Set partitions with circular successions

Toufik Mansour, Augustine O. Munagi

Department of Mathematics, University of Haifa, 3498838 Haifa, Israel
School of Mathematics, University of the Witwatersrand, 2050 Johannesburg, South Africa

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ABSTRACT

We consider the enumeration of partitions of a finite set according to the number of consecutive elements inside a block under the assumption that the elements are arranged around a circle. This statistic, commonly known as circular succession, continues to play a significant role in many combinatorial problems involving combinations of a set following its first appearance in a paper of Irving Kaplansky in the 1940s. In this paper we obtain interesting formulas for the number of partitions avoiding a circular succession and the number of partitions containing a specified number of circular successions. Our methods include both elementary combinatorial reasoning and the application of ordinary and exponential power series generating functions. Several new combinatorial identities are also stated.

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1. Introduction

A partition of \([n] = \{1, 2, \ldots, n\}\) is a decomposition of \([n]\) into nonempty subsets called blocks. A partition into \(k\) blocks is also called a \(k\)-partition and is denoted by \(H_1/H_2/\cdots/H_k\), where the blocks are arranged in standard order: \(\min(H_1) < \cdots < \min(H_k)\).

A (circular) succession is an ordered pair of elements \((a, b)\) in a subset or combination of \([n]\) which satisfies \(b - a \equiv 1 (\text{mod } n)\). In other words a succession is a pair of consecutive integers or an occurrence of the pair \((n, 1)\). For emphasis when the latter is excluded, the other successions are referred to as linear.

E-mail addresses: tmansour@univ.haifa.ac.il (T. Mansour), augustine.munagi@wits.ac.za, amunagi@yahoo.com (A.O. Munagi).

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A partition of \([n]\) is said to contain a succession if any of its blocks does. For example the partition 1259/367/48 contains three successions, namely, \((9, 1), (1, 2)\) and \((6, 7)\).

The enumeration of combinations according to the number of successions originated with the works of Kaplansky and Riordan in the 1940s (see [8, 17]). Subsequent works have followed in the form of generalizations, refinements and variations of the underlying sequences. The interested reader is referred to [1,5,22,10] and [6, Chap. 2.3]. On the other hand, the enumeration of set partitions with respect to linear successions was first considered in rudimentary form in [15]. More extensive results and variations were subsequently proved in [16]. See also the related papers [4,19,13,14]. Even though the classification of combinations by the numbers of both purely circular and linear successions is well established, there does not seem to be a commensurate extension of the circular statistic to set partitions.

This paper fills the gap.

We remark that set partitions have been studied under a similar statistic from an “externalized” viewpoint. In [11] the authors define a block connector of a \(k\)-partition of \([n]\) to be a pair of integers \((a, b)\) such that \(b - a \equiv 1(\text{mod } n)\) and \(a \in H_i, b \in H_j\) with \(j - i \equiv 1(\text{mod } k)\). Further variations and a few generalized results on such block connected partitions have since appeared in [12].

This topic also fits into the general enumeration problems for partitions under special “internal” patterns which is presently a very active area of research. For example, Sagan [18], Klazar [9], and Jelinek and Mansour [7] have studied pattern avoidance in set partitions from different perspectives (see also the references therein). Our paper provides yet another class of restrictions on the partitions of a finite set.

Let \(c_r(n, k)\) denote the number of partitions of \([n]\) containing \(r\) circular successions, so that \(c_0(n, k)\) enumerates the partitions that avoid circular successions. Our main goal is to find formulas for \(c_0(n, k)\) and \(c_r(n, k)\). We will denote the number of \(k\)-partitions of \([n]\), that is, the Stirling number of the second kind, by \(S(n, k)\). So the \(n\)th Bell number \(B(n)\) is defined by \(B(n) = \sum_k S(n, k)\).

