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The Structure of Modules over Hereditary Rings

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Abstract. Let A be a bounded hereditary Noetherian prime ring. For an A-module M_A, we prove that M is a finitely generated projective $A/r(M)$-module if and only if M is a π-projective finite-dimensional module, and either M is a reduced module or A is a simple Artinian ring. The structure of torsion or mixed π-projective A-modules is completely described.

Key words: hereditary ring, projective module, π-projective module.

All rings are assumed to be associative and with nonzero identity element. Expressions such as "a Noetherian ring" mean that the corresponding right and left conditions hold. A module M is said to be *π-projective* if for any two of its submodules U and V with $U+V=M$, there exists an endomorphism f of M such that $f(M)\subseteq U$ and $(1-f)(M)\subseteq V$ (see [1, p. 359]). A module M is said to be *skew-projective* if for every epimorphism $h\colon M\to\overline{M}$ and every endomorphism f^* of the module $\overline{M}$, there exists an endomorphism f of M with $f^*h=hf$. A module is said to be *reduced* if it does not have a nonzero injective direct summand. A module is said to be *finite-dimensional* (in the sense of Goldie) if it does not contain an infinite direct sum of nonzero submodules. By $r_A(N)$, we denote the annihilator in the ring A of a subset N of a right module M_A, with the subscript sometimes omitted when it is clear what ring is meant. By $\mathrm{Sing}(M)$, we denote the *singular submodule* of a module M_A over the ring A (i.e., the set of all of the elements $m\in M$ such that $r(m)$ is an essential right ideal of A). A module M_A is said to be *nonsingular* (*singular*) if $\mathrm{Sing}(M)=0$ ($\mathrm{Sing}(M)=M$).

In [2], the following theorem A is proved.

Theorem A [2, Theorem 4]. *Assume that M is a module over a hereditary Noetherian prime ring A, and the ring A is not right primitive. Then M is a finitely generated projective module $\iff$ M is a π-projective reduced finite-dimensional nonsingular module.*

A ring A is said to be *left* (resp. *right*) *bounded* if every its essential left (resp. right) ideal contains a nonzero ideal of A. Note that every hereditary prime nonprimitive ring is a bounded ring [3]. The main results of this paper are Theorems 1, 2, and 3.

Theorem 1. *Let M be a module over a bounded hereditary Noetherian prime ring A. Then M is a finitely generated projective $A/r(M)$-module $\iff$ M is a π-projective finite-dimensional module, and either M is a reduced module or A is a simple Artinian ring.*

Theorem 2. *Let M be a module over a bounded hereditary Noetherian prime ring A, and let M satisfy* $\mathrm{Sing}(M)\neq 0$. *Then M is a π-projective module $\iff$ M is a skew-projective module $\iff$ one of the following conditions holds*:

(i) *M is a singular module such that every primary component of M is either an indecomposable injective module or a projective module over the factor ring of A with respect to the annihilator of this primary component*;

(ii) *$M=T\oplus F$, where F is a finitely generated projective module, and T is a singular injective module such that every primary component of T is an indecomposable module.*

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Theorem 3. *Let M be a module over a bounded hereditary Noetherian prime ring A. Then M is a π-projective nonsingular finite-dimensional module $\iff$ one of the following conditions holds*:

(i) *M is a finitely generated projective module*;

(ii) *A is a serial ring and $M = T \oplus F$, where T is an injective indecomposable nonsingular module, and F is a finitely generated projective module*;

(iii) *there exists a positive integer n such that the ring A is isomorphic to the ring of all of $(n \times n)$ matrices over a complete uniserial Noetherian domain D and $M = T \oplus F$, where T is an injective finite-dimensional nonsingular module and F is a finitely generated projective module.*

