Diagonal flips in Hamiltonian triangulations on the projective plane

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Abstract

In this paper, we shall prove that any two triangulations on the projective plane with \( n \) vertices can be transformed into each other by at most \( 8n - 26 \) diagonal flips, up to isotopy. To prove it, we focus on triangulations on the projective plane with contractible Hamilton cycles.

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1. Introduction

For a graph \( G \), let \( V(G) \) and \( E(G) \) denote the vertex set and edge set of \( G \), respectively. A \( k \)-cycle means a cycle of length \( k \). For two graphs \( H \) and \( K \), let \( H + K \) denote the graph obtained from \( H \) and \( K \) by joining each vertex of \( H \) to all vertices of \( K \).

A triangulation on a closed surface \( F^2 \) is a simple graph on \( F^2 \) such that each face is bounded by a 3-cycle. A diagonal flip is an operation which replaces an edge \( e \) in the quadrilateral \( D \) formed by two faces sharing \( e \) with another diagonal of \( D \) (see Fig. 1). If the resulting graph is not simple, then we do not apply it.

Wagner proved that any two triangulations on the plane with the same number of vertices can be transformed into each other by a sequence of diagonal flips, up to isotopy [9]. This result has been extended to the torus [1], the projective plane and the Klein bottle [7]. Moreover, Negami has proved that for any closed surface \( F^2 \), there exists a positive integer

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Fig. 1. Diagonal flip.

$N(F^2)$ such that any two triangulations $G$ and $G'$ on $F^2$ with $|V(G)| = |V(G')| \geq N(F^2)$ can be transformed into each other by a sequence of diagonal flips, up to homeomorphism [4]. There are many papers concerning with diagonal flips in triangulations and they are described in [6] for details.

In this paper, we focus on the minimum number of diagonal flips needed to transform two given triangulations on a closed surface $F^2$. Negami’s argument in [5] shows that for the minimum number of diagonal flips needed to transform two triangulations with $n$ vertices on a closed surface $F^2$, there is a quadratic bound with respect to $n$. However, if we restrict $F^2$ to the sphere, then there is a linear bound $6n - 30$ for it, as shown in [3].

In this paper, we shall prove the following theorem:

**Theorem 1.** Any two triangulations on the projective plane with $n$ vertices can be transformed into each other by at most $8n - 26$ diagonal flips, up to isotopy.

This is the first result giving a linear bound for the minimum number of diagonal flips in triangulations on a closed surface other than the sphere.

For a graph $G$, a Hamilton cycle of $G$ is a cycle passing through each vertex of $G$ exactly once. A cycle $C$ of $G$ embedded in a closed surface $F^2$ is said to be contractible if $C$ bounds a 2-cell on $F^2$. In order to prove Theorem 1, we show the following theorem for triangulations on the projective plane with a contractible Hamilton cycle, as in the spherical case in [3].

**Theorem 2.** Let $G$ and $G'$ be two triangulations on the projective plane with $n$ vertices, each of which has a contractible Hamilton cycle. Then $G$ and $G'$ can be transformed into each other by at most $6n - 12$ diagonal flips, preserving their Hamilton cycles.

2. Triangulations with contractible Hamilton cycles

In this section, we deal only with triangulations which have contractible Hamilton cycles. Clearly, a contractible Hamilton cycle in a triangulation $G$ on the projective plane separates
$G$ into two spanning subgraphs of $G$. One is a maximal outer-plane graph, denoted by $G_P$, and the other is a triangulation of the Möbius band, denoted by $G_M$, in which all vertices appear on the boundary of the Möbius band. We call it a Catalan triangulation on the Möbius band.

**Lemma 3.** Let $P$ be a maximal outer-plane graph with $n \geq 3$ vertices and let $v$ be a vertex of degree $k \geq 2$ in $P$. Then $P$ can be transformed into a maximal outer-plane graph in which the degree of $v$ is exactly $n - 1$, by exactly $n - k - 1$ ($\leq n - 3$) diagonal flips, through maximal outer-plane graphs.

