet constraint (19). Likewise, since both c^2 and $|d|^2$ are nonzero positive, from equations (22) and (23) it follows that a^2 must be greater than 1. But for such a there are no b to meet the constraint (18). Therefore, the coefficient setting in [16] is incorrect. Here we rely only on conditions (16) and (17).

To remotely prepare state (15) for Charlie, Alice performs a vNM on qubit A in the basis $\{|\varphi\rangle_A, |\varphi_\perp\rangle_A\}$:

$$|\varphi\rangle_A = \frac{a|0\rangle_A + b|1\rangle_A}{\sqrt{a^2+|b|^2}}, \qquad |\varphi_\perp\rangle_A = \frac{b^*|0\rangle_A - a|1\rangle_A}{\sqrt{a^2+|b|^2}}. \quad (24)$$

Alice is able to do that because she knows the values of both a and b. As for Bob, he performs a vNM on qubit B in the basis $\{|\psi\rangle_B, |\psi_\perp\rangle_B\}$:

$$|\psi\rangle_B = \frac{c|0\rangle_B + d|1\rangle_B}{\sqrt{c^2+|d|^2}}, \qquad |\psi_\perp\rangle_B = \frac{d^*|0\rangle_B - c|1\rangle_B}{\sqrt{c^2+|d|^2}}. \quad (25)$$

Bob is able to do that because he knows the values of both c and d.

In terms of $|\varphi\rangle_A, |\varphi_\perp\rangle_A, |\psi\rangle_B$ and $|\psi_\perp\rangle_B$ the quantum channel (2) reads

$$|\mathrm{GHZ}\rangle_{ABC} = \frac{1}{\sqrt{(a^2+|b|^2)(c^2+|d|^2)}} \times [|\varphi\rangle_A|\psi\rangle_B(ac\alpha|0\rangle + b^*d^*\beta|1\rangle)_C + |\varphi\rangle_A|\psi_\perp\rangle_B(ad\alpha|0\rangle - b^*c\beta|1\rangle)_C + |\varphi_\perp\rangle_A|\psi\rangle_B(bc\alpha|0\rangle - ad^*\beta|1\rangle)_C + |\varphi_\perp\rangle_A|\psi_\perp\rangle_B(bd\alpha|0\rangle + ac\beta|1\rangle)_C], \quad (26)$$

which for $\alpha = \beta = 1/\sqrt{2}$ simplifies to

$$|\mathrm{GHZ}\rangle_{ABC} = \frac{1}{\sqrt{2(a^2+|b|^2)(c^2+|d|^2)}} \times [|\varphi\rangle_A|\psi\rangle_B(ac|0\rangle + b^*d^*|1\rangle)_C + |\varphi\rangle_A|\psi_\perp\rangle_B(ad|0\rangle - b^*c|1\rangle)_C + |\varphi_\perp\rangle_A|\psi\rangle_B(bc|0\rangle - ad^*|1\rangle)_C + |\varphi_\perp\rangle_A|\psi_\perp\rangle_B(bd|0\rangle + ac|1\rangle)_C]. \quad (27)$$

Clearly, from equation (27), when Alice finds $|\varphi_\perp\rangle_A$ and Bob finds $|\psi_\perp\rangle_B$, with a probability of $1/[2(a^2+|b|^2)(c^2+|d|^2)]$, Charlie is able to obtain the desired state by applying a σ_x on C. In particular, it can be easily checked that, when $a = \cos\theta, b = \sin\theta, c = 1$ and $d = \exp(\mathrm{i}\phi)$, the situations corresponding to the two last lines in equation (27) give rise to a success, with the total probability of $2/[2(a^2+|b|^2)(c^2+|d|^2)] = 1/2$, as it should be.

For $\alpha^2 > \beta^2 > 0$ (see equation (26)) the method presented in [16] applies, but the correct success probability must be $\beta^2/[(a^2+|b|^2)(c^2+|d|^2)] \neq \beta^2$ because $(a^2+|b|^2)(c^2+|d|^2) \neq 1$ as discussed above.

