Approximating the $F$ distribution via a general version of the modified signed log-likelihood ratio statistic

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Abstract

A simple normal approximation for the cumulative distribution function of the $F$ distribution is obtained via a general version of the modified signed log-likelihood ratio statistic. This approximation exhibits remarkable accuracy even when the degrees of freedom are small. Using the same methodology, but with a simpler set up, simple and accurate normal approximations to the cumulative distribution functions of the Student $t$ and $\chi^2$ distributions can also be obtained.

1. Introduction

A large number of studies on the approximation to the cumulative distribution function (cdf) of the $F_{u,v}$ distribution is comprehensively reviewed in Johnson and Kotz (1994). As Johnson and Kotz stress, the value of an approximation lies not only in its accuracy, but also in its simplicity. The aim of this paper is to obtain a simple and accurate approximation for the cdf of the $F_{u,v}$ distribution. As a consequence, approximations for the cdfs of the Student $t_u$ and $\chi^2_u$ distributions can also be obtained.

Let $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ be independent samples from $N(0, \sigma^2_x)$ and $N(0, \sigma^2_y)$ respectively. It is well known that
\[
\frac{\sum_{i=1}^m x_i^2/m \sigma^2_x}{\sum_{i=1}^n y_i^2/n \sigma^2_y}
\]

is distributed as $F_{m,n}$ distribution. In this paper, a general version of the modified signed log-likelihood statistic is applied to obtain the inference for the ratio of the variances, $\frac{\sigma^2_x}{\sigma^2_y}$. As a result, a normal approximation for the cdf of the $F_{u,v}$ distribution is obtained. This approximation is not only simple but also very accurate even for very small degrees of freedom. Theoretically, the proposed method has an accuracy of $O(n^{-3/2})$. By applying the proposed method to the one sample problem for inference concerning the mean parameter and the variance parameter separately will result in simple and accurate normal approximations for the cdfs of the Student $t_u$ and $\chi^2_u$ distributions respectively.

A general version of the modified signed log-likelihood ratio method is given in Section 2. In Section 3 the methodology is applied to obtain the inference for the ratio of variances from two independent normal distributions which results in the approximation for the cdf of the $F_{u,v}$ distribution. With a slightly different but much simpler set
up, approximations for the cdfs of the Student $t$ and $\chi^2$ distributions are presented in Section 4. Some concluding remarks are given in Section 5.

2. Main result

Let $(t_1, \ldots, t_n)$ be a random sample from a model with log-likelihood function $\ell(\theta)$, which satisfies all the regularity conditions given in DiCiccio et al. (1990). The overall maximum likelihood estimate (mle), $\hat{\theta}$, can be obtained by solving

$$\ell_\theta(\hat{\theta}) = \left. \frac{\partial \ell(\theta)}{\partial \theta} \right|_{\theta = \hat{\theta}} = 0.$$  

It is well known that as $n \to \infty$, $\hat{\theta}$ is asymptotically distributed as a multivariate normal distribution with mean $\theta$ and variance–covariance matrix $i^{-1}(\theta)$ where

$$i(\theta) = E \left[ -\frac{\partial^2 \ell(\theta)}{\partial \theta \partial \theta'} \right] = E[-\ell_{\theta\theta}(\theta)]$$

is the Fisher expected information matrix. Obtaining $i(\theta)$ can be complicated, however, it can easily be approximated by

$$j_{\theta\theta}(\hat{\theta}) = -\ell_{\theta\theta}(\hat{\theta}),$$

which is referred to as the observed information matrix evaluated at $\hat{\theta}$.