The proofs of Theorems 1, 2, and 3 are decomposed into a series of lemmas. Let us present the necessary notation and definitions. For a module M, we denote by $\mathrm{End}(M)$ and $J(M)$ the endomorphism ring and the Jacobson radical of M, respectively. A module is said to be *hereditary* (resp. *semihereditary*) if all its submodules (resp. all its finitely generated submodules) are projective. A module is said to be *uniserial* if any two of its submodules are comparable with respect to inclusion. A direct sum of uniserial modules is called a *serial* module. A uniserial Noetherian domain A is said to be *complete* if the ring A is complete with respect to the $J(A)$-adic topology. A module M is said to be *projective with respect to a module N* (or *N-projective*) if for every epimorphism $h\colon N \to \overline{N}$ and each homomorphism $\overline{f}\colon M \to \overline{N}$, there exists a homomorphism $f\colon M \to N$ with $hf = \overline{f}$. A module is said to be *quasi-projective* if it is projective with respect to itself. For a module M, a submodule of a factor module of M is called a *subfactor* of M. A module is said to be *infinite-dimensional* if it is not finite-dimensional. A right finite-dimensional ring with the maximum condition on right annihilators is called a *right Goldie* ring. An element a of a ring A is called a *regular* element if a is not a left or right zero-divisor. For a module M, we denote by $T(M)$ the set of all of the elements in M whose annihilators contain regular elements. A module M is said to be *torsion* (resp. *mixed*, *torsion-free*) if $T(M) = M$ (resp. $0 \neq T(M) \neq M$, $T(M) = 0$). A module is said to be *uniform* if any two of its nonzero submodules have nonzero intersection. A submodule N of a module M is said to be *essential* (in M) if N has nonzero intersection with any nonzero submodule of the module M. In this case, we say that M is an *essential extension* of the module N. A submodule V of a right module U over a ring S is said to be a *pure* submodule in U_S if for every left S-module M, a natural group homomorphism $V \otimes_S M \to U \otimes_S M$ is a monomorphism. A module X_A is said to be *pure-injective* if for every module M_A and any pure submodule N of M, all homomorphisms $N \to X$ can be extended to homomorphisms $M \to X$. A module M is said to be *finitely faithful* if there exists a positive integer n such that the module M^n contains a free cyclic submodule. A ring A is said to be *right pure-semisimple* if all right A-modules are pure-injective. A ring A is said to be a ring of *finite representation type* if A is a Artinian ring, and there exists only finitely many of nonisomorphic indecomposable finitely generated (right or left) A-modules.

Let A be a ring with the classical ring of quotients Q, and let B be an ideal of A. The ideal B is said to be *invertible* if there exists a subbimodule B^{-1} of the bimodule ${}_AQ_A$ with $BB^{-1} = B^{-1}B = A$. A maximal element of the set of all of the proper invertible ideals of the ring A is called a *maximal invertible ideal* of A. The set of all of the maximal invertible ideals of a ring A is denoted by $\mathcal{P}(A)$. If M is a A-module and $P \in \mathcal{P}(A)$, then the submodule $\{m \in M \mid mP^n = 0, n = 1, 2, \dots\}$ is called the *P-primary component* of M and is denoted by $M(P)$.

1. Proof of Theorem 1

Lemma 1.1. *Let M be a module over a ring A.*

(1) *M is a quasi-projective (π-projective; skew-projective) A-module $\iff$ M is a quasi-projective (π-projective; skew-projective) $A/r(M)$- module.*

(2) *If M is a quasi-projective module, then M is skew-projective.*

(3) *If all idempotent endomorphisms of all of the factor modules of the module M can be lifted to endomorphisms of M, then M is π-projective.*

(4) *If M is a skew-projective module, then M is π-projective.*

(5) *If M is a quasi-projective module, then M is π-projective.*

(6) *If M is a uniserial module, then M is π-projective.*

(7) *If T is a submodule of M, then T is pure in M $\Longleftrightarrow$ for any two positive integers n and k, the system of equations $\sum_{j=1}^{k} X_j a_{ij} = t_i$ $(i = 1, \dots, n,\ a_{ij} \in A,\ t_i \in T)$, which has a solution $x_1, \dots, x_k \in M$, also has a solution in T.*

Proof. (1) The proof follows from the fact that any two of the subfactors N and P of M are $A/r(M)$-modules and $\mathrm{Hom}_{A/r(M)}(N, P) = \mathrm{Hom}_A(N, P)$.

