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Homotopy perturbation transform method for time-fractional Newell-Whitehead-Segel equation containing Caputo-Prabhakar fractional derivative

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ABSTRACT: The main aim of the current article is to find the solution for Newell-Whitehead-Segel equations with constant coefficients containing Caputo-Prabhakar fractional derivative using the homotopy perturbation transform method. The convergence analysis of the obtained solution for the proposed fractional order model is presented. Four examples are presented to illustrate the efficiency and applicability and accurateness of the proposed numerical technique.

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1. Introduction

Fractional calculus (FC) is as good instrument to explain the hereditary, physical phenomena, diffusion problems, applied mathematics, sciences, nonlinear phenomenas and engineering [30, 23, 11]. Fractional differential operators is as extensions of classical differential operators of integer that they play an important role due to their wide spread applications in science, engineering, nonlinear optics, viscoelasticity, fluid flow, model various physical processes , biology, telecommunication, quantum mechanic, signal processing and other areas [14, 17, 19, 21, 22, 23, 30, 31, 32, 5, 6, 7, 8, 9]. In the recent years, much notice and attempt have been given to nonlinear fractional differential equations containing different fractional operators such as the Riemann-Liouville integral, the Caputo and the Riemann-Liouville derivatives because of their advantage to model anomalous phenomena of differential equations in many scientific and engineering fields and also, most of the models in different research domain of science and engineering applications are nonlinear. The analytical solution of the nonlinear fractional differential equations are usually impossible or difficult. For this reason, we have to solve fractional differential equations by appling various kinds of analytical such as, the variational iteration method [1, 27], the homotopy analysis method [37], the homotopy analysis Sumudu transform method [28], the homotopy perturbation method [38, 29], the Laplace transform method and the Adomian decomposition method(ADM) [20, 4]. The Newell-Whitehead-Segel equation is one of the significant concepts pattern formation theory. It describes the occurrence of stationary spatial stripe samples in a two-dimensional equation also the dynamic behavior near the bifurcation point of the Rayleigh-Benard

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convection of binary fluid mixtures [33]. Frequently two types of patterns are considered: First is the roll pattern in which cylinders form by fluid stream lines that these cylinders may be bend and form spiral like patterns. Second pattern is the hexagonal in which liquid flow is divided into honey comb cells. For instance, stripes patterns can be considered in human fingerprints, on zebra skin and in a visual cortex. Also, another pattern is Hexagonal pattern which obtained from the propagation of laser beams through a non-linear medium and in systems with chemical reaction and diffusion species [18]. In this paper, consider the time-fractional Newell-Whitehead-Segel equations as:

$$\begin{aligned} {}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(x,t) &= ku_{xx} + au - bu^q,\ 0<\mu\le 1,\\ u(x,0) &= u^0(x), \end{aligned} \tag{1}$$

where a, b are real numbers and k, q are positive integers and ${}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}$ denotes the Caputo-Prabhakar fractional derivative of order μ and it is defined as follows:

$$\begin{aligned} {}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(x,t) &= \int_0^t (t-\tau)^{-\mu}E^{-\gamma}_{\rho,1-\mu}(\omega(t-\tau)^{\rho})\dot{u}(x,\tau)d\tau\\ &= \mathbf{E}^{-\gamma}_{\rho,1-\mu,\omega,0^+}\frac{d}{dt}u(x,t),\ 0<\mu\le 1. \end{aligned} \tag{2}$$

Here, $\mathbf{E}$ is given by:

$$(\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+}u)(x,t) = \int_0^t (t-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-\tau)^{\rho})u(x,\tau)d\tau, \tag{3}$$

where $E^{\gamma}_{\rho,\mu}$ denotes the three-parameter Mittag-Leffler function [26] and which is as follows:

$$E^{\gamma}_{\rho,\mu}(t) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!\Gamma(\rho n+\mu)}t^n,\ \Re(\rho),\Re(\mu)>0,\ \gamma>0,\ t\in\mathbb{C}. \tag{4}$$

and symbolize $(\gamma)_n$ denotes the Pochhammer symbol and presented as follows[16]:

$$(\gamma)_0 = 1,\ (\gamma)_n = \gamma(\gamma+1)\dots(\gamma+n-1),\ n\in\mathbb{N}. \tag{5}$$

Most of our interest in studying three-parameter Mittag-Leffler functions are related to their importance in diverse areas of a model of complex susceptibility in the response of disordered materials and heterogeneous systems [25], response in anomalous dielectrics of Havriliak-Negami type [14, 34], in fractional viscoelasticity [12], in the discuss of stochastic processes [10], in probability theory [13], in the description of dynamical models of spherical stellar systems [3],fractional Poisson process [2] and fractional or integral differential equations [24, 5, 6, 7, 8, 9]. Furthermore, in Eq. (1), ${}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(x,t)$ is the variation of $u(x,t)$ with respect to temporal variable t at a set position and ku_{xx} display the variation of $u(x,t)$ with respect to spatial variable x at a specific time t and $au - bu^q$ is a nonlinear source term for $q>1$. Here, symbolize the unknown function $u(x,t)$ can be supposed to be the nonlinear distribution of temperature in a thin and infinitely long rod or the velocity of fluid flow in a tube of unlimited length with a small diameter. In this paper, we consider an approximation method based on the homotopy perturbation transform method(HPTM) which is the combination of two remarkably powerful methods, namely, the Laplace transform method and the homotopy analysis method and it investigated in [32, 36]. In the present investigation, we apply HPTM including Caputo-Prabhakar fractional derivative for solving Eq. (1) with $0<\mu\le 1$. For this purpose, this paper is organized as follows. Some necessary definitions and mathematical preliminaries of the fractional calculus are introduced in Section 2. In Section 3, we introduce an approximation method based on the HPTM, also, in this section, the sufficient conditions for the convergence of the proposed method and its error estimate is introduced. In Section 4, the proposed methods are applied to some numerical test examples to verify the validity and applicability of the suggested method.

2. Preliminaries

In this section, we study some important and basic properties of fractional calculus theory such as Laplace transform , definitions and lemmas which are applied in the next sections.

Definition 2.1. *[23, 30]. Let $0<\alpha\le 1$, $u\in L^1[a,b]$ and $0<t<b\le\infty$. Then the left-sided and the right-sided Riemann-Liouville fractional integrals and derivatives of order α are defined as:*

$$I^{\alpha}_{a^+}u(x,t) = \frac{1}{\Gamma(\alpha)}\int_a^t u(x,\tau)(t-\tau)^{\alpha-1}d\tau, \tag{6}$$

$$I^{\alpha}_{b^-}u(x,t) = \frac{1}{\Gamma(\alpha)}\int_t^b u(x,\tau)(\tau-t)^{\alpha-1}d\tau. \tag{7}$$

Lemma 2.2. *Let $\rho, \mu, \omega, \gamma \in \mathbb{C}$ and $\Re(\mu) > 0, \Re(\rho) > 0$. Then the Laplace transform of Eq. (2) for $m-1 < \mu \leq m$ is given by [9]:*

$$\mathcal{L}\Big({}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(x,t);s\Big) = s^{\mu}(1-\omega s^{-\rho})^{\gamma}U(x,s) - \sum_{k=0}^{m-1} s^{\mu-k-1}(1-\omega s^{-\rho})^{\gamma}u^{(k)}(0^+), \tag{8}$$

where $U(s) = \int_0^\infty e^{-st}u(t)dt$. Also, the Laplace transformation of the three-parameter Mittag-Leffler function which is defined by Eq. (4), is given by [17, 16]:

$$\mathcal{L}\Big(t^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega t^{\rho});s\Big) = s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}. \tag{9}$$