Let $\psi = \psi(\theta)$ be a scalar parameter of interest. The $p$-value function of $\psi$ can then be obtained by a Wald-type statistic:

$$p(\psi) = \Phi \left( \frac{\hat{\psi} - \psi}{\sqrt{\text{var}(\hat{\psi})}} \right)$$

where $\Phi()$ is the cdf of $N(0, 1)$, $\hat{\psi} = \psi(\hat{\theta})$, and $\text{var}(\hat{\psi})$ is the variance of $\hat{\psi}$. The exact $\text{var}(\hat{\psi})$ may be difficult to obtain, and hence Delta’s method is frequently applied to obtain an approximated value of $\text{var}(\hat{\psi})$. Note that this Wald-type statistic based method has accuracy $O(n^{-1/2})$ and it is not invariant to reparameterization.

An alternate large sample asymptotic method is based on the signed log-likelihood ratio statistic:

$$r(\psi) = \text{sgn}(\hat{\psi} - \psi) \sqrt{2[\ell(\hat{\theta}) - \ell(\hat{\theta}_{\psi})]}$$

where $\hat{\theta}_{\psi}$ is the constrained mle for a given $\psi$, which is obtained by maximizing $\ell(\theta)$ subject to the constraint $\psi(\theta) = \psi$. It is well known that as $n \to \infty$, $r(\psi)$ is asymptotically distributed as $N(0, 1)$ with accuracy $O(n^{-1/2})$. Hence, the $p$-value function of $\psi$ can then be obtained by the signed log-likelihood ratio statistic:

$$p(\psi) = \Phi(r(\psi)).$$

The advantage of this method is that it is invariant to reparameterization. Doganaksoy and Schmee (1993) showed that the signed log-likelihood ratio statistic based method has better coverage properties than the Wald-type statistic based method in the family of models they considered.

In the literature, various asymptotic methods have been proposed to improve the accuracy of the existing $O(n^{-1/2})$ methods. Reid (1996) gave a detailed overview of this development. One of these methods is the modified signed log-likelihood ratio method by Barndorff-Nielsen (1986, 1991), who considered the exponential family model with canonical parameter $\theta = (\psi, \lambda)'$, where $\lambda$ is the nuisance parameter. He then defined the modified signed log-likelihood ratio statistic as

$$r^*(\psi) = r(\psi) + \frac{1}{r(\psi)} \log \frac{r(\psi)}{q(\psi)}$$

where $q(\psi)$ is the $q$-function of $\psi$.
where \( r(\psi) \) is the signed log-likelihood ratio statistic given in (1), and

\[
q(\psi) = (\psi - \psi) \left( \frac{|j_{\psi \theta}(\hat{\theta})|}{|j_{\psi \phi}(\hat{\phi})|} \right)^{1/2}
\]

(3)
can be viewed as a standardized mle departure of the parameter of interest \( \psi \), with

\[
j_{\psi \phi}(\hat{\phi}) = -\ell_{\psi \phi}(\hat{\phi})
\]

being the observed nuisance information matrix evaluated at \( \hat{\phi} \). Barndorff-Nielsen (1986, 1991) showed that as \( n \to \infty \), \( r^*(\psi) \) is also asymptotically distributed as \( N(0, 1) \) but with accuracy \( O(n^{-3/2}) \). Thus the \( p \)-value function of \( \psi \) is

\[
p(\psi) = \Phi(r^*(\psi)).
\]

We will now generalize the methodology to any exponential family model with log-likelihood function \( \ell(\theta) \) and canonical parameter \( \varphi = \varphi(\theta) \). The aim is to obtain a simple form of \( r^*(\psi) \) for this type of model.

Since the signed log-likelihood ratio statistic method is invariant to reparameterization, \( r(\psi) \) remains unchanged as defined in (1). \( q(\psi) \), however, is not invariant to reparameterization. Thus, all we have to do is to express \( q(\psi) \) in the canonical parameter, \( \varphi \), scale.