**Proof.** Let $xy$ be an edge of $P$ not in its outer cycle and let $uxy$ and $vxy$ be two faces sharing $xy$. Since $\deg_P(v) = k$, the number of vertices not adjacent to $v$ is $n - k - 1$. Since $P$ has no subgraph isomorphic to $K_4$, $u$ and $v$ are not adjacent in $P$. Therefore, we can flip $xy$ without making multiple edges. Hence, we can increase the degree of $v$ one by one, by diagonal flips. Therefore, the lemma follows. □

In [2], the Catalan triangulations on the Möbius band with $n$ vertices were enumerated and it was proved that any two of them can be transformed into each other by diagonal flips, but the number of diagonal flips had never been estimated yet.

Let $M^2$ denote the Möbius band and let $\partial M^2$ denote the boundary of $M^2$. Let $K$ be a Catalan triangulation on $M^2$ with $m$ vertices. Let $v_1, v_2, \ldots, v_m$ be the vertices of $K$ lying on $\partial M^2$ in this cyclic order. An edge $v_i v_j$ is said to be trivial if cutting along $v_i v_j$ separates a disk $D$ from $M^2$. Clearly, the subgraph of $K$ induced by the vertices on $D$ is a maximal outer-plane graph, which is said to be bounded by $v_i v_j$. Edges of $K$ which are not trivial are said to be essential.

Suppose that a Catalan triangulation $K$ on the Möbius band $M^2$ has no trivial edge. An essential edge $e$ of $K$ incident to a vertex of degree 3 is called a spoke. The subgraph of $K$ induced by the essential edges which are not spokes is said to be the zigzag frame of $K$, which is uniquely taken. It is easy to see that the zigzag frame of $K$ is a cycle of an odd length homotopic to the center line of $M^2$. Moreover, if $K$ has no trivial edge and no spoke, then $K$ is 4-regular.

**Lemma 4.** Let $G$ be a triangulation on the projective plane with $n \geq 7$ vertices. If $G$ has a contractible Hamilton cycle $C$, then $G$ can be transformed into $K + K_1$ by at most $n - 1$ diagonal flips, where $K$ is some Catalan triangulation on the Möbius band.

**Proof.** Let $G_P$ and $G_M$ be the maximal outer-plane graph and the Catalan triangulation on the Möbius band, each of which is a spanning subgraph of $G$ with boundary $C$.

We shall make a vertex of degree 2 in $G_M$ by at most three diagonal flips, without breaking the simpleness of $G$. If $G_M$ has a trivial edge $xy$, then $xy$ bounds an outer-plane graph $L$. It is easy to see that $L$ has a vertex of degree 2 other than $x$ and $y$. Thus, we have nothing to do, and hence we may suppose that $G_M$ has no trivial edge.

First, if $G_M$ has no trivial edge and no spoke, then $G_M$ is 4-regular. Since $G_P$ is outer-planar, $G_P$ has a vertex of degree 2 in $G_P$, say $v$ with two neighbors $p$ and $s$. Suppose that $G_M$ has faces $pqv$, $qrv$ and $rsu$ meeting at $v$, and faces $urs$, $rts$ and $tus$ meeting at $s$ in
Fig. 2. Two diagonal flips making a vertex of degree 2.

$G_M$. (See the left-hand side of Fig. 2.) Observe that since $\deg_{\overline{G_P}}(v) = 2$, any diagonal flip in $G_M$ increasing the degree of $v$ yields no edge forming multiple edges with an edge in $G_P$. Moreover, since $n \geq 7$, we have $vt, vu \notin E(G_M)$; otherwise, we would have $u = q$ and $p = t$. Therefore, $rs$ can be replaced with $vt$, and next $st$ can be replaced with $vu$. Now $s$ has degree 2 in the resulting graph on $M^2$, which is obtained by two diagonal flips. (See the right-hand side of Fig. 2.)