Lemma 2.3. *[23]. Let $\rho, \mu, \omega, \gamma \in \mathbb{C}$ and $\Re(\mu) > 0, \Re(\rho) > 0$. Then*

$$\int_0^t (t-y)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-y)^{\rho})y^{\nu-1}dy = \Gamma(\nu)t^{\mu+\nu-1}E^{\gamma}_{\rho,\mu+\nu}(\omega t^{\rho}). \tag{10}$$

Lemma 2.4. *[23]. Let $\rho, \mu, \nu, \omega, \sigma, \gamma \in \mathbb{C}$ and $\Re(\mu) > 0, \Re(\rho) > 0, \Re(\nu) > 0$. Then the following relation is hold for any summable function $u \in L(a,b)$:*

$$\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+}\mathbf{E}^{\sigma}_{\rho,\nu,\omega,0^+}u = \mathbf{E}^{\gamma+\sigma}_{\rho,\mu+\nu,\omega,0^+}u. \tag{11}$$

Also, by substituting $\sigma = -\gamma$, in Eq. (11) the following relation is obtained:

$$\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+}\mathbf{E}^{-\gamma}_{\rho,\nu,\omega,0^+}u = \mathbf{E}^{0}_{\rho,\mu+\nu,\omega,0^+}u = I^{\mu+\nu}_{0^+}u. \tag{12}$$

3. The HPTM to solve nonlinear fractional order differential equations

The purpose of this section is to obtain an approximation method based on the HPTM for obtaining the solutions of the fractional differential equations as:

$$\begin{aligned}
&{}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(x,t) + \mathbf{A}u(x,t) + \mathbf{N}u(x,t) = h(x,t),\\
&u(x,0) = u^0,
\end{aligned} \tag{13}$$

where $\mathbf{A}u(x,t)$ is a linear operator and the notation $\mathbf{N}u(x,t)$ is a nonlinear operator and $h(x,t)$ is a continuous function. For this aim, using Eq. (8) for $m = 1$ and taking the Laplace transform on both sides of Eq. (13), we obtain:

$$\begin{aligned}
&\mathcal{L}\Big({}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(x,t) + \mathbf{A}u(x,t) + \mathbf{N}u(x,t) - h(x,t);s\Big) = 0,\\
&s^{\mu}(1-\omega s^{-\rho})^{\gamma}U(x,s) - s^{\mu-1}(1-\omega s^{-\rho})^{\gamma}u(0^+) + \mathcal{L}\Big(\mathbf{A}u(x,t) + \mathbf{N}u(x,t);s\Big) = H(x,s),\\
&U(x,s) = s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}H(x,s) + \frac{1}{s}u(0) - (s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}) \times \Big[\mathcal{L}\Big(\mathbf{A}u(x,t) + \mathbf{N}u(x,t);s\Big)\Big].
\end{aligned} \tag{14}$$

Applying the Laplace inverse transform on Eq. (14), we obtain:

$$u(x,t) = \mathcal{L}^{-1}\Big[s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}H(x,s) + \frac{1}{s}u(0);t\Big] - \mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}) \times \Big[\mathcal{L}\Big(\mathbf{A}u(x,t) + \mathbf{N}u(x,t);s\Big)\Big];t\Big], \tag{15}$$

using the HPTM for obtaining the solution of Eq. (15). To obtain the solution of Eq. (15), we assuming that the solution of Eq. (1) is $u(t)$ which can be expressed as the following infinite series:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{16}$$

where $u_n(x,t)$ for $n = 0, 1, 2, \ldots$ are known functions. Also, the nonlinear part $\mathbf{N}u(x,t)$ can be represented as the infinite series as follows:

$$\mathbf{N}u(x,t) = \sum_{n=0}^{\infty} p^n \mathbf{H}_n(u(x,t)), \tag{17}$$

where $\mathbf{H}_n(u(x,t)) = \left\{\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\Big(\sum_{i=0}^{\infty}p^i u_i\Big)\right]_{p=0}\right\}$ are the Adomian polynomials [15]. Now, substituting Eqs. (16) and (17) into Eq. (15), we obtain:

$$\sum_{n=0}^{\infty}p^n u_n(x,t) = H(x,t) - \mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma})\times\Big[\mathcal{L}\Big(\mathbf{A}\sum_{n=0}^{\infty}p^n u_n(x,t) + \mathbf{N}\sum_{n=0}^{\infty}p^n u_n(x,t);s\Big)\Big];t\Big]$$
$$= H(x,t) - \mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma})\times\Big[\mathcal{L}\Big(\mathbf{A}\sum_{n=0}^{\infty}p^n u_n(x,t) + \sum_{n=0}^{\infty}p^n\mathbf{H}_n(u(x,t));s\Big)\Big];t\Big], \tag{18}$$

where $H(x,t) = \mathcal{L}^{-1}\Big[s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}H(x,s) + \frac{1}{s}u(0);t\Big]$. By equalling the coefficients on powers of p on the both sides of Eq. (18), we obtain the series solution as follows:

$$\begin{aligned}
&p^0 : u_0(x,t) = H(x,t),\\
&p^1 : u_1(x,t) = -\mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma})\times\Big[\mathcal{L}\Big(\mathbf{A}u_0(x,t) + \mathbf{H}_0(u(x,t));s\Big)\Big];t\Big],\\
&p^2 : u_2(x,t) = -\mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma})\times\Big[\mathcal{L}\Big(\mathbf{A}u_1(x,t) + \mathbf{H}_1(u(x,t));s\Big)\Big];t\Big],\\
&\vdots\\
&p^{n+1} : u_{n+1}(x,t) = -\mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma})\times\Big[\mathcal{L}\Big(\mathbf{A}u_n(x,t) + \mathbf{H}_n(u(x,t));s\Big)\Big];t\Big].
\end{aligned} \tag{19}$$

Then the solution of Eq. (1) can be written as follows:

$$u(x,t) = \sum_{n=0}^{\infty}u_n(x,t). \tag{20}$$

Now, we focus on the convergence of the proposed approximation method applied to Eq. (1).

Theorem 3.1. *[35] (Banach's fixed point theorem). Let $\mathbb{X}$ be a Banach space and $\mathbb{A} : \mathbb{X} \to \mathbb{X}$ is a nonlinear operator such that $\parallel \mathbb{A}(u) - \mathbb{A}(v) \parallel \leq \varrho \parallel u - v \parallel,\ u, v \in \mathbb{X}$ for some $\varrho < 1$. Then $\mathbb{A}$ has a unique fixed point. The sequence is defined by:*

$$u_{n+1} = \mathbb{A}(u_n),$$

with an arbitrary choice of $u_0 \in \mathbb{X}$, converges to the fixed point of $\mathbb{X}$ and

$$\parallel u_n - u_m \parallel \leq \parallel u_1 - u_0 \parallel \sum_{i=m-1}^{n-2}\varrho^i.$$

Theorem 3.2. *Suppose $u(x,t)$ be the exact solutions of Eq.* (1)*. If there exists a constant $\varrho \in (0,1)$ such that $\parallel u_{n+1}(x,t) \parallel \leq \varrho \parallel u_n(x,t) \parallel$ for each $n \in \mathbb{N}\bigcup\{0\}$. Then, the series solution defined in Eq.* (20) *is convergent to $u(x,t)$.*