Denote \( \varphi_\theta(\theta) = \frac{\partial \varphi(\theta)}{\partial \theta} \). \( \varphi(\theta) \) be the row of \( \varphi_\theta^{-1}(\theta) \) that corresponds to \( \psi \), and \( \|\varphi(\theta)\|^2 \) is the square length of \( \varphi(\theta) \). Let \( \chi(\theta) \) be a rotated coordinate of \( \varphi = \varphi(\theta) \) that agrees with \( \psi(\theta) \) at \( \hat{\theta}_\phi \). Therefore,

\[
\chi(\theta) = -\frac{\varphi(\hat{\theta}_\phi)}{\|\varphi(\hat{\theta}_\phi)\|} \varphi(\theta).
\]

(4)

Thus, \( (\hat{\psi} - \psi) \) in \( \varphi \) scale is

\[
\text{sgn}(\hat{\psi} - \psi)|\chi(\hat{\theta}) - \chi(\hat{\theta}_\phi)|,
\]

which can be considered as the mle departure of \( \psi \) in \( \varphi \) scale.

Moreover, we have \( \ell(\theta) = \ell(\varphi) \). It is easy to show that the determinant of the observed information matrix at \( \hat{\theta} \) in \( \varphi \) scale is

\[
|j_{\psi \theta}(\hat{\theta})| = |j_{\psi \theta}(\hat{\theta})||\varphi_\theta(\hat{\theta})|^{-2}
\]

(5)

and similarly, the determinant of the nuisance information matrix evaluated at \( \hat{\theta}_\phi \) in \( \varphi \) scale is

\[
|j_{\psi \phi}(\hat{\phi})| = |j_{\psi \phi}(\hat{\phi})||\varphi_\phi(\hat{\phi})|^2.
\]

(6)

Thus, the standardized maximum likelihood departure of \( \psi \) in \( \varphi \) scale is

\[
q(\psi) = \text{sgn}(\hat{\psi} - \psi)|\chi(\hat{\theta}) - \chi(\hat{\theta}_\phi)| \left( \frac{|j_{\psi \theta}(\hat{\theta})|}{|j_{\psi \phi}(\hat{\phi})|} \right)^{1/2}.
\]

(7)

Hence, \( r^*(\psi) \) can be obtained from (2).

3. **Approximation of the \( F_{u,v} \) distribution**

Consider two independent samples, \( (x_1, \ldots, x_m) \) and \( (y_1, \ldots, y_n) \), from \( N(0, \sigma_x^2) \) and \( N(0, \sigma_y^2) \) respectively. Let the parameter of interest be the ratio of the variances, \( \psi = \sigma_x^2/\sigma_y^2 \). Then, the log-likelihood function can be written as:

\[
\ell(\theta) = \ell(\sigma_x^2, \psi)
\]

\[
= -\frac{m}{2} \log \psi - \frac{m}{2} \log \sigma_y^2 - \frac{1}{2 \psi \sigma_y^2} \sum_{i=1}^{m} x_i^2 - \frac{n}{2} \log \sigma_y^2 - \frac{1}{2 \sigma_y^2} \sum_{i=1}^{n} y_i^2.
\]

(8)
which is an exponential family model with canonical parameter

\[ \varphi = \varphi(\theta) = \left( \frac{1}{\psi \sigma^2_y}, \frac{1}{\sigma^2_y} \right). \]

By solving \( \ell_\theta(\hat{\theta}) = 0 \), we have the overall mle

\[ \hat{\theta} = (\hat{\sigma}^2_y, \hat{\psi})' = \left( \frac{\sum_{i=1}^n \gamma_i^2}{n}, f \right)' \]

where

\[ f = \frac{\sum_{i=1}^m x_i^2 / m}{\sum_{i=1}^n y_i^2 / n}. \]

Also, the determinant of the observed information matrix evaluated at \( \hat{\theta} \) is

\[ |j_{\theta\theta}(\hat{\theta})| = \frac{mn}{4\hat{\psi}^2 \hat{\sigma}^2_y}. \]

The exact distribution of \( \frac{\sum_{i=1}^m x_i^2 / m}{\sum_{i=1}^n y_i^2 / n} \) is the \( F_{m,n} \) distribution. Hence, the exact \( p \)-value function for \( \psi \) is

\[ p(\psi) = P\left( F_{m,n} \leq \frac{1}{\psi} f \right). \]

In particular,

\[ p(\psi = 1) = P\left( F_{m,n} \leq f \right) \]

which is the cdf of the \( F_{m,n} \) distribution. Thus, \( \psi = 1 \) is of particular interest.