It is worthwhile to stress the difference between enumeration by linear successions and enumeration by circular successions. The latter statistic provides a balanced and more complete analysis in a nontrivial way. This will become evident in the course of deriving several results in the following sections. Table 1 shows some values of the number \(c_r(n) = \sum_k c_r(n, k)\). Note that \(c_{n-1}(n) = 0\) and \(c_n(n) = 1\) as indicated in the last two entries of each row, \(n > 1\). But for linear successions these values are reversed because the succession \((n, 1)\) in the \(1\)-block partition \([n]\) is omitted. One may recognize the first column as the sequence [21, A000296]: number of partitions of an \(n\)-set into blocks of size \(>1\),

\[ n \geq 2: 1, 1, 4, 11, 41, 162, 715, 3425, 17722, 98253, 580317, \ldots . \]

The connection with circular successions follows from the known equidistribution of circular successions and singletons in all partitions of \([n]\), see for example [2,3]. Thus the number of partitions that avoid circular successions corresponds to the number of partitions of \([n]\) without singletons.

In what follows, we first concentrate on using elementary combinatorial techniques to obtain explicit recursive and exact formulas for the enumeration functions \(c_0(n, k)\) and \(c_r(n, k)\) (see Sections 2 and 3). This is followed with a brief discussion of enumeration results for detached or isolated successions in Section 4. Then we apply generating function methods on the recursive formulas to discover
equivalent but more compact results (see Section 5). The two approaches are theoretically useful in highlighting the source of certain strengths and weaknesses of combinatorial formulas. Whereas the main appeal of the generating function approach lies in the brevity of the obtained results, the presence of sign-alternating summands often obscures a natural path to a combinatorial formula.

The final section (Section 6) is devoted to a small collection of combinatorial identities arising from previous sections.

2. Recursive and exact formulas

The formulas will be found by refining the process of enumerating partitions with a given number of linear successions while accounting for the possible occurrence of the distinguished succession (n, 1).

The proofs will use the following result on the enumeration of separated sets with respect to linear successions. An ordered set of integers \( \{v_1, \ldots, v_m\} \) with \( v_1 < \cdots < v_m \) is called separated if it contains a separation, that is, a pair \((v_i, v_{i+1})\) such that \( v_{i+1} - v_i > 1 \).

**Theorem 0** ([16, Theorem 7]). Let \( V \) be an ordered \( m \)-set of positive integers having \( x \) separations, \( m > 0 \). Then the number \( f((m, x), k) \) of \( k \)-partitions of \( V \) into subsets of nonconsecutive integers is given by

\[
f((m, x), k) = \sum_{j=0}^{x} \binom{x}{j} S(m - j - 1, k - 1). \tag{1}
\]

We will adopt the convention of denoting enumeration functions by lower-case letters and the enumerated sets by corresponding upper-case letters where necessary.

**Theorem 1.** Given integers \( n, k, n \geq 4, 2 \leq k \leq n \), the following relation holds:

\[
\begin{align*}
c_0(n, k) &= c_0(n - 1, k - 1) + c_0(n - 2, k - 1) + (k - 1)c_0(n - 2, k) + (k - 2)c_0(n - 1, k), \\
c_0(n, 1) &= \delta_{n, 1}, \quad c_0(3, 2) = 0, \quad c_0(2, 2) = c_0(3, 3) = 1. \tag{2}
\end{align*}
\]

**Proof.** The number of \( k \)-partitions of \([n]\) without linear successions is composed of two parts. These are the partitions in which 1 and \( n \) lie in one block, and in two different blocks, enumerated respectively by \( \ell_1(n, k) \) and \( \ell_2(n, k) \). Note the easily proved relation:

\[
\ell_1(n, k) = \ell_2(n - 1, k), \quad 2 \leq k < n, \quad \ell_1(1, 1) = 1. \tag{3}
\]

By definition \( \ell_2(n, k) = c_0(n, k) \).

We consider the number of partitions \( \pi \) obtained by putting \( n \) into a partition \( \beta \) of \([n - 1]\) which contains no successions without creating a succession.

Insert the singleton \([n]\) into \( \beta \) in \( \ell_1(n - 1, k - 1) \) or \( \ell_2(n - 1, k - 1) \) ways depending on whether 1 and \( n - 1 \) lie in one block or not. Put \( n \) into a block of \( \beta \in L_j(n - 1, k), j = 1, 2 \) as follows. If 1 and \( n - 1 \) belong to the same block, put \( n \) into any block except that block, so the number of partitions \( \pi \) obtained is \((k - 1)\ell_1(n - 1, k)\). If 1 and \( n - 1 \) belong to two different blocks, put \( n \) into any block except those two blocks, so the number of partitions \( \pi \) obtained is \((k - 2)\ell_2(n - 1, k)\).