Finally suppose that $G_M$ has spokes but no trivial edges. We first suppose that $G_M$ has two consecutive spokes $pq$ and $pr$ such that $q$ and $r$ are adjacent on $C$ and $\deg_{G_M}(q) = \deg_{G_M}(r) = 3$. Let $pq$, $pqs$ and $prt$ be three faces meeting at $p$. It is easy to see that a diagonal flip can replace an edge $pq$ with $sr$ without making multiple edges in $G_M$, but $G_P$ might already have an edge $sr$. In this case, by the planarity of $G_P$, $G$ does not have an edge $qt$ because of the obstruction of $sr$. Therefore, we can make $r$ have degree 2 by one diagonal flip.

Now consider the case when the vertices of degree 3 in $G_M$ are independent. Since $n \geq 7$, the zigzag frame of $G_M$ has length at least 5. (For otherwise, i.e., if the zigzag frame has length 3 and all vertices of degree 3 are independent, then we have $n \leq 6$, a contradiction.) Let $pq$ be a spoke with $\deg_{G_M}(q) = 3$ and shared by two faces $pqs$ and $pq$. Note that $4 \leq \deg_{G_M}(s), \deg_{G_M}(t) \leq 5$. Apply a diagonal flip of $pq$ to make a vertex of degree 2 in $G_M$. If impossible, $G_P$ already has an edge $st$. (Here, if $G$ is assumed to be 4-connected, then this does not happen, because $G - \{p,t\}$ must be connected.) If $G_P$ has an edge $st$, then we can make $q$ have degree 5 or 6 and $s$ have degree 2 by at most three diagonal flips, flipping the edges incident to $s$ in $G_M$, not on $\partial M^2$, to make them be incident to $q$, similarly to the case when $G_M$ is 4-regular. (Note that only the final case requires at most three diagonal flips to make a vertex of degree 2 and it does not happen in the 4-connected case. Hence this proves the following remark.)

We turn our attention to $G_P$. Let $G'_M$ denote a Catalan triangulation with a vertex $v$ of degree 2 obtained from $G_M$ by at most three diagonal flips. Then we can apply any diagonal flip in $G_P$ increasing the degree of $v$, without making multiple edges with an edge of $G_M$. Observe that $\deg_{G_P}(v) \geq 3$, since every vertex of a triangulation on a closed surface has degree at least 3. Therefore, at most $n - 4$ diagonal flips can make $v$ have degree $n - 1$ in $G_P$, by Lemma 3. In the resulting graph, $v$ is adjacent to all other vertices, and the graph with $v$ removed is obviously a Catalan triangulation with $n - 1$ vertices.
As shown in the above proof, we have the following remark.

**Remark 5.** In Lemma 4, if we assume the 4-connectedness of \( G \), then the number of diagonal flips can be improved to \( n - 2 \).

Consider a Catalan triangulation on the Möbius band shown in the left-hand side of Fig. 3, which is a unique Catalan triangulation with five vertices isomorphic to \( K_5 \). Let \( e = v_4v_5 \) be an edge of the Catalan triangulation \( K_5 \) lying on the boundary of the Möbius band. Subdivide \( e \) by \( m \) vertices as shown in the right-hand side of Fig. 3, where the Möbius band is obtained by identifying the arrows indicated in the left-hand and the right-hand sides of the rectangles. The resulting graph is called the standard form of the Catalan triangulations and denoted by \( \Gamma_m \).

The following is the most essential argument in this paper.

**Lemma 6.** Every Catalan triangulation \( K \) on the Möbius band with \( n \) vertices can be transformed into the standard form \( \Gamma_{n-5} \) by at most \( 2n - 3 \) diagonal flips.

**Proof.** Suppose that \( K \) has \( p \) trivial edges. Then it is easy to see that the unique sub-Catalan triangulation, denoted by \( K' \), of \( K \) with no trivial edges is obtained from \( K \) by successively removing a vertex of degree 2. Clearly, \( K' \) has exactly \( n - p \) vertices.