For \( \psi = 1 \), the constrained mle is obtained by maximizing \( \ell(\sigma^2_y, \psi = 1) \) and we have

\[ \hat{\theta}_{\psi=1} = \left( \tilde{\sigma}^2_y, 1 \right)' \]

where \( \tilde{\sigma}^2_y = (\sum_{i=1}^m x_i^2 + \sum_{i=1}^n y_i^2)/(m + n) \). Hence, the determinant of the nuisance information matrix evaluated at \( \hat{\theta}_{\psi=1} \) is

\[ |j_{\lambda\lambda}(\hat{\theta}_{\psi=1})| = \frac{m + n}{2\tilde{\sigma}^2_y}. \]

Thus, the signed log-likelihood ratio statistic is

\[ r(\psi = 1) = \text{sgn}(f - 1) \left\{ (m + n) \log \frac{mf + n}{m + n} - m \log f \right\}^{1/2}. \]

Therefore, based on the signed log-likelihood ratio statistic, the cdf of \( F_{m,n} \) distribution can be approximated by \( \Phi(r(\psi = 1)) \) with accuracy \( O(n^{-1/2}) \).

Since \( \varphi(\theta) \) is known, we have

\[ \varphi_{\theta}(\theta) = \begin{pmatrix} -\frac{1}{\psi \sigma^2_y} & -\frac{1}{\psi^2 \sigma^2_y} \\ -\frac{1}{\sigma^4_y} & 0 \end{pmatrix}, \quad \varphi_{\theta}^{-1}(\theta) = \begin{pmatrix} 0 & -\sigma^4_y \\ -\psi^2 \sigma^2_y & \psi \sigma^2_y \end{pmatrix}. \]
and
\[ \varphi_\theta^\psi(\theta) = (-\psi^2 \sigma_y^2, \psi \sigma_y^2). \]

Thus, from (4),
\[ \chi(\theta) = \frac{1}{\sqrt{2} \sigma_y^2} \left( 1 - \frac{1}{\psi} \right). \]

Furthermore, from (5) and (6), we have
\[ |j_{(\theta \theta)}(\hat{\theta})| = \frac{mn \hat{\psi}^2 \hat{\sigma}_y^4}{4}, \]
\[ |j_{(\lambda \lambda)}(\hat{\theta}_{\psi=1})| = \frac{(m + n) \tilde{\sigma}_y^4}{4} \]
and the standardized maximum likelihood departure of \( \psi \) in \( \varphi \) scale from (7) is:
\[ q(\psi = 1) = \frac{f - 1}{mf + n} \left\{ \frac{mn(m + n)}{2} \right\}^{1/2}. \]

Finally, based on the general version of the modified signed log-likelihood ratio statistic,
\[ p(\psi = 1) = \Phi(r^*(\psi = 1)) = \Phi \left( r(\psi = 1) - \frac{1}{r(\psi = 1)} \log \frac{r(\psi = 1)}{Q(\psi = 1)} \right) \]
(9)
is an approximation of the cdf of the \( F_{m,n} \) distribution with accuracy \( O(n^{-3/2}) \).