3. Protocol 2

In protocol 1 described in the preceding section a difficulty may be associated with availability of the GHZ state. Technically, generation of multipartite entangled states is more demanding than that of bipartite entangled ones. In this sense, one asks: 'can the same JRSP task be performed using only bipartite entangled states?' The answer is 'yes' and in this section we present a new protocol for the JRSP task using a pair of EPR states, instead of a single GHZ state.

Let Alice (Bob) and Charlie share an EPR state of the form

$$|\mathrm{EPR}\rangle_{AC_1} = (\alpha|00\rangle + \beta|11\rangle)_{AC_1} \tag{28}$$

$$\left(|\mathrm{EPR}\rangle_{BC_2} = (\alpha|00\rangle + \beta|11\rangle)_{BC_2}\right) \tag{29}$$

of which qubit $A(B)$ belongs to Alice (Bob) while qubits C_1 and C_2 are hold by Charlie. We also assume real normalization coefficients $\alpha, \beta : \alpha^2+\beta^2 = 1$. When $\alpha = \beta = 1/\sqrt{2}$ the EPR states are maximally entangled but when $\alpha^2 > \beta^2 > 0$ their entanglement is nonmaximal.

To remotely prepare state (1), exactly as in protocol 1, here Alice and Bob also begin with measuring her/his own qubit in basis (3) and (4), respectively. In terms of $|\Phi\rangle_A, |\Phi_\perp\rangle_A, |\Psi\rangle_B$ and $|\Psi_\perp\rangle_B$ the quantum channels (28) and (29) read

$$\begin{aligned}
&|\mathrm{EPR}\rangle_{AC_1}|\mathrm{EPR}\rangle_{BC_2} \\
&\quad = \tfrac{1}{\sqrt{2}}\big[|\Phi\rangle_A|\Psi\rangle_B(\alpha\cos\theta|0\rangle + \beta\sin\theta|1\rangle)_{C_1}(\alpha|0\rangle \\
&\qquad + \beta\,\mathrm{e}^{-\mathrm{i}\phi}|1\rangle)_{C_2} - |\Phi_\perp\rangle_A|\Psi\rangle_B(\beta\cos\theta|1\rangle \\
&\qquad - \alpha\sin\theta|0\rangle)_{C_1}(\alpha|0\rangle + \beta\,\mathrm{e}^{-\mathrm{i}\phi}|1\rangle)_{C_2} + |\Phi\rangle_A|\Psi_\perp\rangle_B \\
&\qquad \times(\alpha\cos\theta|0\rangle + \beta\sin\theta|1\rangle)_{C_1}(\alpha\,\mathrm{e}^{\mathrm{i}\phi}|0\rangle - \beta|1\rangle)_{C_2} \\
&\qquad - |\Phi_\perp\rangle_A|\Psi_\perp\rangle_B(\beta\cos\theta|1\rangle \\
&\qquad - \alpha\sin\theta|0\rangle)_{C_1}(\alpha\,\mathrm{e}^{\mathrm{i}\phi}|0\rangle - \beta|1\rangle)_{C_2}\big].
\end{aligned} \tag{30}$$

As is clear from equation (30), for whatever measurement outcome of Alice and Bob, neither of Charlie's qubits C_1 and C_2 can be converted to the desired one by an elementary operation $U \in \{I, \sigma_x, \sigma_y, \sigma_z\}$. Fortunately, a further application by Charlie of a CNOT on qubits C_1 and C_2 followed by measuring C_2 proves useful. Under the action of $\mathrm{CNOT}_{C_1C_2}$ equation (30) becomes