Suppose that \( K' \) has \( q \) spokes and let \( r = n - p - q \). Then the zigzag frame \( v_1 \ldots v_r \), of \( K' \) has an odd length \( r \geq 3 \). Let \( q_i \) be the number of spokes of \( K' \) incident to \( v_i \), for \( i = 1, \ldots, r \). We may suppose that \( q_1 + q_3 + \cdots + q_r \geq q_2 + q_4 + \cdots + q_{r-1} \). (For otherwise, we can replace \( v_i \) by \( v_{i-1} \) for each \( i \), because the subscripts are cyclic and taken modulo \( r \).)

Let \( q_2 + q_4 + \cdots + q_{r-1} = m \) and hence we have \( 2m \leq q \). (See Fig. 4.)

If \( r = 3 \), then by the simpleness of graphs, we have \( q_2, q_3 \geq 1 \). Hence, we can flip an edge \( v_2v_3 \) to make the zigzag frame have length 5. So, suppose that \( r \geq 5 \). Apply diagonal flips to all \( m \) spokes incident to \( v_2, v_4, \ldots, v_{r-1} \) to make them trivial one by one. The number of diagonal flips we did is exactly

\[
q_2 + q_4 + \cdots + q_{r-1} = m. \tag{1}
\]
Fig. 4. $K'$ with a zigzag frame $v_1v_2 \ldots v_r$.

Fig. 5. Reducing the length of the zigzag frame.

Note that even if $r = 3$, the estimation (1) is true. Though we need one more diagonal flip of $v_2v_3$ to increase the length of the zigzag frame, this diagonal flip decreases $q_2$ and $q_3$ by one, respectively.

Next reduce the length of the zigzag frame from $r$ to 5. In particular, we first apply a diagonal flip of $v_4v_5$, secondly flip $q_5$ spokes incident to $v_5$, and finally flip $v_5v_6$. (See Fig. 5(1).) The number of diagonal flips we did is $q_5 + 2$. In the resulting graph, the zigzag frame has length $r - 2$, and exactly one new trivial edge $v_3v_7$ appears. As far as the length of the zigzag frame is at least 7, we apply these operations. If its length is exactly 5, then we apply $q_1 + q_r$ diagonal flips, as shown in Fig. 5(2). Then the total number of
diagonal flips we did is
\[
(q_5 + 2) + (q_7 + 2) + \cdots + (q_{r-2} + 2) + q_1 + q_r \leq (q - m) + 2 \left( \frac{r - 5}{2} \right). \tag{2}
\]

Let \( H' \) be the current Catalan triangulation obtained from \( K' \). The zigzag frame of \( H' \) has length exactly 5, and all spokes of \( H' \) are incident to \( v_3 \). Moreover, \( H' \) has \( \frac{1}{2} (r - 5) + m \) trivial edges, since all \( m \) spokes incident to \( v_2, v_4, \ldots, v_{r-1} \) in \( K' \) are replaced with trivial edges of \( H' \), and since decreasing the length of the zigzag frame of \( K' \) by two yields exactly one new trivial edge. Let \( H \) be the Catalan triangulation consisting of \( H' \) and all trivial edges of \( K' \). Then \( H \) has exactly \( p + \frac{1}{2} (r - 5) + m \) trivial edges.

Now, renaming vertices, we put \( H \) with the zigzag frame \( u_1 u_2 u_3 u_4 u_5 \) as shown in Fig. 6, where \( u_1 = v_1, u_3 = v_3 \) and \( u_5 = v_5 \). The four triangular faces \( u_1 u_2 u_5, u_1 u_3 u_5, u_3 u_4 u_5 \) and \( u_4 u_5 u_1 \) of \( H \) come from \( K' \). Let \( R_i \) denote the outer-plane graph bounded by an edge \( u_{i-1} u_{i+1} \) and containing the edge \( u_{i-1} u_{i+1} \), for \( i \neq 3 \). (Note that \( R_i \) might be just an edge.)