(2) Let $h\colon M \to \overline{M}$ be an epimorphism, and let f^* be an endomorphism of the module $\overline{M}$. Set $\overline{f} \equiv f^* h \in \mathrm{Hom}(M, \overline{M})$. Since M is a quasi-projective module, there exists an endomorphism f of M such that $hf = \overline{f} = f^* h$. Therefore, M is a skew-projective module.

(3) Let U and V be two submodules of M with $U + V = M$, $\overline{M} \equiv M/(U \cap V)$, and let $h\colon M \to \overline{M}$ be a natural epimorphism. Since $\overline{M} = h(U) \oplus h(V)$, there exists a natural projection $\overline{f}\colon \overline{M} \to h(U)$ with the kernel $h(V)$. By assumption, there exists an endomorphism f of the module M with $\overline{f}h = hf$. Therefore, $(1_{h(M)} - \overline{f})h = h - \overline{f}h = h(1_M - f)$. Then $hf(M) = \overline{f}(\overline{M}) = h(U)$ and $h(1-f)(M) = h(V)$. Therefore, $f(M) \subseteq U + U \cap V = U$, $(1-f)(M) \subseteq V + U \cap V = V$, and M is a π-projective module.

(4) The proof follows from (3).

(5) The proof follows from (2) and (4).

(6) By (3), it is sufficient to prove that every nonzero idempotent endomorphism f^* of any factor module $\overline{M}$ of the uniserial module M can be lifted to an endomorphism of M. Since $\overline{M}$ is a uniserial module, f^* is the identity automorphism. Therefore, f^* can be lifted to the identity automorphism of M.

(7) The assertion is proved in [1, 34.5]. □

Lemma 1.2. *Let M be a module over a ring A, and let $\{Y_i\}_{i \in I}$ be a set of A-modules.*

(1) *If all the modules Y_i are projective with respect to M, then the module $\oplus_{i \in I} Y_i$ is M-projective.*
(2) *Assume that Y is a subfactor of the module $\oplus_{i \in I} Y_i$, X is a module which is projective with respect to all of the modules Y_i, and either I is a finite set or X is a finitely generated module. Then the module X is Y-projective.*
(3) *If the module $\oplus_{i \in I} Y_i$ is π-projective, then Y_i is Y_j-projective for any distinct subscripts $i, j \in I$.*
(4) *If A is a right Artinian ring, then M is a quasi-projective module $\Longleftrightarrow$ M is a projective $A/r(M)$-module.*

Proof. The proofs of (1), (2), and (3) follow from [1, 18.1, 18.2, 41.14].

The proof of (4) follows from [4, Theorem 2.3]. □

Lemma 1.3. *Let a module M be a direct sum of finitely generated modules M_i $(i \in I)$. The following conditions are equivalent:*

(1) *M is a quasi-projective module;*
(2) *M is a π-projective module, and all the modules M_i are quasi-projective;*
(3) *M_i is a M_j-projective module for all subscripts i, $j \in I$.*

Proof. The implication (1) $\Longrightarrow$ (2) follows from Lemma 1.1 (5) and the fact that direct summands of quasi-projective modules are quasi-projective.

The implication (2) $\Longrightarrow$ (3) follows from the quasi-projectivity of the modules M_i and the fact that by Lemma 1.2 (3), M_i is an M_j-projective module for any distinct subscripts i, $j \in I$.

The implication (3) $\Longrightarrow$ (1) follows from Lemmas 1.2 (1) and (2). □

Lemma 1.4.