Proof. The Theorem 3.1 is a sufficient condition to discussion the convergence of HPTM for some partial differential equations. Then we define the following sequence $\{\mathbb{S}_n\}_{n=0}^{\infty}$:

$$\begin{aligned}
&\mathbb{S}_0 = u_0(x,t),\\
&\mathbb{S}_1 = u_0(x,t) + u_1(x,t),\\
&\mathbb{S}_2 = u_0(x,t) + u_1(x,t) + u_2(x,t),\\
&\vdots\\
&\mathbb{S}_n = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots + u_n(x,t).
\end{aligned} \tag{21}$$

To prove convergence, we show that the sequence $\{\mathbb{S}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $\mathbb{X}$. So, for each $n \in \mathbb{N}\bigcup\{0\}$, from $\parallel u_{n+1}(x,t) \parallel \leq \varrho \parallel u_n(x,t) \parallel$ we have:

$$\begin{aligned}
&\parallel u_1(x,t) \parallel \leq \varrho \parallel u_0(x,t) \parallel = \varrho \parallel u^0(x) \parallel,\\
&\parallel u_2(x,t) \parallel \leq \varrho \parallel u_1(x,t) \parallel \leq \varrho\varrho \parallel u^0(x) \parallel,\\
&\vdots\\
&\parallel u_{n+1}(x,t) \parallel \leq \varrho \parallel u_n(x,t) \parallel \leq \varrho^{n+1} \parallel u^0(x) \parallel .
\end{aligned} \tag{22}$$

Then from Eq. (22), the following inequality is obtained:

$$\| \mathbb{S}_{n+1}(x,t) - \mathbb{S}_n(x,t) \| = \| u_{n+1}(x,t) \| \le \varrho^{n+1} \| u^0(x) \|, \tag{23}$$

then for $n \ge m,\ m,n \in \mathbb{N}$, we have:

$$\begin{aligned} \| \mathbb{S}_n(x,t) - \mathbb{S}_m(x,t) \| &= \| (\mathbb{S}_n(x,t) - \mathbb{S}_{n-1}(x,t)) + (\mathbb{S}_{n-1}(x,t) - \mathbb{S}_{n-2}(x,t)) + \ldots + (\mathbb{S}_{m+1}(x,t) - \mathbb{S}_m(x,t)) \| \\ &\le \| \mathbb{S}_n(x,t) - \mathbb{S}_{n-1}(x,t) \| + \| \mathbb{S}_{n-1}(x,t) - \mathbb{S}_{n-2}(x,t) \| + \ldots + \| \mathbb{S}_{m+1}(x,t) - \mathbb{S}_m(x,t) \| \\ &\le \varrho^n \| u_0(x,t) \| + \varrho^{n-1} \| u_0(x,t) \| + \ldots + \varrho^{m+1} \| u_0(x,t) \| \\ &\le \frac{\varrho^{m+1}(1-\varrho^{n-m})}{1-\varrho} \| u_0(x,t) \| . \end{aligned} \tag{24}$$

Therefore, $\| \mathbb{S}_n(x,t) - \mathbb{S}_m(x,t) \| \to 0$ when $m,n \to \infty$. So, the sequence $\{\mathbb{S}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $\mathbb{X}$ and it results that the series $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ is converges.

□

Theorem 3.3. *Suppose $u(x,t)$ be the exact solutions of Eq.* (1) *and the series introduced in Eq.* (20) *is convergent to the solution $u(x,t)$. If the truncated series $\mathbb{S}_m(x,t) = \sum_{i=0}^{m} u_i(x,t)$ is used as an approximation to the solution of Eq.* (20)*. Then the maximum error is calculated as follows:*

$$\| u(x,t) - \sum_{i=0}^{m} u_i(x,t) \| \le \frac{\varrho^{m+1}}{1-\varrho} \| u^0(x) \| .$$

Proof. From Eq. (24) for $n \ge m$, we have:

$$\| \mathbb{S}_n(x,t) - \mathbb{S}_m(x,t) \| \le \frac{\varrho^{m+1}(1-\varrho^{n-m})}{1-\varrho} \| u_0(x,t) \|, \tag{25}$$

as $n \to \infty,\ 0 < \varrho < 1$, then $\lim_{n\to\infty} \mathbb{S}_n(x,t) = u(x,t)$, from Eq. (25) is obtained:

$$\begin{aligned} \| u(x,t) - \sum_{i=0}^{m} u_i(x,t) \| &\le \frac{\varrho^{m+1}(1-\varrho^{n-m})}{1-\varrho} \| u_0(x,t) \| \\ &\le \frac{\varrho^{m+1}}{1-\varrho} \| u_0(x,t) \|,\ 1-\varrho^{n-m} < 1, \end{aligned} \tag{26}$$

which completes the proof.

□

Table 1: The absolute errors for various values of μ and $k = 1, \rho = 0.5, \omega = 1, \gamma = 0.75$ for Example 4.1.

x	$\mu = 0.5$	$\mu = 0.75$	$\mu = 0.85$	$\mu = 0.95$	$\mu = 0.99$
0.0	4.3581×10^{-10}	1.8395×10^{-10}	1.0111×10^{-10}	3.0494×10^{-11}	5.8377×10^{-12}
0.05	4.5815×10^{-10}	1.9338×10^{-10}	1.0629×10^{-10}	3.2057×10^{-11}	6.1370×10^{-12}
0.1	4.8164×10^{-10}	2.0329×10^{-10}	1.1174×10^{-10}	3.3701×10^{-11}	6.4516×10^{-12}
0.15	5.0633×10^{-10}	2.1372×10^{-10}	1.1747×10^{-10}	3.5429×10^{-11}	6.7824×10^{-12}
0.2	5.3230×10^{-10}	2.2467×10^{-10}	1.2349×10^{-10}	3.7245×10^{-11}	7.1301×10^{-12}
0.25	5.5959×10^{-10}	2.3619×10^{-10}	1.2982×10^{-10}	3.9155×10^{-11}	7.4957×10^{-12}
0.3	5.8828×10^{-10}	2.4830×10^{-10}	1.3648×10^{-10}	4.1162×10^{-11}	7.8800×10^{-12}
0.35	6.1844×10^{-10}	2.6103×10^{-10}	1.4348×10^{-10}	4.3273×10^{-11}	8.2840×10^{-12}
0.4	6.5015×10^{-10}	2.7442×10^{-10}	1.5083×10^{-10}	4.5491×10^{-11}	8.7088×10^{-12}
0.5	6.8348×10^{-10}	2.8849×10^{-10}	1.5857×10^{-10}	4.7824×10^{-11}	9.1553×10^{-12}