The region \( F_i \) of the zigzag frame of \( H \) is the union of the faces bounded by the two edges \( u_{i-1} u_1, u_{i+1} u_{i+1} \) and the path on \( \partial M^2 \) connecting \( u_{i-1} \) and \( u_{i+1} \), for each \( i \), where the subscripts are taken modulo 5. Now we shall transform \( H \) into a Catalan triangulation in which all the regions of the zigzag frame, except one corresponding to \( F_3 \), consists of just one face.

Here we focus on the outer-plane graph \( R_2 \) and first suppose that \( |V(R_2)| \geq 3 \). By Lemma 3, we can make \( u_1 \) have degree \( |V(R_2)| - 1 \) by at most \( |V(R_2)| - 3 \) diagonal flips. Let \( u_1, x_1, \ldots, x_l, u_3 \) be the vertices of \( R_2 \) lying on \( \partial M^2 \) in this order. Apply five diagonal flips of \( u_1 u_3, u_1 u_2, u_1 x_1, u_1 u_5 \) and \( u_4 u_5 \) in this order, if \( l \geq 2 \). (See Fig. 7.) If \( l = 1 \), then apply three diagonal flips of \( u_1 u_3, u_1 u_2, u_4 u_5 \) in this order. In the resulting graph, each of two regions of the zigzag frame corresponding to \( F_2 \) and \( F_3 \) is just a face. The number of diagonal flips we did is at most \( |V(R_2)| - 3 + 5 \).

Secondly we suppose that \( |V(R_2)| = 2 \). If we also have \( |V(R_3)| = 2 \), then we have nothing to do for \( F_2 \) and \( F_3 \). So, suppose that \( |V(R_3)| \geq 3 \). Similarly to the above case, at most \( |V(R_3)| - 3 \) diagonal flips make \( u_4 \) have degree \( |V(R_3)| - 1 \) in \( R_3 \) and we apply two
diagonal flips of \( u_1u_4 \) and \( u_4u_5 \). In the resulting graph, the two regions corresponding to \( F_2 \) and \( F_5 \) are just faces. Then the number of diagonal flips we did is at most \(|V(R_5)| - 3 + 2\). Note that by the above operations, the number of trivial edges decreases by one, if \(|V(R_2)| \geq 3 \) or \(|V(R_5)| \geq 3 \).

We can do the same procedures for the regions \( R_1 \) and \( R_4 \). Let \( L \) denote the resulting graph in which exactly four regions are just faces. Hence, the number of diagonal flips transforming \( H \) into \( L \) is at most

\[
\max\{|V(R_2)| + 2, |V(R_5)| - 1\} + \max\{|V(R_4)| + 2, |V(R_1)| - 1\} \\
\leq p + \frac{1}{2}(r - 5) + m + 8,
\]

(3)

since

\[
(|V(R_1)| - 2) + (|V(R_2)| - 2) + (|V(R_4)| - 2) + (|V(R_5)| - 2) \\
\leq p + \frac{1}{2}(r - 5) + m.
\]

Note that we can assume that the number of trivial edges of \( L \) is at most \( p + \frac{1}{2}(r - 5) + m - 1 \), since we may suppose that at least one of \( R_1, R_2, R_4 \) and \( R_5 \) has at least three vertices. (For otherwise, we do not need to add (3) to the estimation of the maximum number of diagonal flips, and this case requires a few number of diagonal flips.)

Finally we flip all trivial edges of \( L \), all of which are incident to \( u_3 \). Since the number of trivial edges of \( L \) is at most \( p + \frac{1}{2}(r - 5) + m - 1 \), the number of diagonal flips transforming \( L \) into the standard form is at most

\[
p + \frac{1}{2}(r - 5) + m - 1 = p + \frac{r}{2} + m - \frac{7}{2}.
\]

(4)

Therefore, by (1)–(4), the total number of diagonal flips is at most

\[
m + (q - m + r - 5) + \left(p + \frac{r}{2} + m + \frac{11}{2}\right) + \left(p + \frac{r}{2} + m - \frac{7}{2}\right) \\
= 2p + q + 2m + 2r - 3 \\
\leq 2(p + q + r) - 3 = 2n - 3,
\]

since \( q \geq 2m \). Therefore, the lemma follows. \( \square \)
3. Triangulations with contractible Hamilton cycles

In the previous section, we described only the result on triangulations with contractible Hamilton cycles. In this section, we shall mention how we can obtain triangulations with contractible Hamilton cycles from any triangulations.