In summary, we have two approximations of the cdf of the \( F_{u,v} \) distribution. The first approximation is an \( O(n^{-1/2}) \) method based on the signed log-likelihood ratio statistic:
\[ P(F_{u,v} \leq f) = \Phi(r) \]
(10)
where
\[ r = \text{sgn}(f - 1) \left\{ (u + v) \log \frac{uf + v}{u + v} - u \log f \right\}^{1/2}. \]
(11)
The second approximation is an \( O(n^{-3/2}) \) method based on the general version of the signed log-likelihood ratio statistic:
\[ P(F_{u,v} \leq f) = \Phi(r^*) = \Phi \left( r - \frac{1}{r} \log \frac{r}{q} \right) \]
(12)
where \( r \) is defined in (11) and
\[ q = \frac{f - 1}{uf + v} \left\{ uf(u + v) \right\}^{1/2}. \]
(13)

Both approximations, (10) and (12), can easily be implemented into any statistical softwares.

A direct saddlepoint method to obtain the cdf of \( F_{u,v} \) distribution can be obtained by the method described in Daniels (1983). The method is more complicated but also produced an \( O(n^{-3/2}) \) approximation.

To illustrate the accuracy of the approximations, Fig. 1 plots approximations obtained from (10) and (12), and Daniels (1983), and the exact cdf of \( F_{u,v} \) distribution. Moreover, Fig. 2 plots the corresponding percentage relative errors for the three approximations. It is obvious that approximation by (10) is not satisfactory at all. On the other hand, approximations by (12) and Daniels (1983) are almost indistinguishable from the exact cdf for any degree of freedom.
4. Approximation of the Student $t_u$ and $\chi^2_u$ distributions

For a simpler set up, let $(y_1, \ldots, y_n)$ be a random sample from $N(\mu_y, \sigma^2_y)$. The log-likelihood function is

$$\ell(\theta) = -\frac{n}{2} \log \sigma^2_y - \frac{1}{2\sigma^2_y} \sum_{i=1}^{n} (y_i - \mu_y)^2$$

and the canonical parameter is $\varphi = \varphi(\theta) = \left(\frac{\mu_y}{\sigma^2_y}, \frac{1}{\sigma^2_y}\right)$.

Consider the parameter of interest be $\psi = \psi(\theta) = \mu_y$. Then,

$$t = \frac{\bar{y} - \mu_y}{s_y/\sqrt{n}}$$

is distributed as Student $t_{n-1}$ distribution. Applying the proposed procedure to this problem, we can obtain

$$r(\psi) = \text{sgn}(t) \left\{ n \log \left[ 1 + \frac{t^2}{n-1} \right] \right\}^{1/2}$$

and $r^*(\psi)$ is given in (2). Hence the cdf of the Student $t_{n-1}$ distribution, $p(\psi)$, can be approximated by either $\Phi(r(\psi))$ or $\Phi(r^*(\psi))$ with accuracy $O(n^{-1/2})$ and $O(n^{-3/2})$ respectively. Note that these two approximations are independent of $\psi$. By re-indexing (14) and (15), the cdf of Student $t_u$ distribution can be approximated by

$$P(t_u \leq t) = \Phi(r)$$

Fig. 1. Approximating the cdf of the $F(u, v)$ distribution.
(a) $F(1, 2)$.

(b) $F(10, 2)$.

(c) $F(3, 4)$.

(d) $F(5, 30)$.

Fig. 2. Percentage relative error of the approximated cdf of the $F(u, v)$ distribution.

Fig. 3. Percentage relative error for the approximations in Table 1.

with accuracy $O(n^{-1/2})$, or by

$$P(t_u \leq t) = \Phi(r^*) = \Phi\left(r - \frac{1}{r} \log \frac{r}{q}\right)$$  \hspace{1cm} (17)
with accuracy $O(n^{-3/2})$, where
\[
    r = \text{sgn}(t) \left\{ (u + 1) \log \left[ 1 + \frac{t^2}{u} \right] \right\}^{1/2}
\]
\[
    q = \sqrt{u(u + 1)} \left\{ \frac{t}{u + t^2} \right\}. 
\]