$$\begin{aligned}
&\mathrm{CNOT}_{C_1C_2}|\mathrm{EPR}\rangle_{AC_1}|\mathrm{EPR}\rangle_{BC_2} \\
&\quad = \tfrac{1}{\sqrt{2}}\big\{|\Phi\rangle_A|\Psi\rangle_B\big[\mathrm{i}\sigma_y(\beta^2\,\mathrm{e}^{-\mathrm{i}\phi}\sin\theta|0\rangle - \alpha^2\cos\theta|1\rangle)_{C_1}|0\rangle_{C_2} \\
&\qquad + \mathrm{e}^{-\mathrm{i}\phi}\alpha\beta(\cos\theta|0\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_{C_1}|1\rangle_{C_2}\big] \\
&\qquad - |\Phi_\perp\rangle_A|\Psi\rangle_B\big[\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\phi}\sigma_y(\beta^2\cos\theta|0\rangle \\
&\qquad + \alpha^2\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_{C_1}|0\rangle_{C_2} - \alpha\beta(\,\mathrm{e}^{-\mathrm{i}\phi}\sin\theta|0\rangle \\
&\qquad - \cos\theta|1\rangle)_{C_1}|1\rangle_{C_2}\big] - |\Phi\rangle_A|\Psi_\perp\rangle_B\big[\mathrm{e}^{\mathrm{i}\phi}\sigma_x(\beta^2\,\mathrm{e}^{-\mathrm{i}\phi}\sin\theta|0\rangle \\
&\qquad - \alpha^2\cos\theta|1\rangle)_{C_1}|0\rangle_{C_2} + \alpha\beta\sigma_z(\cos\theta|0\rangle \\
&\qquad + \mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_{C_1}|1\rangle_{C_2}\big] + |\Phi_\perp\rangle_A|\Psi_\perp\rangle_B\big[\sigma_x(\beta^2\cos\theta|0\rangle \\
&\qquad + \alpha^2\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_{C_1}|0\rangle_{C_2} - \alpha\beta\,\mathrm{e}^{\mathrm{i}\phi}\sigma_z \\
&\qquad \times(\mathrm{e}^{-\mathrm{i}\phi}\sin\theta|0\rangle - \cos\theta|1\rangle)_{C_1}|1\rangle_{C_2}\big]\big\},
\end{aligned} \tag{31}$$

where σ_x, σ_y and σ_z act only on qubit C_1.

First we consider the case of maximally entangled quantum channels with $\alpha = \beta = 1/\sqrt{2}$ for which equation (31) reduces to

$$\begin{aligned}
&\mathrm{CNOT}_{C_1C_2}|\mathrm{EPR}\rangle_{AC_1}|\mathrm{EPR}\rangle_{BC_2} \\
&\quad = \tfrac{1}{2\sqrt{2}}\big\{|\Phi\rangle_A|\Psi\rangle_B\big[\mathrm{i}\sigma_y|q_\perp\rangle_{C_1}|0\rangle_{C_2} + \mathrm{e}^{-\mathrm{i}\phi}|q\rangle_{C_1}|1\rangle_{C_2}\big] \\
&\qquad - |\Phi_\perp\rangle_A|\Psi\rangle_B\big[\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\phi}\sigma_y|q\rangle_{C_1}|0\rangle_{C_2} - |q_\perp\rangle_{C_1}|1\rangle_{C_2}\big] \\
&\qquad - |\Phi\rangle_A|\Psi_\perp\rangle_B\big[\mathrm{e}^{\mathrm{i}\phi}\sigma_x|q_\perp\rangle_{C_1}|0\rangle_{C_2} + \sigma_z|q\rangle_{C_1}|1\rangle_{C_2}\big] \\
&\qquad + |\Phi_\perp\rangle_A|\Psi_\perp\rangle_B\big[\sigma_x|q\rangle_{C_1}|0\rangle_{C_2} - \mathrm{e}^{\mathrm{i}\phi}\sigma_z|q_\perp\rangle_{C_1}|1\rangle_{C_2}\big]\big\}.
\end{aligned} \tag{32}$$

Clearly, if the measurement outcomes of Alice, Bob and Charlie are $\{|\Phi\rangle_A, |\Psi\rangle_B, |1\rangle_{C_2}\}$, $\{|\Phi_\perp\rangle_A, |\Psi\rangle_B, |0\rangle_{C_2}\}$, $\{|\Phi\rangle_A, |\Psi_\perp\rangle_B, |1\rangle_{C_2}\}$ or $\{|\Phi_\perp\rangle_A, |\Psi_\perp\rangle_B, |0\rangle_{C_2}\}$, each happens with an equal probability of 1/8, the state of qubit C_1 can be transformed to $|q\rangle_{C_1}$ (possibly up to an unimportant global phase factor) by an elementary operation $U = I, \sigma_y, \sigma_z$ or σ_x, respectively. The total success probability of JRSP in this case is thus $4\times 1/8 = 1/2$.