Hence, the total number of partitions \( \pi \) is

\[
\ell_1(n - 1, k - 1) + \ell_2(n - 1, k - 1) + (k - 1)\ell_1(n - 1, k) + (k - 2)\ell_2(n - 1, k).
\]

Since \( \ell_2(n, k) = c_0(n, k) = \ell_1(n + 1, k) \), we obtain

\[
c_0(n - 2, k - 1) + c_0(n - 1, k - 1) + (k - 1)c_0(n - 2, k) + (k - 2)c_0(n - 1, k),
\]

which gives the main result. The initial values may be verified separately.

We next obtain an explicit formula for \( c_0(n, k) \). The following result may be justified in a fairly tedious manner by showing that it satisfies the recurrence in **Theorem 1**. But we give a direct constructive proof.
**Theorem 2.** We have $c_0(n, 1) = \delta_{1,n}$, and when $2 \leq k \leq n$ we have

$$c_0(n, k) = \sum_{j=2}^{\frac{n+2}{2}} \binom{n-j}{j-2} \sum_{i=0}^{j-2} \binom{j-2}{i} S(n-j-i, k-2).$$

**Proof.** Since $c_0(n, k) = \ell_1(n+1, k)$, it will suffice to find a formula for $\ell_1(n, k)$. We claim that

$$\ell_1(n, k) = \sum_{j=2}^{\frac{n+2}{2}} \binom{n-j-1}{j-2} \sum_{i=0}^{j-2} \binom{j-2}{i} S(n-j-i-1, k-2). \tag{4}$$

To see this, let us construct a partition $\pi$ enumerated by $\ell_1(n, k)$, say $\pi = H_1/\cdots/H_k$. Then $|H_1| = j \geq 2$, since 1, $n \in H_1$, and $H_1$ assumes the form $\{x_1, x_2, \ldots, x_j\}$ with $x_1 = 1, x_j = n$ and $x_{i+1} - x_i > 1 \, \forall \, i$. Thus $H_1$ can be built in as many ways as there are subsets of $\{2, 3, \ldots, n-2, n-1\} \setminus \{2, n-1\}$ consisting of $j-2$ non-consecutive elements. Denote the number of ways by $nc(n-4, j-2)$. Next we find the other blocks $H_2, \ldots, H_k$ by partitioning $[n] \setminus B_1$, having exactly $j-2$ separations, into $k-1$ blocks, a task which can be performed in $f((n-j, j-2), k-1)$ ways. Hence for each $j$, the number of partitions $\pi$ is $nc(n-4, j-2)f((n-j, j-2), k-1)$. Hence we obtain

$$\ell_1(n, k) = \sum_{j=0}^{u} nc(n-4, j-2)f((n-j, j-2), k-1),$$

where $nc(m, k) = \binom{m-k+1}{k}$. Lastly, note that the set $\{3, 4, \ldots, n-3, n-2\}$ admits a choice of at most $\frac{n-3}{2}$ and $\frac{n-4}{2}$ nonconsecutive elements if $n$ is odd and even, respectively. Thus $0 \leq j-2 \leq \lfloor \frac{n-3}{2} \rfloor$. Eq. (4) now follows from the definitions of the associated functions. \qed

**Corollary 3.** Let $c_0(n) = \sum_k c_0(n, k)$. Then for all integers $n > 1$,

$$c_0(n) = \sum_{j=2}^{\frac{n+2}{2}} \binom{n-j}{j-2} \sum_{i=0}^{j-2} \binom{j-2}{i} B(n-j-i).$$

3. Formulas for $c_r(n, k)$

The general recurrence is obtained by extending the technique used in Section 2.