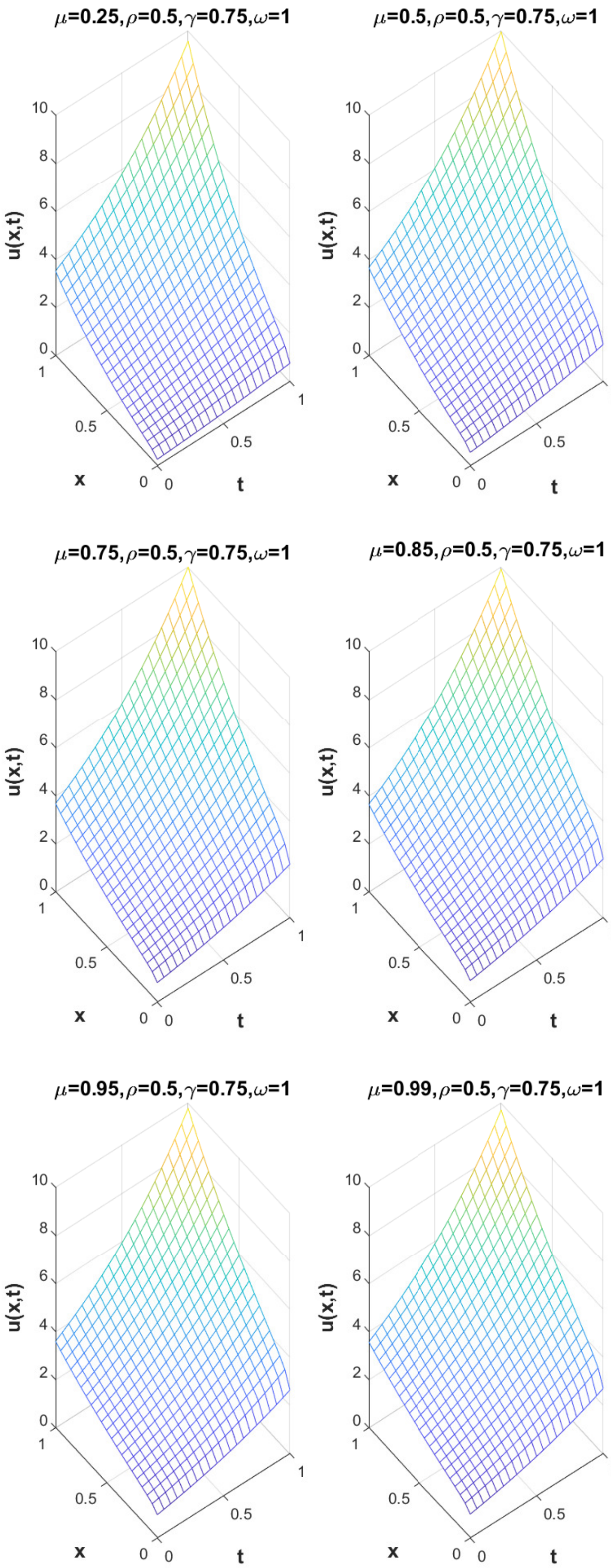

Figure 1: Graphs of approximation solution for Example 4.1.

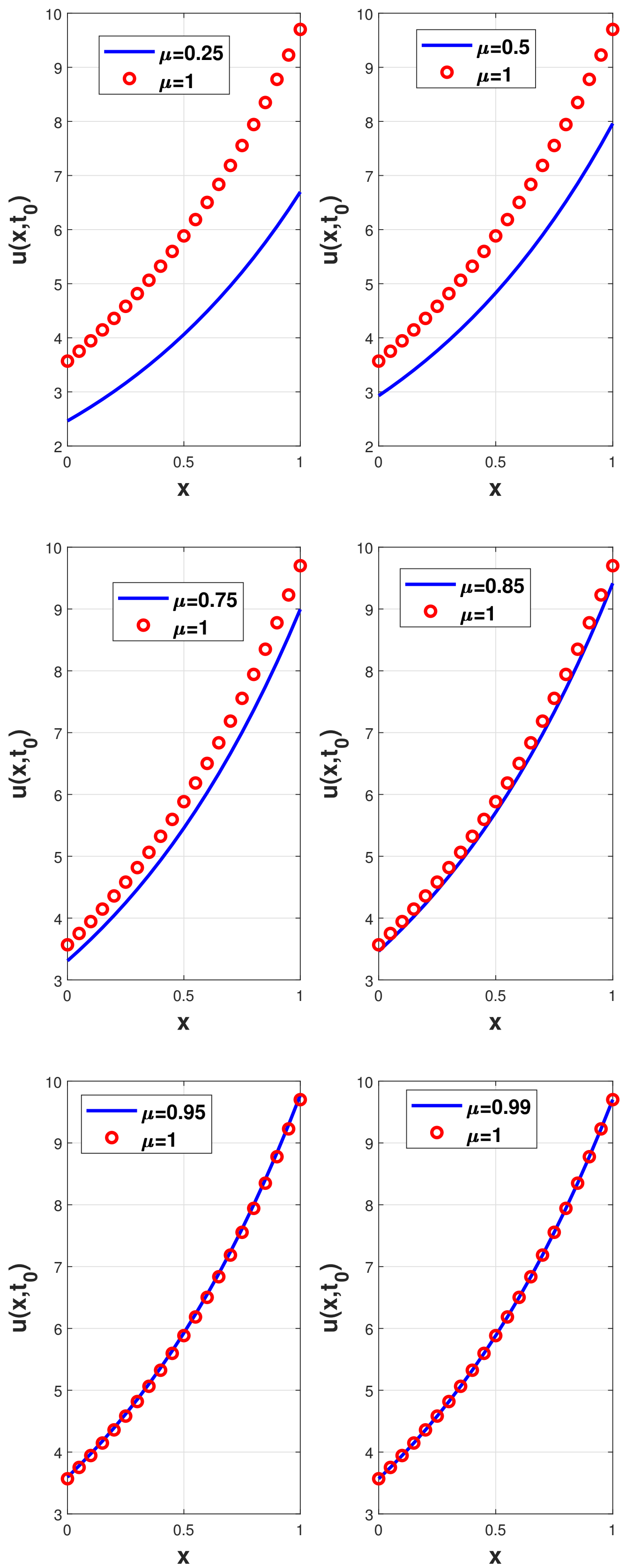

Figure 2: Plots of the exact solutions $u(x,t_0)$ and the approximate solutions $u_n(x,t_0)$ for $t_0 = 0.5$ at various values of $\mu = 0.25, 0.5, 0.75, 0.85, 0.95, 0.99, 1$ and $k = 1, n = 4$ for Example 4.1.