The following gives an important sufficient condition for a graph on the projective plane to have a contractible Hamilton cycle.

**Lemma 7 (Thomas and Yu [8]).** Every 4-connected graph on the projective plane has a contractible Hamilton cycle.

The following lemma is essential.

**Lemma 8.** Let \( G \) be a triangulation on the projective plane with \( n \) vertices. Then \( G \) can be transformed into a 4-connected triangulation by at most \( n - 6 \) diagonal flips.

**Proof.** Observe that a triangulation on the projective plane has no separating 3-cycle if and only if it is 4-connected. We first show that \( G \) has at most \( n - 6 \) separating 3-cycles, by induction on \( n \). It is well-known that the smallest triangulation on the projective plane is the unique triangular embedding of \( K_6 \), which has no separating 3-cycle. Therefore, the lemma holds when \( n = 6 \).

When \( n \geq 7 \), we may assume that \( G \) has a separating 3-cycle \( C = xyz \), and it is innermost, that is, there is no separating 3-cycle in the 2-cell bounded by \( C \). Cutting along \( C \), we can decompose \( G \) into a plane triangulation \( G_1 \) with no separating 3-cycle and a triangulation \( G_2 \) on the projective plane. By induction hypothesis, \( G_2 \) has at most \( |V(G_2)| - 6 \) separating 3-cycles. Let \( M \) denote the number of separating 3-cycles in \( G \). Then we have

\[
M \leq |V(G_2)| - 6 + 1 = (n - |V(G_1)| + 3) - 5 \leq n - 6,
\]

since \( |V(G_1)| \geq 4 \).

Now we shall show that there is a diagonal flip decreasing the number of separating 3-cycles by at least one. Let \( C = xyz \) be a separating 3-cycle in \( G \) and \( e = xy \). Let \( xayb \) be the quadrilateral formed by two triangular faces sharing \( e \), where \( a \) lies in the 2-cell region bounded by \( C \). Consider the diagonal flip of \( e \) replacing \( xy \) with \( ab \). In the resulting graph \( G' \), the separating cycle \( C \) in \( G \) has disappeared.

We shall show that no new separating 3-cycle arises in \( G' \), by possibly re-choosing \( e \). Suppose that \( G' \) has a new separating 3-cycle \( C' \). Then \( C' \) contains both \( a \) and \( b \); otherwise, \( C' \) would be contained in \( G \). We must have \( C' = abz \), where we assume that \( x \) is contained in the 2-cell region bounded by \( C' \) in \( G' \). This means that \( V(G_1) = \{ x, y, z, a \} \) since \( C \) is innermost in \( G \). In this case, the edge \( yz \) can be flipped to destroy a 3-cycle \( byz \) and make no new separating 3-cycle, because \( byz \) separates \( a \) and other vertices outside \( byz \). Therefore, at most \( n - 6 \) diagonal flips can make the graph be 4-connected. \( \Box \)

4. Proof of theorems

It is well-known that the smallest triangulation on the projective plane is the unique triangular embedding of \( K_6 \). Let \( xy \) be one of its edges, and suppose that two faces \( xyz \)
and \(xwy\) share \(xy\). Subdivide \(xy\) by \(m\) vertices \(v_1, \ldots, v_m\) and add \(2m\) edges \(v_i z, v_i w\) for \(i = 1, \ldots, m\). The resulting graph with \(m + 6\) vertices is called the standard form of triangulations on the projective plane and denoted by \(\Psi_m\). (See Fig. 8.) Clearly, we obtain the standard form \(\Psi_m\) from the standard form \(\Sigma_{m-1}\) of Catalan triangulations of the Möbius band \(M^2\) by pasting a disk along \(\partial M^2\), placing a vertex \(v\) at its center and joining \(v\) to all vertices of \(\Sigma_m\).