**Theorem 4.** The number $c_r(n, k)$ fulfills the following recurrence relation for all positive integers $n, k, r, 1 \leq r \leq n, 2 \leq k \leq n-2$:

$$c_r(n, k) = c_r(n-2, k-1) + c_r(n-1, k-1) - c_{r-1}(n-2, k-1) + (k-1)c_r(n-2, k) + (k-2)(c_r(n-1, k) - c_{r-1}(n-2, k)) + 2c_{r-1}(n-1, k) - c_{r-2}(n-2, k),$$

$c_r(n, 1) = \delta_{r,n}, \ c_r(n, n-1) = n\delta_{r,1}$ if $n > 1$ and $r \geq 1$, $c_r(n, n) = \delta_{r,0}$ and $c_0(n, k) = (\text{Theorem 2}).$

**Proof.** The number of $k$-partitions of $[n]$ containing $r > 0$ linear successions is given by $p_1(n, k, r) + p_2(n, k, r)$, where the summands enumerate partitions in which 1 and $n$ lie in one block and in two different blocks, respectively. The following relation holds:

$$p_1(n, k, r) = p_1(n-1, k, r-1) + p_2(n-1, k, r), \quad 2 \leq k < n. \tag{5}$$

Indeed, deleting $n$ from a partition counted by $p_1(n, k, r)$ either gives a partition counted by $p_2(n-1, k, r)$, or a partition counted by $p_1(n-1, k, r-1)$ (since we lose the succession $n-1, n$).

Note that we have, by the definitions,

$$c_r(n, k) = p_1(n, k, r-1) + p_2(n, k, r). \tag{6}$$
Thus using (5) we can also write

\[ c_r(n, k) = p_1(n + 1, k, r). \] (7)

We first consider the number of partitions \( \pi \) obtained by putting \( n \) into a partition \( \beta \) of \([n - 1]\) which contains \( r \) circular successions without creating a succession.

Insert the singleton \([n]\) into \( \beta \) in \( p_1(n - 1, k - 1, r) \) or \( p_2(n - 1, k - 1, r) \) ways depending on whether \( 1 \) and \( n - 1 \) lie in one block or not. (Note that a partition of \([n]\) enumerated by \( p_1(n - 1, k - 1, r) \) contains \( r \) successions (strictly linear) since the presence of \([n]\) nullifies the previous (purely circular) succession \( n - 1, 1\).) Put \( n \) into a block of \( \beta \in P_j(n - 1, k, r), j = 1, 2 \) as follows. If \( 1 \) and \( n - 1 \) belong to the same block, put \( n \) into any block except that block, so the number of partitions \( \pi \) obtained is \((k - 1)p_1(n - 1, k, r)\). If \( 1 \) and \( n - 1 \) belong to different blocks, put \( n \) into any block except those two blocks, so the number of partitions \( \pi \) obtained is \((k - 2)p_2(n - 1, k, r)\).

Thus the total number of partitions \( \pi \) is

\[ p_1(n - 1, k - 1, r) + p_2(n - 1, k - 1, r) + (k - 1)p_1(n - 1, k, r) + (k - 2)p_2(n - 1, k, r). \] (8)

Since \( p_2(n, k, r) = p_1(n + 1, k, r) - p_1(n, k, r - 1) \), \( c_r(n, k) - c_{r-1}(n - 1, k) \), (8) becomes

\[

c_r(n - 2, k - 1) + c_r(n - 1, k - 1) - c_{r-1}(n - 2, k - 1) + (k - 1)c_r(n - 2, k) \\
+ (k - 2)(c_r(n - 1, k) - c_{r-1}(n - 2, k)).
\] (9)

Next we consider the number of partitions \( \pi \) obtained by putting \( n \) into a partition \( \beta \) of \([n - 1]\) such that \( n \) forms a part of a circular succession.

If \( 1 \) and \( n - 1 \) belong to the same block, put \( n \) into that block; so the number of partitions \( \pi \) obtained is \( p_1(n - 1, k, r - 2) \). (Note that this introduces the new succession \((n - 1, n)\) and replaces \((n - 1, 1)\) by \((n, 1)\), with \((n, 1)\) now counting.) If \( 1 \) and \( n - 1 \) belong to two different blocks, put \( n \) into either of the two blocks, so the number of partitions \( \pi \) obtained is \( 2p_2(n - 1, k, r - 1) \).

The total number of partitions \( \pi \) obtained is \( p_1(n - 1, k, r - 2) + 2p_2(n - 1, k, r - 1) \), that is,

\[ 2c_{r-1}(n - 1, k) - c_{r-2}(n - 2, k). \] (10)

The main result follows from addition of (9) and (10). The initial values may be verified separately.