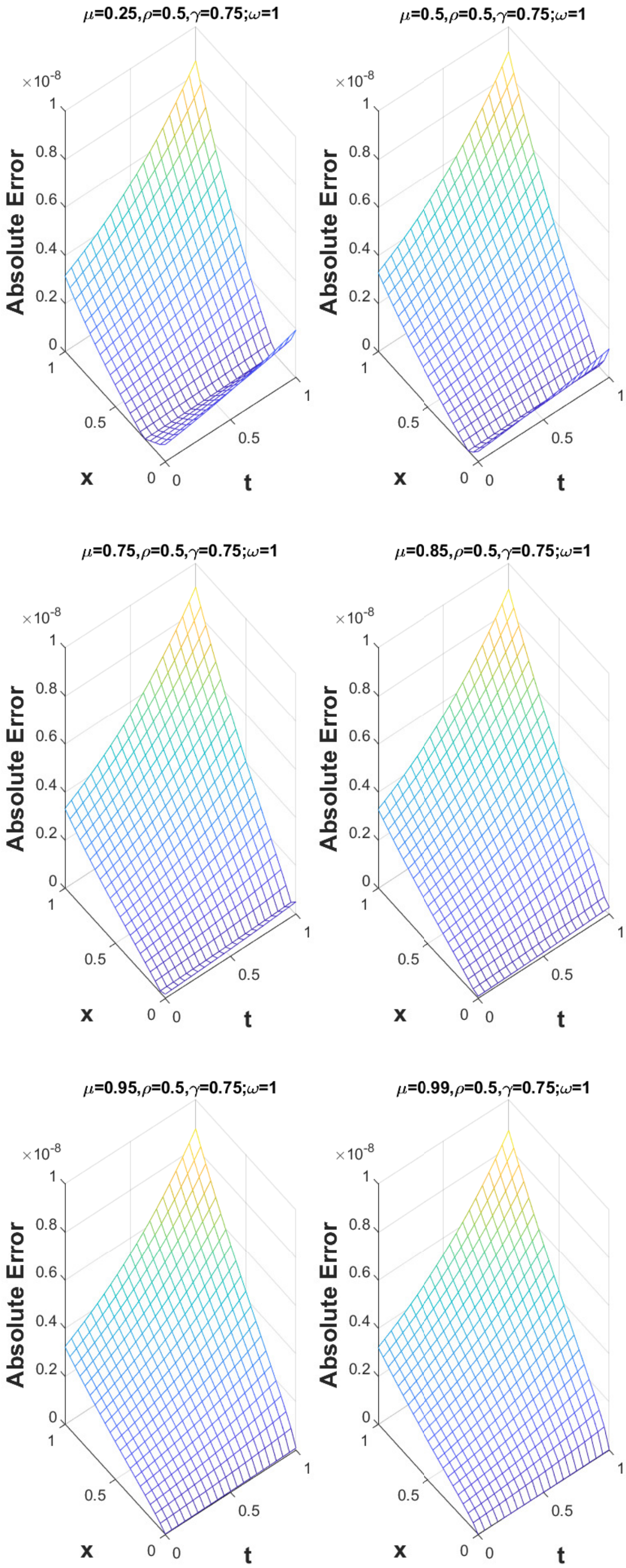

Figure 3: The analysis of absolute errors by the HPTM for Example 4.1.

4. Numerical Results

In this section, we are applying the HPTM to solve some test examples by considering $\mathbf{A}u(x,t) = -(ku_{xx} + au), \mathbf{N}u(x,t) = bu^q$ in Eq. (1) in order to show the application and effectiveness of the presented technique.

Example 4.1. *In this Example, we consider the following 2-dimensional linear time-fractional Newell-Whitehead-Segel equation of order* $0 < \mu \leq 1$*:*

$$\begin{aligned} {}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+}u(x,t) &= ku_{xx} - 2u,\ 0 < \mu \leq 1, \\ u(x,0) &= e^x, \end{aligned} \tag{27}$$

for $\mu = 1$ *the analytical solution of this equation is given by* $u(x,t) = e^{x-t}$*. Thus we solve the current test problem with the HPTM to find the solution of this kind of Eq. (27), then we have:*

$$\begin{aligned} p^0 &: u_0(x,t) = \mathcal{L}^{-1}\Big[s^{-\mu}(1-\omega s^{-\rho})^{-\gamma} \times 0 + \frac{1}{s}u(0);t\Big] = e^x, \\ p^1 &: u_1(x,t) = -\mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}) \times \Big[\mathcal{L}\Big(-k(u_0(x,t))_{xx} + 2u_0(x,t) + u_0^2;s\Big)\Big];t\Big] \\ &= \Big((k-2)e^x - e^{2x}\Big)\mathcal{L}^{-1}\Big[\frac{(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma})}{s}\Big] = \Big((k-2)e^x - e^{2x}\Big)t^{\mu}E^{\gamma}_{\rho,1+\mu}(\omega t^{\rho}), \\ p^2 &: u_2(x,t) = -\mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}) \times \Big[\mathcal{L}\Big(-k(u_1(x,t))_{xx} + 2u_1(x,t) + 2u_0u_1);s\Big)\Big];t\Big] \\ &= \Big[(k-2)^2e^x - 2(3k-2)e^{2x} + 3e^{3x}\Big]t^{2\mu}E^{\gamma}_{\rho,1+2\mu}(\omega t^{\rho}), \\ p^3 &: u_3(x,t) = -\mathcal{L}^{-1}\Big[(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}) \times \Big[\mathcal{L}\Big(-k(u_2(x,t))_{xx} + u_1^2(x,t) + 2u_0u_2);s\Big)\Big];t\Big], \\ &\vdots \end{aligned} \tag{28}$$

We use Eq. (9) on Eq. (28) and $\mathcal{L}^{-1}\Big[\frac{(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma})}{s}\Big]$ *is obtained as:*

$$\mathcal{L}^{-1}\Big[\frac{(s^{-\mu}(1-\omega s^{-\rho})^{-\gamma})}{s}\Big] = t^{\mu}E^{\gamma}_{\rho,1+\mu}(\omega t^{\rho}).$$

Finally, the solution of Eq. (27) are expressed by:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{29}$$

The approximation solution is expressed in Figure 1 for different values of μ *where* $\mu = 0.25, 0.5, 0.75, 0.85, 0.99, 1$*. Figure 2 shows the behavior of the analytical solution* $u(x,t_0)$ *and approximation solution* $u_n(x,t_0)$ *at different values of* μ*. Also, Figure 3 presents the absolute error between the exact solution and the approximation solution by the HPTM for Example 4.1. In Table 1 the absolute errors is showed for various values* μ*.*

Table 2: The absolute errors for various values of μ and $k = 1, \rho = 0.5, \omega = 1, \gamma = 0.75$ for Example 4.2.

x	$\mu = 0.5$	$\mu = 0.75$	$\mu = 0.85$	$\mu = 0.95$	$\mu = 0.99$
0.05	3.5197×10^{-9}	2.2526×10^{-9}	1.8138×10^{-9}	1.4375×10^{-9}	1.3049×10^{-9}
0.1	2.9282×10^{-9}	2.1640×10^{-9}	1.8546×10^{-9}	1.5671×10^{-9}	1.4600×10^{-9}
0.15	2.7308×10^{-9}	2.1790×10^{-9}	1.9321×10^{-9}	1.6910×10^{-9}	1.5981×10^{-9}
0.2	2.6568×10^{-9}	2.2280×10^{-9}	2.0207×10^{-9}	1.8103×10^{-9}	1.7273×10^{-9}
0.25	2.6397×10^{-9}	2.2935×10^{-9}	2.1144×10^{-9}	1.9269×10^{-9}	1.8514×10^{-9}
0.3	2.6546×10^{-9}	2.3689×10^{-9}	2.2114×10^{-9}	2.0421×10^{-9}	1.9727×10^{-9}
0.35	2.6902×10^{-9}	2.4512×10^{-9}	2.3112×10^{-9}	2.1569×10^{-9}	2.0927×10^{-9}
0.4	2.7403×10^{-9}	2.5390×10^{-9}	2.4137×10^{-9}	2.2723×10^{-9}	2.2126×10^{-9}
0.5	2.8017×10^{-9}	2.6318×10^{-9}	2.5191×10^{-9}	2.3889×10^{-9}	2.3332×10^{-9}