Next, we consider the case of nonmaximally entangled quantum channels with $\alpha^2 > \beta^2 > 0$. From equation (31) one can see that, whenever the measurement outcomes of Alice, Bob and Charlie are $\{|\Phi\rangle_A, |\Psi\rangle_B, |1\rangle_{C_2}\}$ or $\{|\Phi\rangle_A, |\Psi_\perp\rangle_B, |1\rangle_{C_2}\}$, each happens with an equal probability of $\alpha^2\beta^2/2$, the state of qubit C_1 collapses automatically into $|q\rangle_{C_1}$ or can be σ_z-corrected to be in $|q\rangle_{C_1}$. However, the measurement outcomes $\{|\Phi_\perp\rangle_A, |\Psi\rangle_B, |0\rangle_{C_2}\}$ or $\{|\Phi_\perp\rangle_A, |\Psi_\perp\rangle_B, |0\rangle_{C_2}\}$ project C_1 onto the (unnormalized) state

$$\tfrac{1}{\sqrt{2}}\sigma_y(\beta^2\cos\theta|0\rangle + \alpha^2\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_{C_1} \tag{33}$$

or

$$\tfrac{1}{\sqrt{2}}\sigma_x(\beta^2\cos\theta|0\rangle + \alpha^2\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_{C_1}, \tag{34}$$

which are not yet the desired one. Here a similar trick as in protocol 1 can be invoked. Namely, first Charlie applies σ_y or σ_x on C_1 to have it in the state $\propto (\beta^2\cos\theta|0\rangle + \alpha^2\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_{C_1}$. Then Charlie introduces an ancillary qubit C' in state $|0\rangle_{C'}$ and lets the qubits C_1 and C' go through a two-qubit gate $U_{C_1C'}$ followed by measuring qubit C'. The gate $U_{C_1C'}$ has the explicit form

$$U_{C_1C'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\beta^2/\alpha^2 & \sqrt{1-\beta^4/\alpha^4} & 0 \\ 0 & \sqrt{1-\beta^4/\alpha^4} & \beta^2/\alpha^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{35}$$

in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}_{C_1C'}$. This gate $U_{C_1C'}$ can be constructed in a similar way as for $V_{CC'}$, i.e., $U_{C_1C'} = \mathrm{CNOT}_{C_1C'}\ \mathrm{CROT}_{\ C'C_1}\ \mathrm{CNOT}_{C_1C'}$ except that now we have $\cos\vartheta = \beta^2/\alpha^2$.

Because

$$U_{C_1C'}\left[\frac{1}{\sqrt{2}}(\beta^2\cos\theta|0\rangle + \alpha^2\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|1\rangle)_{C_1}|0\rangle_{C'}\right]$$

$$= \frac{\beta^2}{\sqrt{2}}|q\rangle_{C_1}|0\rangle_{C'} + \sqrt{\frac{\alpha^4-\beta^4}{2}}\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta|0\rangle_{C_1}|1\rangle_{C'} \tag{36}$$

when the outcome of Charlie's measuring C' is $|0\rangle_{C'}$ the state of qubit C_1 becomes $|q\rangle_{C_1}$ with a probability of $\beta^4/2$. Consequently, the overall success probability of protocol 2 in the case of nonmaximally entangled quantum channels is $2(\alpha^2\beta^2/2+\beta^4/2)=\beta^2$.

To remotely prepare state (1), exactly as in protocol 1, here Alice and Bob also begin with measuring her/his own qubit in the basis (24) and (25), respectively. In terms of $|\varphi\rangle_A, |\varphi_\perp\rangle_A, |\psi\rangle_B$ and $|\psi_\perp\rangle_B$, the expression for $\mathrm{CNOT}_{C_1C_2}|\mathrm{EPR}\rangle_{AC_1}|\mathrm{EPR}\rangle_{BC_2}$ reads