We next obtain a second type of recurrence for \( c_r(n, k) \). The next theorem can be proved by showing that the right-hand side fulfills the recurrence in Theorem 4. But we provide a direct proof.

**Theorem 5.** Given integers \( n, k, r, 0 \leq r < n, 2 \leq k \leq n \), then

\[ c_r(n, k) = \binom{n}{r} c_0(n - r, k). \]

**Proof.** There are \( n \) possible successions \((j, j + 1), j = 1, \ldots, n\), which correspond to the terms of the cyclic sequence \( U_n = (1, 2, \ldots, n) \), where \( n + 1 \) is reduced modulo \( n \). The occurrence of \( r \) successions in a partition \( \pi \) then corresponds to a choice of \( r \) terms from \((1, 2, \ldots, n)\), in \( \binom{n}{r} \) ways. Once the corresponding successions are fixed we treat each contiguous sequence of \( t \leq r \) successions (consisting of \( t + 1 \) consecutive elements) as a single object. Thus \( U_n \) is transformed into a sequence of \( n - r \) “coded” terms, say \( V_{n-r} \).

To see this replace the terms in \( V_{n-r} \), assumed to be \( x \) in number, with their lengths, say \( c_i \), to obtain a composition of \( n \), that is, \((c_1, c_2, \ldots, c_x)\), \( c_1 + \cdots + c_x = n \). Since a term of length \( t \) contains \( t - 1 \) successions, we have \( r = (c_1 - 1) + \cdots + (c_x - 1) = n - x \).

For example if \( n = 9, r = 4 \), then a choice of \((2, 3, 6, 8)\) gives \( V_5 = ((1), (2, 3, 4), (5), (6, 7), (8, 9)) \) and a choice of \((2, 4, 6, 9)\) gives \( V_5 = ((9, 1), (2, 3), (4, 5), (6, 7), (8)) \).

Lastly, since the required number of successions has been fixed, the construction of \( \pi \) will be complete on partitioning \( V_{n-r} \) into \( k \) blocks of nonconsecutive terms, a task which returns \( \ell_2(n - r, k) = c_0(n - r, k) \) possible partitions. Hence the total number of partitions \( \pi \) is

\[ \binom{n}{r} \ell_2(n - r, k) = \binom{n}{r} c_0(n - r, k). \]

The next formula follows from Theorems 5 and 2.
Theorem 6. Given integers \( n, k, r \), \( 0 \leq r < n, \ 2 \leq k < n \), then
\[
c_r(n, k) = \binom{n}{r} \sum_{j=2}^{\frac{n-r+2}{2}} \binom{n-r-j}{j-2} \sum_{i=0}^{j-2} \binom{j-2}{i} \ S(n-r-j-i, k-2).
\]

Corollary 7. Let \( c_r(n) = \sum_k c_r(n, k) \). For all integers \( n, r \), \( 0 \leq r < n \), we have
\[
c_r(n) = \binom{n}{r} \sum_{j=2}^{\frac{n-r+2}{2}} \binom{n-r-j}{j-2} \sum_{i=0}^{j-2} \binom{j-2}{i} \ B(n-r-j-i).
\]

4. A remark on detached successions

If \( \pi \) is enumerated by \( c_r(n, k) \), then the \( r \) successions in \( \pi \) are called detached if no block of \( \pi \) contains a \( t \)-string of consecutive elements with \( t > 2 \). Let \( d_r(n, k) \) denote the number of \( k \)-partitions of \( [n] \) containing \( r \) detached circular successions. Then \( d_0(n, k) = c_0(n, k) \) and \( d_1(n, k) = c_1(n, k) \). For example, the following two partitions of \( [9] \) are enumerated by \( d_3(9, 3) \):

\[
1349/267/58 \quad \text{with successions (9, 1), (3, 4), (6, 7)}; \quad \text{and}
\]
\[
1267/3589/4 \quad \text{with successions (1, 2), (6, 7), (8, 9)}.
\]

However, \( 1259/367/48 \) is not since it contains the successions (9, 1), (1, 2), (6, 7) of which the first two are formed by a sequence of three consecutive elements (modulo 9): 9, 1, 2.