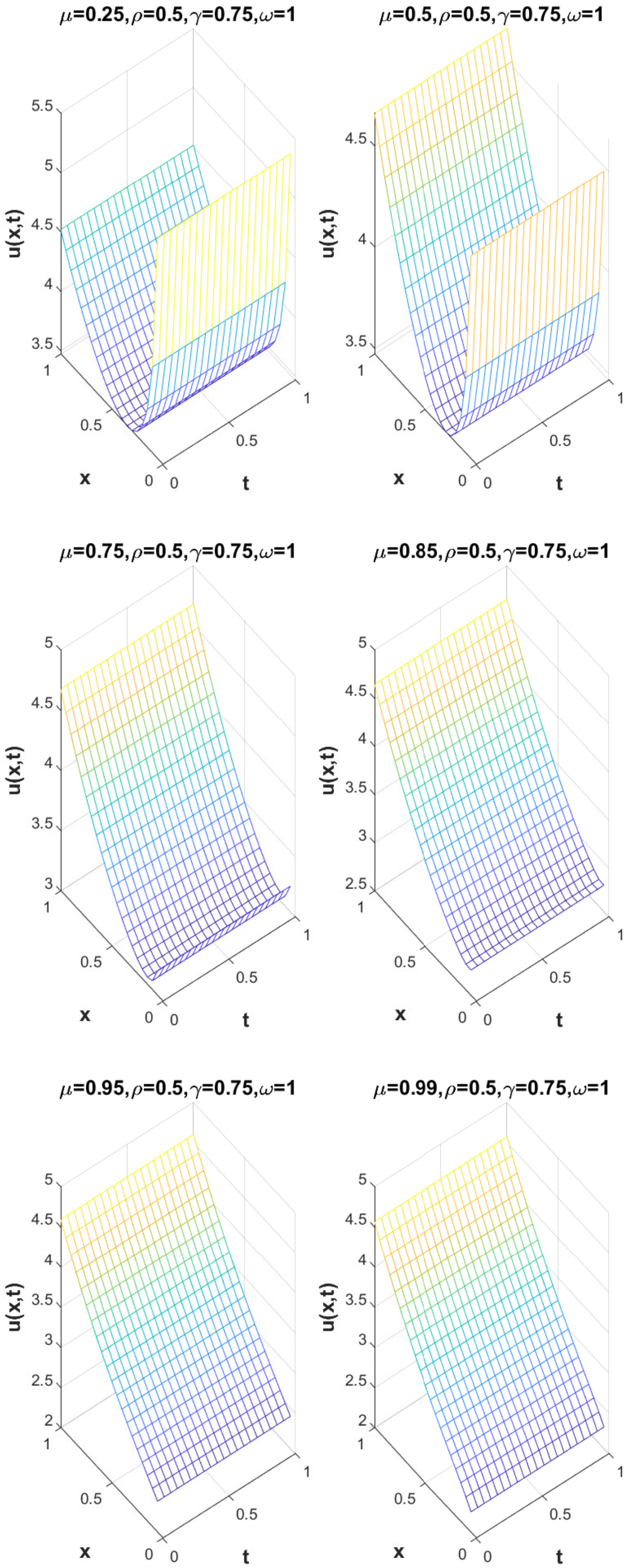

Figure 4: Graphs of approximation solution for Example 4.2.

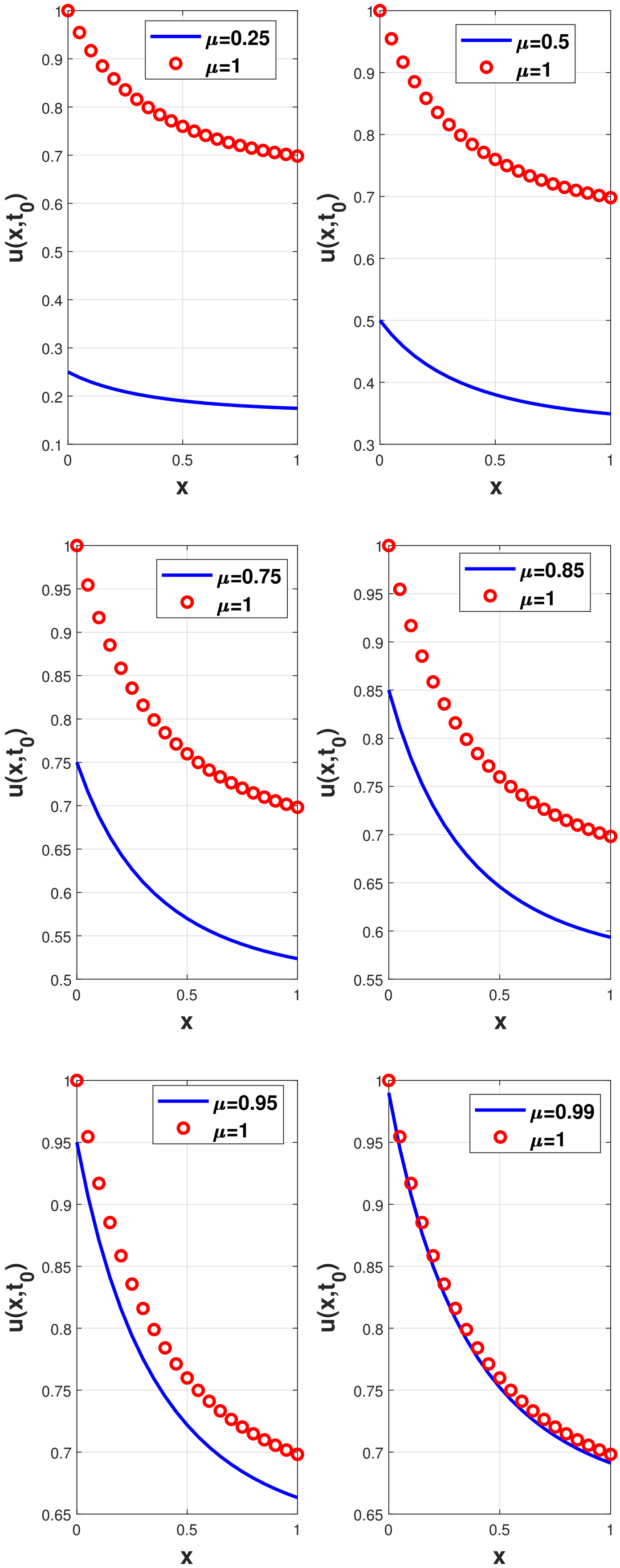

Figure 5: Plots of the exact solutions $u(x,t_0)$ and the approximate solutions $u_n(x,t_0)$ for $t_0 = 0.5$ at various values of $\mu = 0.25, 0.5, 0.75, 0.85, 0.95, 0.99, 1$ and $\lambda = 1, n = 4$ for Example 4.2.

Example 4.2. *In this Example we express the following nonlinear time-fractional Newell-Whitehead-Segel equation of order* $0 < \mu \leq 1$*:*

$$\begin{aligned} {}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+} u(x,t) &= u_{xx} + 2u - 3u^2, \\ u(x,0) &= \lambda, \end{aligned} \tag{30}$$

for $\mu = 1$ *the analytical solution for the corresponding conditions is* $u(x,t) = \frac{\frac{-2}{3}\lambda e^{2t}}{\frac{-2}{3}+\lambda-\lambda e^{2t}}$. *We solve the current test problem with the same manner presented in Example 4.1. The approximation solution is expressed in Figure 4 for different values of* μ *where* $\mu = 0.25, 0.5, 0.75, 0.85, 0.99, 1$. *Figure 5 shows the behavior of the analytical solution* $u(x,t_0)$ *, approximation solution* $u_n(x,t_0)$ *at different values of* μ. *In Table 2 the absolute errors is showed for various values* μ.