$$\begin{aligned}\mathrm{CNOT}_{C_1C_2}|\mathrm{EPR}\rangle_{AC_1}|\mathrm{EPR}\rangle_{BC_2} &= \frac{1}{\sqrt{(a^2+|b|^2)(c^2+|d|^2)}}\\ &\times\big\{|\varphi\rangle_A|\psi\rangle_B\big[(\alpha^2ac|0\rangle+\beta^2b^*d^*|1\rangle)_{C_1}|0\rangle_{C_2}\\ &+\alpha\beta(ad^*|0\rangle+cb^*|1\rangle)_{C_1}|1\rangle_{C_2}\big]+|\varphi\rangle_A|\psi_\perp\rangle_B\big[(\alpha^2ad|0\rangle\\ &-\beta^2b^*c|1\rangle)_{C_1}|0\rangle_{C_2}-\alpha\beta(ac|0\rangle-b^*d|1\rangle)_{C_1}|1\rangle_{C_2}\big]\\ &+|\varphi_\perp\rangle_A|\psi\rangle_B\big[(\alpha^2bc|0\rangle-\beta^2ad^*|1\rangle)_{C_1}|0\rangle_{C_2}\\ &+\alpha\beta(bd^*|0\rangle-ac|1\rangle)_{C_1}|1\rangle_{C_2}\big]\\ &+|\varphi_\perp\rangle_A|\psi_\perp\rangle_B\big[(\alpha^2bd|0\rangle+\beta^2ac|1\rangle)_{C_1}|0\rangle_{C_2}\\ &-\alpha\beta(bc|0\rangle+ad|1\rangle)_{C_1}|1\rangle_{C_2}\big]\big\},\end{aligned} \tag{37}$$

which in the case of $\alpha=\beta=1/\sqrt{2}$ simplifies to

$$\begin{aligned}\mathrm{CNOT}_{C_1C_2}|\mathrm{EPR}\rangle_{AC_1}|\mathrm{EPR}\rangle_{BC_2} &= \frac{1}{2\sqrt{(a^2+|b|^2)(c^2+|d|^2)}}\\ &\times\big\{|\varphi\rangle_A|\psi\rangle_B\big[(ac|0\rangle+b^*d^*|1\rangle)_{C_1}|0\rangle_{C_2}\\ &+(ad^*|0\rangle+cb^*|1\rangle)_{C_1}|1\rangle_{C_2}\big]+|\varphi\rangle_A|\psi_\perp\rangle_B\\ &\times\big[(ad|0\rangle-b^*c|1\rangle)_{C_1}|0\rangle_{C_2}-(ac|0\rangle-b^*d|1\rangle)_{C_1}|1\rangle_{C_2}\big]\\ &+|\varphi_\perp\rangle_A|\psi\rangle_B\big[(bc|0\rangle-ad^*|1\rangle)_{C_1}|0\rangle_{C_2}\\ &+(bd^*|0\rangle-ac|1\rangle)_{C_1}|1\rangle_{C_2}\big]\\ &+|\varphi_\perp\rangle_A|\psi_\perp\rangle_B\big[(bd|0\rangle+ac|1\rangle)_{C_1}|0\rangle_{C_2}\\ &-(bc|0\rangle+ad|1\rangle)_{C_1}|1\rangle_{C_2}\big]\big\}.\end{aligned} \tag{38}$$

From equation (38) it is evident that, if the measurement outcomes of Alice, Bob and Charlie are $\{|\varphi_\perp\rangle_A, |\psi_\perp\rangle_B, |0\rangle_{C_2}\}$, which occurs with a probability of $1/[4(a^2+|b|^2)(c^2+|d|^2)]$, the state of qubit C_1 can be transformed to $|Q\rangle_{C_1}$ by application of σ_x. In particular, if $a=\cos\theta$, $b=\sin\theta$, $c=1$ and $d=\exp(\mathrm{i}\phi)$, all the outcomes $\{|\varphi\rangle_A, |\psi\rangle_B, |1\rangle_{C_2}\}$, $\{|\varphi_\perp\rangle_A, |\psi\rangle_B, |0\rangle_{C_2}\}$, $\{|\varphi\rangle_A, |\psi_\perp\rangle_B, |1\rangle_{C_2}\}$ and $\{|\varphi_\perp\rangle_A, |\psi_\perp\rangle_B, |0\rangle_{C_2}\}$ are fine and the total success probability adds to be $4/[4(a^2+|b|^2)(c^2+|d|^2)]=1/2$, as it should be.