(1) *If M is a direct sum of finitely generated quasi-projective modules, then M is a quasi-projective module $\Longleftrightarrow$ the module M is π-projective.*
(2) *If $X \oplus N$ is a π-projective module and Y is a subfactor of the module N, then the module X is projective with respect to the module Y.*
(3) *If A is a serial Artinian ring, then A is a ring of finite representation type, and every A-module is a direct sum of cyclic uniserial quasi-projective modules.*
(4) *If M is a module over a serial Artinian ring, then M is a π-projective module $\Longleftrightarrow$ M is a quasi-projective module $\Longleftrightarrow$ M is a projective $A/r(M)$-module.*

Proof. The proof of (1) follows from Lemma 1.3.
The proof of (2) follows from Lemmas 1.2 (3) and (2).
The proof of (3) follows from [1, 55.16, 53.6].
The proof of (4) follows from (1), (3), and Lemma 1.2 (4). □

Lemma 1.5. *Let A be a hereditary Noetherian prime ring.*

(1) *Every proper factor ring of A is a serial Artinian ring, and the ring A is either bounded or primitive. (If A is a bounded primitive ring, then A is a simple Artinian ring.)*
(2) *If M is a right A-module and $r(M) \neq 0$, then M is a π-projective module $\iff$ M is a quasi-projective module $\iff$ M is a projective $A/r(M)$-module.*

Proof. The proof of (1) follows from [5, Theorem 25.5.1] and [3].
The proof of (2) follows from (1) and Lemmas 1.1 (1) and 1.4 (4). □

Lemma 1.6. *Let A be a semiprime right Goldie ring.*

(1) *A is a right nonsingular ring, A has the semisimple Artinian classical right ring of quotients Q, Q_A is an injective hull of the module A_A, and the set of all of the essential right ideals of A coincides with the set of all of the right ideals of A containing regular elements. In addition, if the module Q_A is Noetherian, then $A = Q$, whence A is a semisimple Artinian ring.*
(2) *The class of all of the singular right A-modules coincides with the class of all of the torsion right A-modules, and the class of all of the nonsingular right A-modules coincides with the class of all of the right torsion-free A-modules.*
(3) *Every essential extension of a torsion right A-module is a torsion module.*
(4) *If M is a right A-module, then $\operatorname{Sing}(M)$ is a singular torsion module and $M/\operatorname{Sing}(M)$ is a nonsingular torsion-free module.*
(5) *If M is a right A-module and M is not torsion, then M contains a nonzero torsion-free submodule which is isomorphic to a right ideal of the ring A.*
(6) *Every uniform right A-module is either a torsion-free module or a torsion module.*

Proof. The proof of (1) follows from [6, 5.9, 5.48 (1), 5.31 (1)].
The proofs of (2), (3), and (4) follow from (1).
(5) By (1), there exists an element $m \in M \setminus \operatorname{Sing}(M)$. Then the right ideal $r(m)$ of A is not essential. Therefore, there exists a nonzero right ideal B of A with $B \cap r(m) = 0$. If $f\colon A_A \to mA$ is a natural epimorphism with the kernel $r(m)$, then $B \cap \operatorname{Ker}(f) = 0$, $B \cong f(B) \subseteq M$ and $f(B)$ is a nonzero torsion-free module.
(6) The proof follows from (5). □

Lemma 1.7. *Let A be a hereditary Noetherian prime ring which is not right primitive, and let M be a nonzero torsion right A-module.*

(1) *If M is a finitely generated module, then $r(M) \neq 0$ and M is a finite direct sum of cyclic uniserial modules of finite length.*
(2) *M has a nonzero uniserial countably generated direct summand.*
(3) *If the module M is not injective, then M has a nonzero cyclic uniserial direct summand of finite length.*
(4) *If M is an indecomposable module, then either M is an injective uniserial module or M is a cyclic uniserial module of finite length and $r(M) \neq 0$.*
(5) *If M is a finite-dimensional module, then M is a finite direct sum of uniserial modules.*
(6) *If M is a reduced finite-dimensional module, then $r(M) \neq 0$ and M is a finite direct sum of cyclic uniserial nonzero modules of finite length.*
(7) *If M is a reduced finite-dimensional module, then M is a π-projective module $\iff$ M is a quasi-projective module $\iff$ M is a projective $A/r(M)$-module.*