Example 4.3. *We introduce the following nonlinear time-fractional Newell-Whitehead-Segel equation:*

$$\begin{aligned} {}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+} u(x,t) - u_{xx}(x,t) &= u(x,t) - u^4(x,t), \\ u(x,0) &= \Big(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}}\Big)^{\frac{2}{3}}. \end{aligned} \tag{31}$$

Using the proposed method in this manuscript, this example is solved for different values of μ *and the numerical results are shown in Figure 6.*

Example 4.4. *Consider the following nonlinear time-fractional Newell-Whitehead-Segel equation:*

$${}^{C}\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+} u(x,t) - u_{xx}(x,t) + \sin(u) = t^2\sin(x) + \frac{2t^{2-\mu}\sin(x)}{\Gamma(3-\mu)} + \sin(t^2\sin(x)), \tag{32}$$

with the following initial and boundary conditions:

$$\begin{aligned} &u(x,0) = 0,\ u_t(x,0) = 0,\ x \in [0,1], \\ &u(0,t) = 0,\ u(1,t) = t^2\sin(1). \end{aligned} \tag{33}$$

The exact solution is $u(x,t) = t^2\sin(x)$. *This example by using the proposed method for different values of* μ *and* $\rho = 0.5,\ \omega = 1,\ \gamma = 0.75$ *is solved and the numerical results are displayed in Fig. 7.*

5. Conclusion

In this paper, the HPTM is applied in order to obtain the approximation solution of the fractional order Newell-Whitehead-Segel equation contain Caputo-Prabhakar fractional derivative. The fractional derivatives in this paper are demonstrated in the Caputo-Prabhakar fractional derivative senses. Finally, The numerical examples are demonstrated to show the ability and the validity and the applicability of the suggested method.

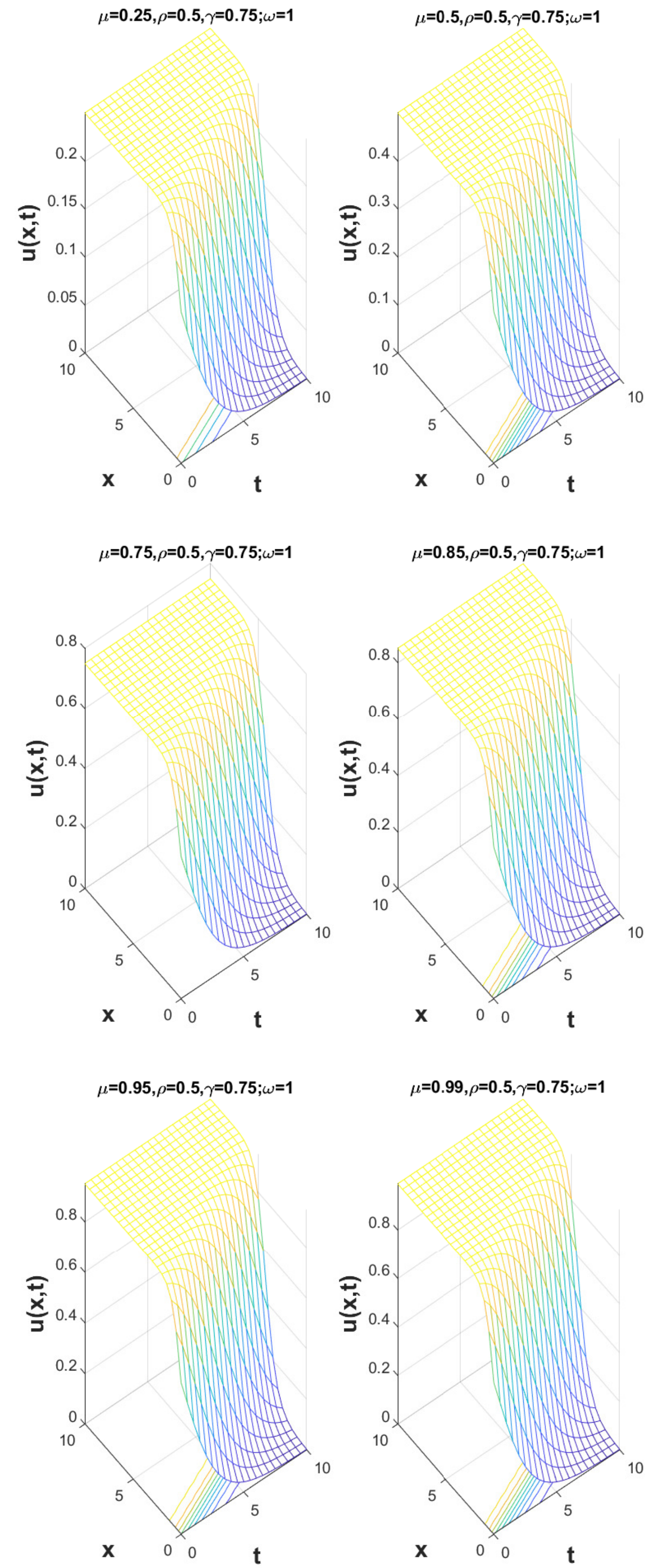

Figure 6: Graphs of approximation solution at various values of $\mu = 0.25, 0.5, 0.75, 0.85, 0.95, 0.99$ for Example 4.3.

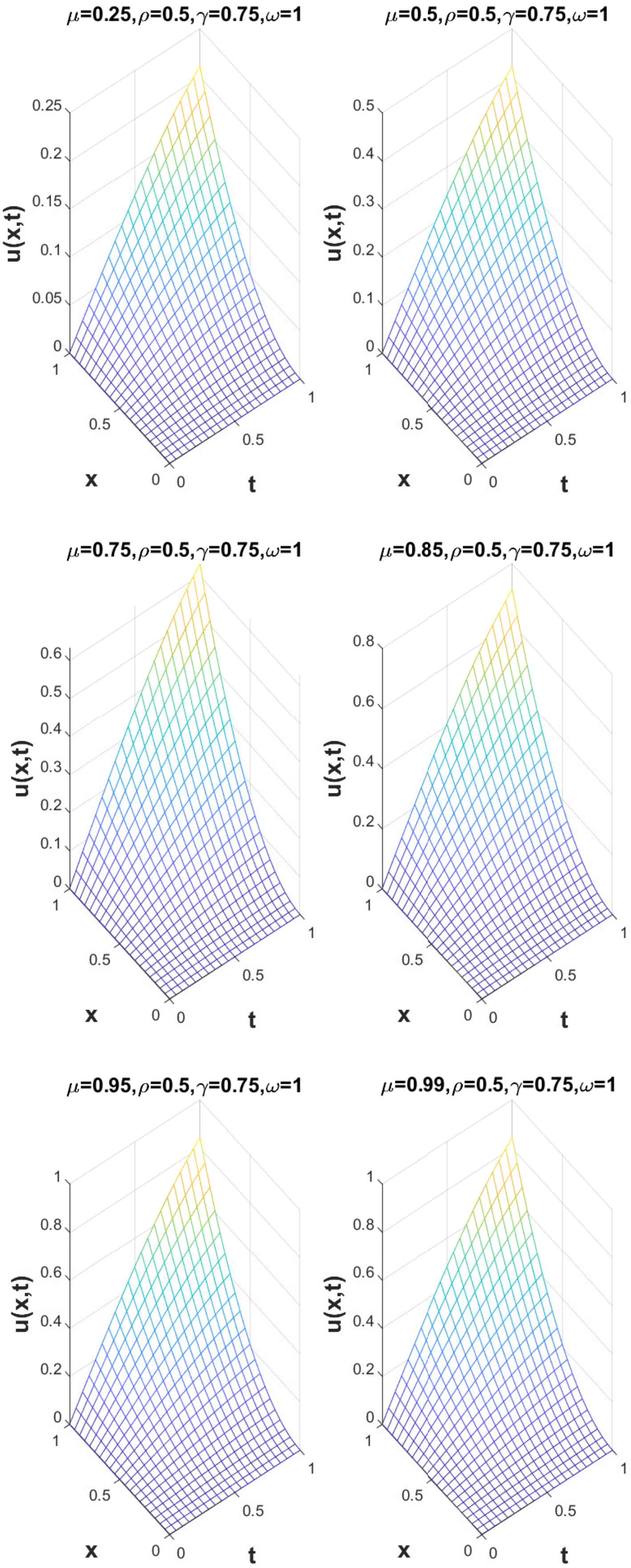

Figure 7: Plots of approximation solution at various values of $\mu = 0.25, 0.5, 0.75, 0.85, 0.95, 0.99$ for Example 4.4.