For $\alpha^2>\beta^2>0$, from equation (37) it follows that, whenever the measurement outcomes are $\{|\varphi_\perp\rangle_A, |\psi_\perp\rangle_B, |0\rangle_{C_2}\}$ the qubit C_1 is projected onto the (unnormalized) state $(\alpha^2bd|0\rangle+\beta^2ac|1\rangle)_{C_1}=\sigma_x(\beta^2ac|0\rangle+\alpha^2bd|1\rangle)_{C_1}$. Charlie then applies σ_x on C_1 followed by using an ancilla C' and the gate $U_{C_1C'}$ (see equations (35) and (36)) as described above in order to obtain the desired state $|Q\rangle_{C_1}$. The probability for such situation is $\beta^4/[(a^2+|b|^2)(c^2+|d|^2)]$.

4. Discussion and conclusion

Before the conclusion, we give some discussions regarding the required amounts of classical communication and quantum entanglement as well as the feasibility of our proposed protocols.

As is well known, classical communication plays a crucial role in any quantum protocol. In fact, without classical communication the quantum channel remains passive. In general, in each of our protocols 2 cbits (one announced by Alice and the other by Bob on their vNM outcome) are needed for Charlie to complete the state preparation at his node. In some cases the cbits can be cleverly managed to instruct Charlie's actions. Let us illustrate this for the state $|q\rangle$ given by equation (1). Let Alice's measurement outcome be $a=\{0,1\}$ if she finds $\{|\Phi\rangle_A, |\Phi_\perp\rangle_A\}$, Bob's measurement outcome be $b=\{0,1\}$ if he finds $\{|\Psi\rangle_B, |\Psi_\perp\rangle_B\}$ and, Charlie's measurement outcome be $c=\{0,1\}$ if she finds $\{|0\rangle_{C_2}, |1\rangle_{C_2}\}$. Then, for protocol 1, $a=0$ means a failure for both maximal and nonmaximal entanglement, $a=1$ indicates a deterministic success without ancillas for maximally entangled quantum channel and a probabilistic success with using ancillas for nonmaximally entangled quantum channel, while the value of b indicates the right correction operator ($b=0\to\sigma_y$, $b=1\to\sigma_x$). As for protocol 2, $a\oplus c=0$ means a failure for both maximal and nonmaximal entanglement, $a\oplus c=1$ signals a deterministic success without ancillas for maximally entangled quantum channel and a probabilistic success for nonmaximally entangled quantum channel (note in the latter case that the value of a provides an additional useful guidance: $a=0$ needs no ancillas, $a=1$ requires using ancillas), while the value of $(a\oplus b, c)$ instructs the right correction operator (namely, $\{(a\oplus b, c)\}=\{(0,0),(0,1),(1,0),(1,1)\}\to\{\sigma_x, I, \sigma_y, \sigma_z\}$).

It is worth noting also that the failure mentioned above (i.e., when $a=0$ in protocol 1 and $a\oplus c=0$ in protocol 2) implies impossibility of joint preparation of the exact state $|q\rangle$ at Charlie's site. However, as can easily be realized from equations (5) and (31), the exact complement state $|q_\perp\rangle=\mathrm{e}^{-\mathrm{i}\phi}\sin\theta|0\rangle-\cos\theta|1\rangle$ can be obtained instead in those failure events. If Charlie's purpose is just to simulate the measurement statistics, he can do that all the time with maximally (or $2\beta^2$ of the time with nonmaximally) entangled quantum channels since he is always able to suitably change his measuring apparatus whenever he gets $|q_\perp\rangle$ (for more detail on this issue see [5]). In this sense the failure events of JRSP are not entirely useless.