Proof. The proof of ((1) follows from [7, Lemma 2] and [8, Lemma 1].
The proofs of (2) and (3) follow from [8, Theorem 10, Lemma 1].
The proof of (4) follows from (2) and (3).

The proof of (5) follows from (4) and the fact that every finite-dimensional module is a finite direct sum of indecomposable modules.

The proof of (6) follows from (4), (5), and (1).

The proof of (7) follows from (6) and Lemma 1.5 (2). □

Lemma 1.8. *Let M be a right module over a ring A, and let X be a pure submodule in M.*

(1) *If Y is a pure submodule in X, then Y is a pure submodule in M.*
(2) *If X is a pure-injective module, then X is a direct summand of M.*
(3) *If B is a proper ideal of A and $h\colon M \to M/MB$ is a natural epimorphism, then $h(X)$ is a pure submodule of the A/B-module $h(M)$.*
(4) *$X \cap MB = XB$ for every left ideal B of A.*

Proof. The proofs of (1) and (2) are direct verifications.

(3) Let n and k be two positive integers, and let $\sum_{j=1}^{k} h(m_j)(a_{ij}+B) = h(x_i)$, where $m_j \in M$, $a_{ij} \in A$, and $x_i \in X$ $(i = 1, \dots, n)$. There exist elements $t_1, \dots, t_n \in MB$ such that

$$\sum_{j=1}^{k} m_j a_{ij} + t_i = x_i, \, i = 1, \dots, n.$$

It follows from Lemma 1.1 (7) that there exist elements $y_j \in X$ and $z_i \in XB$ for which we have

$$\sum_{j=1}^{k} y_j a_{ij} + z_i = x_i, \, i = 1, \dots, n.$$

Then $\sum_{j=1}^{k} h(y_j)a_{ij} = h(x_i)$ $(i = 1, \dots, n)$. By Lemma 1.1 (7), $h(X)$ is a pure submodule in $h(M)$.

The proof of (4) follows from [1, 34.5 and 34.9]. □

Lemma 1.9. *Let B be a proper nonzero ideal of a ring A, X be a pure submodule right A-module M, $XB = 0$, and let $h\colon M \to M/MB$ be a natural epimorphism.*

(1) *$X \cap MB = 0$.*
(2) *If $h(X)$ is a direct summand in $h(M)_{A/B}$, then X is a direct summand in M_A.*
(3) *If the ring A/B is right pure-semisimple, then X is a direct summand of the module M_A.*
(4) *If A/B is a ring of finite representation type, then X is a direct summand of the module M_A.*

Proof. (1) Since X is a pure submodule in M, we see that $X \cap MB = XB = 0$ by Lemma 1.8 (4).

(2) By (1) $X \cap MB = 0$. Let $h(M) = h(X) \oplus h(Y)$, where $MB \subseteq Y \subseteq M$. Then $M = X + Y$ and $X \cap Y = X \cap MB = 0$. Therefore, $M = X \oplus Y$.

(3) By Lemma 1.8 (2), the pure-injective A/B-module $h(X)$ is a direct summand of the A/B-module $h(M)$. By (1), X is a direct summand of the A-module M.