We will use a classical result of Kaplansky [8] to obtain a formula for \( d_r(n, k) \) namely:

The number of circular \( s \)-combinations of \( [m] \) with no two consecutive elements (in which the pair \( (n, 1) \) is treated as consecutive) is
\[
u_s(m, s) = \frac{m}{m-s} \left( \frac{m-s}{s} \right).
\]

As in the proof of Theorem 5 we obtain a recurrence using (11).

Theorem 8. Given integers \( n, k, r \), \( 0 \leq r < n, \ 2 \leq k \leq n \), then
\[
d_r(n, k) = \frac{n}{n-r} \binom{n-r}{r} d_0(n-r, k).
\]

The following explicit result is now a consequence of Theorems 8 and 2.

Corollary 9. Given integers \( n, k, r \), \( 0 \leq r < n, \ 2 \leq k < n \), then
\[
d_r(n, k) = \frac{n}{n-r} \binom{n-r}{r} \sum_{j=2}^{\frac{n-r+2}{2}} \binom{n-r-j}{j-2} \sum_{i=0}^{j-2} \binom{j-2}{i} \ S(n-r-j-i, k-2).
\]

5. Generating function methods

Define the polynomial \( C(n, k; z) = \sum_{r=0}^{n} c_r(n, k)z^r \). Then by Theorem 4 we have
\[
C(n, k; z) - C(n, k; 0) = (1-z)C(n-2, k-1; z) - C(n-2, k-1, 0) + C(n-1, k-1; z) - C(n-1, k-1, 0) + (k-1+z)(1-z)C(n-2, k; z) - (k-1)C(n-2, k, 0) + (k-2+2z)C(n-1, k; z) - (k-2)C(n-1, k, 0).
\]
By Theorem 1 and the fact that $C(n, k; 0) = c_0(n, k)$, we obtain

$$C(n, k; z) = (1 - z)C(n - 2, k - 1; z) + C(n - 1, k - 1; z) + (k - 1 + z)(1 - z)C(n - 2, k; z) + (k - 2 + 2z)C(n - 1, k; z), \quad (12)$$

for all $2 \leq k \leq n - 2$. The initial conditions in the statement of Theorem 4 give $C(k, k; z) = 1$ and $C(k + 1, k; z) = \left(\frac{k}{2}\right) - 1 + (k + 1)z$ if $k \geq 2$.

Let $C_k(x, z)$ be the generating function for the sequence $\{C(n, k; z)\}_{n \geq k}$, that is, $C_k(x, z) = \sum_{n \geq k} C(n, k; z)x^n$. Multiplying $(12)$ by $x^n$ and summing over all $n \geq k + 2$, we obtain

$$C_k(x, z) - C(k, k; z)x^k - C(k + 1, k; z)x^{k+1} = (1 - z)x^2C_{k-1}(x, z) - C(k - 1, k - 1; z)x^{k-1} \quad (13)$$

Thus using the initial conditions $C(k, k; z) = 1$ and $C(k + 1, k; z) = \left(\frac{k}{2}\right) - 1 + (k + 1)z$, we deduce

$$C_k(x, z) = (1 - z)x^2C_{k-1}(x, z) + xC_{k-1}(x, z) + (k - 1 + z)(1 - z)x^2C_k(x, z) + (k - 2 + 2z)xC_k(x, z),$$

which implies

$$C_k(x, z) = \frac{x}{1 - (k - 1)x - xz}C_{k-1}(x, z), \quad k \geq 3.$$

From the initial values, we see that $C(1, 1; z) = 1$ and $C(n, 1; z) = z^n$ for all $n \geq 2$, which lead to

$$C_1(x, z) = x + \frac{x^2z^2}{1 - xz}.$$  

Also, we have that $C(2, 2; z) = 1$, $C(3, 2; z) = 3z$ and $C(n, 2; z) = \frac{1}{2}((z + 1)^n + (z - 1)^n) - z^n$ if $n \geq 4$, which give

$$C_2(x, z) = \frac{x^2}{(1 + x - xz)(1 - xz)(1 - x - xz)}.$$  

Hence, by iterating $(13)$, we can state the following result.