As for the entanglement amount of the quantum channel, it determines the success probability of the JRSP. The entanglement amount of state (2) can be assessed by $\mathcal{C}^{(A|BC)}$, $\mathcal{C}^{(B|CA)}$ and $\mathcal{C}^{(C|AB)}$, while that of states (28)–(29) can be assessed by $\mathcal{C}^{(A|C_1)}$ and $\mathcal{C}^{(B|C_2)}$, with $\mathcal{C}^{(X|Y)}$ the concurrence [17] of the composite system XY with respect to the partition $(X|Y)$. In our protocols $\mathcal{C}^{(A|BC)}=\mathcal{C}^{(B|CA)}=\mathcal{C}^{(C|AB)}=\mathcal{C}^{(A|C_1)}=\mathcal{C}^{(B|C_2)}=2\sqrt{(1-\beta^2)\beta^2}=\mathcal{C}$. Hence, for $\beta^2\leqslant\alpha^2$, i.e., $0\leqslant\beta^2\leqslant1/2$, $\mathcal{C}$ is monotonously increasing with β^2, starting from $\mathcal{C}=0$ for $\beta^2=0$ and approaching $\mathcal{C}=1$ for $\beta^2\to1/2$. In our protocols the calculated success probability is proportional to β^2, implying that sharing a nonzero entanglement amount is compulsorily necessary and, as should be expected, the larger the shared entanglement amount the higher the success probability.

As for feasibility, we stress that our protocols could be implemented iff the following three conditions are satisfied: (i) the participating parties have successfully shared the entangled states (2) or (28)–(29), (ii) perfect measuring devices for vNMs at all the parties' locations and perfect operations of two-qubit gates at Charlie's are available, and (iii) Charlie has quantum memories to store his qubit(s) until receiving the classical communication from Alice and Bob. The condition (i) is associated with the pre-protocol preparation which is practically demanding due to losses and decoherence during distribution of the entangled qubits. Generally, to meet the condition (i), entanglement distillation procedures (see, e.g., [18]) should be carried out, which is beyond the scope of the present paper. As in all other works of similar topics, here we start our protocols with the assumption that states (2) or (28)–(29) have been *a priori* provided. The condition (ii) is ideal since in practice errors appear in both qubit measuring and gate operating. If so, the prepared state will not be faithful. But, in some cases, it can be made optimally close to the desired state by an error-correcting procedure. This issue, however, is not touched upon here. As for the condition (iii), it cannot be avoided because super-luminal signaling is prohibited. This condition is also required in teleportation [2], RSP [5] as well as in many other global quantum tasks.

In conclusion, we have proposed two different new quantum protocols to remotely prepare a secret qubit state whose full classical knowledge is split among two distant parties. Individually neither party can do the task, but working in concert the two parties can. Each party performs a vNM measurement in the basis determined by his/her partial classical knowledge of the state to be prepared. Then they publicly broadcast their measurement outcomes according to which a third remote party is able to reconstruct the desired state. As for the quantum channels, protocol 1 uses a single GHZ state and protocol 2 uses a pair of EPR states. For the state $|Q\rangle$ (see equation (1)) a success is achieved when Alice finds $|\varphi_\perp\rangle_A$ and Bob finds $|\psi_\perp\rangle_B$ in both the protocols. However, protocol 1 yields a success probability of $1/[2(a^2+|b|^2)(c^2+|d|^2)](\beta^2/[(a^2+|b|^2)(c^2+|d|^2)])$ if the quantum channel is maximally (nonmaximally) entangled, whereas the corresponding probability of protocol 2 is $1/[4(a^2+|b|^2)(c^2+|d|^2)](\beta^4/[(a^2+|b|^2)(c^2+|d|^2)])$. For the state $|q\rangle$ (see equation (1)), both protocols 1 and 2 succeed with a total probability of 50% in the case of maximally entangled quantum channels and of β^2 if the shared entanglement is only partial. When there is maximal entanglement no ancillas are needed at all. When this is not so the need of using ancillas appears but it is protocol-dependent: protocol 1 always requires ancillas, while sometimes ancillas are not necessary in protocol 2. The advantage in quantum resources of protocol 2 is that it only utilizes EPR states. But its technical disadvantage is that it requires Charlie to be capable of performing a CNOT. Since nowadays GHZ, EPR states can be produced and various controlled-U gates can be engineered, our protocols could be implemented in the future when the aforementioned conditions (i) and (ii) become realized technologically.

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