(4) The proof follows from (3) and the fact that every ring of finite representation type is a pure-semisimple ring [1, 54.3]. □

Lemma 1.10. *Let N be a module, X be an N-projective module, and let Y be a subfactor of the module N.*

(1) *If there exists an epimorphism $h\colon Y \to X$, then $\mathrm{Ker}(h)$ is a direct summand of the module Y, and X is a quasi-projective module which is isomorphic to a direct summand of the module Y.*
(2) *Let X be a t-generated module, where t is a cardinal number. If there exists a positive integer n such that the module Y^n contains a free submodule F of rank t, then the module X is projective.*
(3) *If Y is a finitely faithful module, then the module X is projective with respect to any finitely generated right A-module.*
(4) *If X is a finitely generated module and Y is a finitely faithful module, then the module X is projective.*
(5) *Assume that Y is an indecomposable module, $X \neq 0$, and there exists an epimorphism $f\colon Y \to X$. Then f is an isomorphism.*

Proof. (1) Since X is projective with respect to Y, we see that for the epimorphism f and the identity map 1_X, there exists a homomorphism $g\colon X \to Y$ with $1_X = fg$. Therefore, $Y = \overline{X} \oplus \operatorname{Ker}(f)$, where $\overline{X} \cong X$. In addition, X is projective with respect to $\overline{X}$. Therefore, the module X is quasi-projective.

(2) By Lemma 1.2 (2), the module X is F-projective. Since there exists an epimorphism $h\colon F \to X$, the module X is isomorphic to a direct summand of the free module F by (1).

(3) There exists a positive integer n such that Y^n contains a free cyclic submodule F. By (1), the module X is Y-projective. By Lemma 1.2 (2), the module X is F-projective. By Lemma 1.2 (2), the module X is projective with respect to any finitely generated free module. Since any finitely generated module S is a homomorphic image of a finitely generated free module, X is projective with respect to any finitely generated module by Lemma 1.2 (2).

(4) The proof follows from (2).

(5) The proof follows from (1). □

Lemma 1.11. *Let A be a prime right Goldie ring, and let Q be the injective hull of the module A_A.*

(1) *There exists a positive integer n such that for every nontorsion right A-module N, the module N^n contains a free cyclic submodule.*
(2) *There exists a positive integer n such that for every infinite-dimensional torsion-free module Y_A, the module Y^n contains a free submodule of infinite rank.*
(3) *If N is a nontorsion right A-module, then every finitely generated N-projective module X is projective.*
(4) *If Y is an infinite-dimensional torsion-free right A-module, then every countably generated Y-projective module X is projective.*
(5) *If N is a nontorsion right A-module and X is a nonzero finitely generated torsion module, then the module $X \oplus N$ is not π-projective.*
(6) *If a right A-module N contains an infinite-dimensional torsion-free submodule Y and X is a nonzero countably generated torsion module, then the module $X \oplus N$ is not π-projective.*
(7) *For any injective nonsingular indecomposable nonzero right A-module E, all injective nonsingular indecomposable nonzero right A-modules are isomorphic to the module E, and there exists a positive integer n with $Q \cong E^n$.*
(8) *If there exists a Noetherian injective nonsingular indecomposable nonzero right A-module E, then A is a simple Artinian ring.*

Proof. (1) By Lemma 1.6 (5), the module N contains a torsion-free submodule which is isomorphic to a nonzero right ideal B of A. By Lemma 1.6 (1), A is a right order in a semisimple Artinian ring Q. Let n be the length of the composition series of the module Q_Q. In the ring Q, every properly descending chain of right annihilators contains at most n inclusions. Therefore, every properly descending chain of right annihilators in A contains at most n inclusions. Therefore, there exist elements $b_1, \dots, b_n \in B$ such that $r(B) = r(b_1, \dots, b_n) = r(b_1) \cap \dots \cap r(b_n)$. Therefore, the module B^n contains a free cyclic submodule. Therefore, the module N^n contains a free cyclic submodule.

(2) By assumption, the module Y contains a submodule $\oplus_{i=1}^{\infty} N_i$, where all the N_i are torsion-free nonzero modules. By (1), there exists a positive integer n such that every module N_i^n contains a free cyclic submodule F_i. Then the module Y^n contains a free submodule $\oplus_{i=1}^{\infty} F_i$ of infinite rank.