We first prove the following theorem.

**Theorem 9.** Let \(G\) be a triangulation on the projective plane with \(n\) vertices which has a contractible Hamilton cycle. Then \(G\) can be transformed into \(\Psi_{n-6}\), preserving the Hamilton cycle, by at most \(3n - 6\) diagonal flips. If \(G\) is 4-connected, then the number of diagonal flips is improved to \(3n - 7\).

**Proof.** We may suppose that \(n \geq 7\). By Lemma 4, \(G\) can be transformed into \(K + K_1\) by at most \(n - 1\) diagonal flips, preserving the Hamilton cycle, where \(K\) is some Catalan triangulation on the Möbius band with \(n - 1\) vertices. (By Remark 5, if \(G\) is 4-connected, the number “\(n - 1\)” of diagonal flips can be replaced with “\(n - 2\)”.

Note that no two vertices of \(K\) are joined by an edge outside \(K\). Therefore, it suffices to prove that \(K\) can be transformed into \(\Sigma_{n-6}\). By Lemma 6, it can be done by at most \(2(n - 1) - 3\) diagonal flips. Therefore, \(G\) can be transformed into \(\Psi_{n-6}\) by at most \(3n - 6\) (\(3n - 7\) when \(G\) is 4-connected) diagonal flips, preserving the Hamilton cycle. \(\square\)

**Theorem 10.** Every triangulation on the projective plane with \(n\) vertices can be transformed into the standard form \(\Psi_{n-6}\) by at most \(4n - 13\) diagonal flips, up to isotopy.
Proof. Let $G$ be a triangulation on the projective plane with $n$ vertices. By Lemma 8, at most $n - 6$ diagonal flips transform $G$ into a 4-connected triangulation, denoted by $H$. By Lemma 7, $H$ has a contractible Hamilton cycle. Then apply Theorem 9. □

Now we shall prove Theorems 1 and 2.

Proof of Theorems 1 and 2. Theorems 1 and 2 follow from Theorems 10 and 9, respectively, via the standard form $\Psi_{n-6}$. □

Finally, we consider two triangulations $G_1$ and $G_2$ on the projective plane with $n$ vertices which need many diagonal flips to transform them into each other. Let $G_1 = \Psi_{n-6}$, and let $G_2$ be a triangulation with maximum degree 6. For example, it is constructed from $K_6$ by putting a triangular mesh shown in Fig. 9 to each face.

The maximum degree of $G_1$ is $n - 1$ and it is attained by two vertices, say $x$ and $y$. To transform $G_1$ into $G_2$, we have to decrease the degree of $x$ and $y$ to six or five. Since each diagonal flip decreases the degree of a fixed vertex at most by one, each of $x$ and $y$ requires at least $(n - 1) - 6$ diagonal flips. Observe that the degree of $x$ and $y$ decrease simultaneously by one diagonal flip, only if this diagonal flip is applied to the edge $xy$. If such diagonal flips are applied at least twice in the process from $G_1$ to $G_2$, then there must be a diagonal flip joining $x$ and $y$, which increases $\text{deg}(x) + \text{deg}(y)$. Therefore, if the number $\text{deg}(x) + \text{deg}(y)$ is non-increasing in the process from $G_1$ to $G_2$, then the edge $xy$ is flipped at most once, and the number of diagonal flips transforming $G_1$ to $G_2$ is at least

$$(n - 1) - 6 + (n - 1) - 6 - 1 = 2n - 13.$$ 

Therefore, the order of our estimation in Theorems 2 and 1 cannot be improved.

Proposition 11. For any natural number $N$, there exists a pair of triangulations $G_1$ and $G_2$ on the projective plane with $n \geq N$ vertices such that at least $2n - 13$ diagonal flips are needed to transform them into each other.
References