References

[1] N. Anjum, J. H. He, Laplace transform: making the variational iteration method easier, Applied Mathematics Letters, 92 (2019) 134-138.

[2] L. Beghin, E. Orsingher, Fractional Poisson processes and related planar random motions, Electronic Journal of Probability, 14 (2009) 1790-1826.

[3] J. An, E. Van Hese, M. Baes, Phase-space consistency of stellar dynamical models determined by separable augmented densities, Monthly Notices of the Royal Astronomical Society, 422(1) (2012) 652-664.

[4] V. Daftardar-Gejji, H. Jafari, An iterative method for solving nonlinear functional equations, Journal of Mathematical Analysis and Applications, 316(2) (2006) 753-763.

[5] M. H. Derakhshan, A. Ansari, Fractional Sturm-Liouville problems for Weber fractional derivatives, International Journal of Computer Mathematics, 96(2) (2019) 217-237.

[6] M. H. Derakhshan, A. Ansari, On Hyers-Ulam stability of fractional differential equations with Prabhakar derivatives, Analysis, 38(1), (2018) 37-46.

[7] M. H. Derakhshan, A. Ansari, Numerical approximation to Prabhakar fractional Sturm-Liouville problem, Computational and Applied Mathematics, 38(2) (2019) 71-90. https://doi.org/10.1007/s40314-019-0826-4.

[8] M. H. Derakhshan, A. Ansari, M. Ahmadi Darani, On asymptotic stability of Weber fractional differential systems, Computational Methods for Differential Equations, 6(1) (2018) 30-39.

[9] M. H. Derakhshan, M. Ahmadi Darani, A. Ansari, R. Khoshsiar Ghaziani, On asymptotic stability of Prabhakar fractional differential systems, Computational Methods for Differential Equations, 4(4) (2016) 276-284.

[10] M. D'Ovidio, F. Polito, Fractional diffusion-telegraph equations and their associated stochastic solutions, (2013) arXiv preprint arXiv:1307.1696.

[11] B. Dumitru, D. Kai, S. Enrico, Fractional calculus: models and numerical methods (Vol. 3). World Scientific (2012).

[12] A. Giusti, I. Colombaro, Prabhakar-like fractional viscoelasticity, Communications in Nonlinear Science and Numerical Simulation, 56 (2018) 138-143.

[13] K. Górska, A. Horzela, L. Bratek, K. A. Penson, G. Dattoli, The probability density function for the Havriliak-Negami relaxation, (2016) arXiv preprint arXiv:1611.06433.

[14] R. Garrappa, F. Mainardi, M. Guido, Models of dielectric relaxation based on completely monotone functions, Fractional Calculus and Applied Analysis, 19(5) (2016) 1105-1160.

[15] A. Ghorbani, Beyond Adomian polynomials: he polynomials, Chaos, Solitons & Fractals, 39(3) (2009) 1486-1492.

[16] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, Mittag-Leffler functions, related topics and applications (Vol. 2). Berlin: Springer (2014).

[17] R. Garra, R. Gorenflo, F. Polito, Ž. Tomovski, Hilfer-Prabhakar derivatives and some applications, Applied Mathematics and Computation, 242 (2014) 576-589.

[18] J. H. He, Homotopy perturbation technique, Computer methods in applied mechanics and engineering, 178(3-4) (1999) 257-262.

[19] M. Hamarsheh, A. I. Ismail, Z. Odibat, An analytic solution for fractional order Riccati equations by using optimal homotopy asymptotic method, Applied Mathematical Sciences, 10(23) (2016) 1131-1150.

[20] H. Jafari, Iterative methods for solving system of fractional differential equations, Doctoral dissertation, Pune University, Pune, India, (2006).

[21] B. Karaagac, Two step Adams Bashforth method for time fractional Tricomi equation with non-local and non-singular Kernel, Chaos, Solitons & Fractals, 128 (2019) 234-241.

[22] P. Karunakar, S. Chakraverty, Solutions of time-fractional third-and fifth-order Korteweg-de-Vries equations using homotopy perturbation transform method. Engineering Computations, 36(7) (2019) 2309-2326. https://doi.org/10.1108/EC-01-2019-0012.

[23] A. A. Kilbas, M. Saigo, R. K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators, Integral Transforms and Special Functions, 15(1) (2004) 31-49.

[24] A. Liemert, T. Sandev, H. Kantz, Generalized Langevin equation with tempered memory kernel, Physica A: Statistical Mechanics and its Applications, 466 (2017) 356-369.

[25] P. Miskinis, The Havriliak-Negami susceptibility as a nonlinear and nonlocal process, Physica Scripta, 2009(T136) (2009) 014019.

[26] T,.R. Prabhakar, A singular integral equation with a generalized Mittag Leffler function in the kernel, Yokohama Mathematical Journal, 19 (1971) 7-15.

[27] A. Prakash, M. Goyal, S. Gupta, Fractional variational iteration method for solving time-fractional Newell-Whitehead-Segel equation, Nonlinear Engineering, 8(1) (2019) 164-171.

[28] A. Prakash, H. Kaur, A New Numerical Method for a Fractional Model of Non-Linear Zakharov-Kuznetsov Equations via Sumudu Transform Methods of Mathematical Modelling: Fractional Differential Equations, (2019) 189-204.

[29] S. A. Pasha, Y. Nawaz, M. S. Arif, The modified homotopy perturbation method with an auxiliary term for the nonlinear oscillator with discontinuity, Journal of Low Frequency Noise, Vibration and Active Control, 38(3-4) (2019) 1363-1373.

[30] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications (Vol. 198). Elsevier (1998).

[31] N. A. Pirim, F. Ayaz, A new technique for solving fractional order systems: Hermite collocation method, Applied Mathematics, 7(18) (2016) 2307-2323.

[32] R. K. Pandey, H. K. Mishra, Homotopy analysis Sumudu transform method for time-fractional third order dispersive partial differential equation, Advances in Computational Mathematics, 43(2) (2017) 365-383.

[33] H. C. Rosu, O. Cornejo-Pérez, Supersymmetric pairing of kinks for polynomial nonlinearities, Physical Review E, 71(4) (2005) 046607. doi: 10.1103/PhysRevE.71.046607.

[34] A. Stanislavsky, K. Weron, Atypical case of the dielectric relaxation responses and its fractional kinetic equation, Fractional Calculus and Applied Analysis, 19(1) (2016) 212-228.

[35] M. Tatari, M. Dehghan, On the convergence of He's variational iteration method, Journal of Computational and Applied Mathematics, 207(1) (2007) 121-128.

[36] J. Vahidi, The combined Laplace-homotopy analysis method for partial differential equations, J. Math. Comput. Sci.JMCS, 16(1) (2016) 88-102.

[37] H. Yépez-Martínez, J. F. Gómez-Aguilar, A new modified definition of Caputo-Fabrizio fractional-order derivative and their applications to the Multi Step Homotopy Analysis Method (MHAM), Journal of Computational and Applied Mathematics, 346 (2019) 247-260.

[38] D. N. Yu, J. H. He, A. G. Garcıa, Homotopy perturbation method with an auxiliary parameter for nonlinear oscillators, Journal of Low Frequency Noise, Vibration and Active Control, 38(3-4) (2019) 1540-1554.

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