(3) The proof follows from (1) and Lemma 1.10 (4).

(4) The proof follows from (2) and Lemma 1.10 (2).

(5), (6) Assume that the module $X \oplus N$ is π-projective. By Lemma 1.4 (2) the module X is projective with respect to any subfactor of the module N. By (3) and (4), the module X is projective. Since every submodule of a free module is a torsion-free module, the module X is torsion-free. Therefore, X is a nonzero torsion torsion-free module; this is a contradiction.

(7) The proof follows from Lemma 1.6 (1) and [2, Lemma 1.19 (1)].

(8) By (7), there exists a positive integer n such that $Q \cong E^n$. Therefore, the module Q is Noetherian. By Lemma 1.6 (1), A is an Artinian prime ring. Therefore, A is a simple ring. □

Lemma 1.12. *Let A be a semihereditary semiprime Goldie ring.*

(1) *Every torsion-free A-module is a flat module.*

(2) *If M is a right A-module, then $M/\operatorname{Sing}(M)$ is a flat torsion-free module, and every pure submodule X of the module $\operatorname{Sing}(M)$ is a pure torsion submodule of M.*

Proof. The proof of (1) follows from [9, p. 60].

(2) By Lemma 1.6 (3), X is a torsion module, and the module $M/\operatorname{Sing}(M)$ is torsion-free. By (1), the module $M/\operatorname{Sing}(M)$ is flat. Therefore, $\operatorname{Sing}(M)$ is a pure submodule in M [9, p.37]. By Lemma 1.8 (1), X is a pure submodule in M. □

Lemma 1.13. *Let M be a module over a hereditary Noetherian prime ring A, and let B be a proper nonzero ideal of A.*

(1) *If X is a pure submodule of M and $XB = 0$, then X is a direct summand of M.*
(2) *If X is a pure submodule of the module $\operatorname{Sing}(M)$ and $XB = 0$, then X is a direct summand of M.*
(3) *If X is a direct summand of the module $\operatorname{Sing}(M)$ and $XB = 0$, then X is a direct summand of M.*

Proof. (1) By Lemma 1.5 (1), A/B is a serial Artinian ring. By Lemma 1.4 (3), A/B is a ring of finite representation type. Now the proof of (1) follows from Lemma 1.9 (4).

(2) The proof follows from (1) and the fact that X is a pure submodule in M by Lemma 1.12 (2).

(3) The proof follows from (2) and the fact that every direct summand is a pure submodule. □

Lemma 1.14. *Let A be a hereditary Noetherian prime ring which is not right primitive, and let M be a mixed A-module.*

(1) *Either $M = T \oplus F$, where T is an injective torsion nonzero module and F is a torsion-free nonzero module, or $M = X \oplus N$, where X is a cyclic uniserial torsion nonzero module and N is a nontorsion module.*
(2) *M has a uniserial nonzero countably generated torsion direct summand which is either injective or cyclic.*
(3) *If M is a reduced module, then $M = X \oplus N$, where X is a cyclic uniserial torsion nonzero module, and N is a nontorsion module.*
(4) *If M is a π-projective module, then $M = T \oplus F = X \oplus N$, where T is a nonzero injective torsion module, F is a nonzero torsion-free module, X is a nonzero uniserial injective countably generated torsion module, N is a nontorsion module.*
(5) *If M is a reduced module, then the module M is not π-projective.*

Proof. (1) Set $T \equiv \operatorname{Sing}(M)$. Without loss of generality, it is sufficient to consider only the case in which T is a noninjective nonzero torsion module. By Lemmas 1.7 (3) and (4), T has a nonzero cyclic uniserial torsion nonzero direct summand X with $r(X) \neq 0$. By Lemma 1.13 (3), there exists a direct decomposition $M = X \oplus N$. Since the module M is mixed, N is a nontorsion module.

(2) and (3) The proofs follow from (1) and Lemma 1.7 (2).