**Theorem 10.** The ordinary generating function $C_k(x, z)$ is given by

$$C_k(x, z) = \frac{x^k}{\prod_{j=0}^{k-1}(1 - (j - 1)x)}, \quad k \geq 2,$$

with $C_0(x, z) = 1$ and $C_1(x, z) = x + \frac{x^2z^2}{1 - xz}$.

**Theorem 10** with $z = 0$ and the fact that $\frac{x^k}{\prod_{j=1}^{k}(1-jx)} = \sum_{n \geq k} S(n, k)x^n$ give

$$C_k(x, 0) = \frac{x^k}{\prod_{j=0}^{k}(1 - (j - 1)x)} = \frac{x}{1 + x} \sum_{i \geq k-1} S(i, k - 1)x^i$$

$$= x \sum_{j \geq 0} \sum_{i \geq k-1} (-1)^i S(i, k - 1)x^{i+j},$$

which implies the result.
Corollary 11. We have

\[ c_0(n, k) = \sum_{j=0}^{n-1} (-1)^j S(n - 1 - j, k - 1), \quad k > 1. \]

We next obtain a formula for the exponential generating function

\[ E_k(x, z) = \sum_{n \geq k} \sum_{r=0}^{n} c_r(n, k) z^r x^n n!. \]

By Theorem 10, we can write

\[ C_k(x, z) = \sum_{j=0}^{k} \frac{a_j}{1 - (j - 1 + z)x}, \]

where

\[ a_j = \lim_{x \to j - 1 + z} (1 - (j - 1 + z)x) C_k(x, z) = \frac{1}{(-1)^{k-j} j! k!} \begin{pmatrix} k \\ j \end{pmatrix}. \]

Thus

\[ C_k(x, z) = \frac{1}{k!} \sum_{j=0}^{k} \frac{(-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix}}{1 - (j - 1 + z)x} = \frac{1}{k!} \sum_{j=0}^{k} \sum_{i \geq 0} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} (j - 1 + z)^i x^i. \]

Hence,

\[ E_k(x, z) = \frac{1}{k!} \sum_{j=0}^{k} \sum_{i \geq 0} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} (j - 1 + z)^i x^i = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} e^{(j-1+z)x} = \frac{1}{k!} (e^x - 1)^k e^{x(z-1)}, \]

which implies that

\[ \sum_{k \geq 2} E_k(x, z)y^k = e^{x(z-1)}(e^{y(x-1)} - y(e^{x} - 1) - 1). \]

It is not hard to see that \( E_0(x, z) = 1 \) and \( E_1(x, z) = e^{xz} - 1 - x(z - 1). \) Therefore we can state:

Theorem 12. The generating function \( E(x, y, z) = \sum_{k \geq 0} \sum_{n \geq k} \sum_{r=0}^{n} c_r(n, k) z^r y^k x^n n! \) is given by

\[ E(x, y, z) = e^{y(x-1)+x(z-1)} + (y-1)(e^{x(z-1)} - 1) - xy(z-1). \]

By Theorem 12 and sequence [21, A000296], we have

\[ E(x, 1, z) = e^{x} - x^{-1 + xz} - x(z - 1) \]

\[ = \sum_{i \geq 0} \left[ (-1)^i \sum_{j=1}^{i} (-1)^{i-j} B(i-j) \right] \frac{x^i}{i!} \sum_{j=0}^{i} \frac{x^i z^j}{j!} - x(z - 1), \]

which implies the following result.

Corollary 13. Let \( n \geq 2. \) The number of partitions of \( [n] \) with exactly \( r \) circular successions is given by

\[ c_r(n) = \binom{n}{r} \left[ (-1)^{n-r} + \sum_{j=1}^{n-r} (-1)^{n-j} B(n - r - j) \right]. \]
Let \( k \geq 2 \), by (14), we have
\[
E_k(x, z) = \frac{1}{k!} (e^x - 1)^k e^{x(z-1)} = \sum_{j \geq k} S(j, k) \frac{x^j}{j!} \sum_{i \geq 0} \frac{x^i(z-1)^i}{i!},
\]
which implies that \( C(n, k; z) = n! \sum_{j=0}^{n-1} \frac{S((k, j; (z-1)^{n-j})}{(n-j)!} \). Therefore,
\[
c_r(n, k) = \sum_{j=k}^{n-r} (-1)^{n-r-j} \binom{n-j}{r} S(j, k).
\]