Jing et al. (2004) applied the saddlepoint method without using the moment generating function to approximate the cdf of the Student $t_u$ distribution. The exact form of their result is very complicated. They provide numerical results for comparing their approximations with the exact Student $t_5$ distribution. Note that their results are for the survivor function rather than the cdf. Table 1 contains the results from Jing et al. (2004) and the results from (16) and (17). Fig. 3 plots the percentage relative errors of the three approximations. From Table 1 and Fig. 3, we observed that (16) does not give satisfactory approximation. Jing et al. (2004) method and approximation from (17) are almost indistinguishable around the center of the distribution but (17) are much better towards the tail of the distribution which is crucial for inference purpose. Fig. 4 plots (16) and (17) and the exact cdf of the Student $t_u$ distribution and Fig. 5 plots the corresponding percentage relative errors of the two approximations for various $u$. Approximation by (16) is not satisfactory; whereas approximation by (17) gives excellent approximation even for the extreme case, Student $t_1$ distribution.

By setting $\mu_y = 0$ and the parameter of interest is $\psi = \psi(\theta) = \sigma_y^2$. Let
\[
    S = \sum_{i=1}^{n} y_i^2. 
\]
Table 1
Comparisons of exact and approximate values for $1 - F_t(t)$

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Fig. 5. Percentage relative error of the approximated cdf of the $t(\nu)$ distribution.

It is well known that $S_{\psi}$ is distributed as a $\chi^2_n$ distribution. In particular,

$$p(\psi = 1) = P(\chi^2_n \leq S)$$
is the cdf for the $\chi^2_n$ distribution. Applying the proposed method and evaluated at $\psi = 1$, we have
\begin{align*}
    r(\psi = 1) &= \text{sgn}(S - n) \left\{ S - n + n \log \frac{n}{S} \right\}^{1/2} \\
    q(\psi = 1) &= \frac{S - n}{\sqrt{2n}}
\end{align*} 
(18)
(19)
and $r^*(\psi = 1)$ is given in (2). Hence, the cdf of the $\chi^2_n$ distribution, $p(\psi = 1)$, can be approximated by either $\Phi(r(\psi = 1))$ or $\Phi(r^*(\psi = 1))$ with accuracy $O(n^{-1/2})$ and $O(n^{-3/2})$ respectively. Thus, the cdf of $\chi^2_u$ distribution can be approximated by
\begin{align*}
P(\chi^2_u \leq S) &= \Phi(r) \\
\text{(20)}
\end{align*}
with accuracy $O(n^{-1/2})$, or by
\begin{align*}
P(\chi^2_u \leq S) &= \Phi \left( r - \frac{1}{r} \log \frac{r}{q} \right) \\
\text{(21)}
\end{align*}
with accuracy $O(n^{-3/2})$, where
\begin{align*}
r &= \text{sgn}(S - u) \left\{ S - u + u \log \frac{u}{S} \right\}^{1/2} \\
q &= \frac{S - u}{\sqrt{2u}}.
\end{align*} 
(22)
(23)
Note that the cumulant generating function exists for the $\chi^2_u$ distribution. Hence saddlepoint method can be applied directly to obtain an approximation of the cdf of the $\chi^2_u$ distribution. In this case, the resulting $r$ and $q$ are exactly the same as those given in (22) and (23) respectively.

Fig. 6 plots (20) and (21) and the exact cdf of $\chi^2_u$ distribution and Fig. 7 plots the percentage relative errors of the two approximations for various $u$. Approximation by (20) is not satisfactory; whereas approximation by (21) gives excellent approximation even for the extreme case, $\chi^2_1$ distribution.

5. Conclusion

Simple and yet highly accurate normal approximations for the cdfs of the $F$, Student $t$ and $\chi^2$ distributions have been derived using a general version of the modified signed log-likelihood ratio statistic regardless of the degree of freedom. Theoretically, these simple approximations have known $O(n^{-3/2})$ accuracy and can easily be programmed into any statistical softwares.

References