Thus we can state:

**Corollary 14.** Let \( n \geq 2 \). The number of partitions of \([n]\) with exactly \( k \) blocks and \( r \) circular successions is given by
\[
c_r(n, k) = \binom{n}{r} \sum_{j=k}^{n-r} (-1)^{n-r-j} \binom{n-r}{j} S(j, k).
\]

Note that by differentiating \( E(x, y, z) \) with respect to \( z \) and then setting \( z = y = 1 \), we obtain that the total number occurrences of circular successions in all partitions of \([n]\) is given by \( nB(n-1) \) for \( n \geq 2 \).

6. Combinatorial identities

In this section we state a few identities arising from the formulas established in Sections 2–5. It will be interesting to discover direct proofs of these identities.

Using the notations introduced in the proof of Theorem 1, one notes that \( c_0(n, k) = \ell_2(n, k) = f((n, 0), k) - \ell_1(n, k) \). Since \( f((n, 0), k) = S(n-1, k-1) \), it follows that \( S(n-1, k-1) = \ell_1(n-1, k) + \ell_1(n, k) \), which, from (4), is equivalent to
\[
S(n, k) = \sum_{j=0}^{n-1} \sum_{t=0}^{n-2-j} \binom{n-2-j}{j} \binom{j}{t} S(n-2-j-t, k-1)
+ \sum_{j=0}^{n-1} \sum_{t=0}^{n-1-j} \binom{n-1-j}{j} \binom{j}{t} S(n-1-j-t, k-1).
\]

Equating the formulas for \( c_r(n, k) \) in Theorem 6 and Corollary 14 leads to the following identity for all integers \( n, k, r \), \( 2 \leq k \leq n, 0 \leq r < n \).
\[
\sum_{j=2}^{\lfloor \frac{n-r+2}{2} \rfloor} \binom{n-r-j}{j-2} \sum_{i=0}^{\lfloor \frac{j-2}{2} \rfloor} \binom{j-2}{i} S(n-r-j-i, k-2) = \sum_{j=k}^{n-r} (-1)^{n-r-j} \binom{n-r}{j} S(j, k).
\]

In particular, Theorem 2 and Corollary 11 give, for all \( 2 \leq k \leq n \),
\[
\sum_{j=2}^{\lfloor \frac{n-j+2}{2} \rfloor} \binom{n-j}{j-2} \sum_{i=0}^{\lfloor \frac{j-2}{2} \rfloor} \binom{j-2}{i} S(n-j-i, k-2) = \sum_{i=0}^{n-1} (-1)^i S(n-1-j, k-1).
\]

Similarly Corollaries 7 and 13 imply:
\[
\sum_{j=2}^{\lfloor \frac{n-r+2}{2} \rfloor} \binom{n-r-j}{j-2} \sum_{i=0}^{\lfloor \frac{j-2}{2} \rfloor} \binom{j-2}{i} B(n-r-j-i) = (-1)^{n-r} + \sum_{j=1}^{n-r} (-1)^{j-1} B(n-r-j).
\]
Note that Theorems 5 and 8 imply the relation:
\[
\binom{n-1}{r} d_r(n, k) = \binom{n-r}{r} c_r(n, k).
\]

Finally we observe that the total number of occurrences of circular successions in all partitions of \([n]\) is given by \(n + \sum_{a < r < n} r c_r(n)\). Thus by Corollary 7 and invoking the remark at the end of Section 5, we obtain the identity \(n + \sum_{a < r < n} r c_r(n) = nB(n - 1)\) or
\[
\sum_{i=1}^{n-1} r \binom{n-r+j}{r} \sum_{j=2}^{n-r+2} \binom{n-r-j}{j-2} \sum_{i=0}^{j-2} \binom{j-2}{i} B(n - r - j - i) = n(B(n - 1) - 1).
\]
(Note: Mark Shattuck [20] has recently found combinatorial proofs of the identities in this section, with some generalizations.)

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References