(4) The proof follows from (1), (2), and Lemma 1.11 (5).

(5) The proof follows from (4). □

End of the proof of Theorem 1. We can assume that A is not a simple Artinian ring. By Lemma 1.5 (1), A is not right primitive. By Lemma 1.6 (2), the following three cases are the only possible cases: 1) the module M is nonsingular; 2) M is a torsion module; 3) the module M is mixed. In the case 1) Theorem 1 follows from Theorem A. In the case 2) Theorem 1 follows from Lemma 1.7 (7). The case 3) is impossible by Lemma 1.14 (5). □

2. Proof of Theorems 2 and 3

Lemma 2.1. *Let A be a bounded hereditary Noetherian prime ring which is not a simple Artinian ring, $\mathcal{P}(A)$ be the set of all of the maximal invertible ideals of A, X be a nonzero torsion A-module, and let $X(P) \equiv \{x \in X \mid xP^n = 0 \text{ for some } n\}$.*

(1) *The ring A is not right or left primitive.*

(2) *X is a direct sum of its primary components $X(P)$ ($P \in \mathcal{P}(A)$).*
(3) *If X is an indecomposable injective module, then $X = X(P)$ for some $P \in \mathcal{P}(A)$ and every injective indecomposable injective torsion module E, we see that either there exist epimorphisms $X \to E$ and $E \to X$ or $E = E(Q)$, where $Q \in \mathcal{P}(A)$ and $P \neq Q$.*
(4) *If X is a cyclic indecomposable module, then X is a uniserial module of finite length, $X = X(P)$ for some $P \in \mathcal{P}(A)$, and there exists a positive integer n such that $XP^n = 0$ and $XP^n \subset XP^{n-1} \subset \cdots \subset XP \subset X$ is a unique composition series for the module X.*
(5) *If X is a cyclic indecomposable module, then $r(X) \neq 0$ and X is a projective $A/r(X)$-module.*
(6) *Let Y and Z be two submodules of the module X, and let $Y \subseteq Z$. Then $Y(P) = Y \cap X(P)$ for every $P \in \mathcal{P}(A)$, there exists a natural isomorphism*

$$Z/Y \to \bigoplus_{P \in \mathcal{P}(A)} Z(P)/Y(P),$$

and we have

$$\operatorname{Hom}\Big(Z(P)/Y(P), \bigoplus_{P \in \mathcal{P}(A)} Z(P)/Y(P)\Big) = \operatorname{Hom}\big(Z(P)/Y(P), Z(P)/Y(P)\big)$$

for all $P \in \mathcal{P}(A)$. In particular, all $Z(P)/Y(P)$ are fully invariant submodules in the module $\bigoplus_{P \in \mathcal{P}(A)} Z(P)/Y(P)$.
(7) *X is a π-projective (resp. quasi-projective, skew-projective) module $\iff$ $X(P)$ is a π-projective (resp. quasi-projective, skew-projective) module for every $P \in \mathcal{P}(A)$.*

Proof. (1) The proof follows from Lemma 1.5 (1).
(2), (3), and (4) See the proofs in [11].
(5) By (4), $r(X) \neq 0$. By Lemma 1.5 (2), X is a projective $A/r(X)$-module.
(6) is verified with the use of (2).
(7) is verified with the use of (6). $\square$

Lemma 2.2. *Let A be a bounded hereditary Noetherian prime ring which is not a simple Artinian ring, $P \in \mathcal{P}(A)$, and let M be an injective indecomposable nonzero P-primary module.*

(1) *M is a uniserial noncyclic module, and all the proper submodules of M are cyclic, have finite length, and form a countable chain $0 = X_0 \subset X_1 \subset \cdots \subset X_k \subset \cdots$, where X_k/X_{k-1} is a simple module for every k and there exists a positive integer n such that $X_j/X_{j-1} \cong X_k/X_{k-1} \iff j-k$ is divided by n.*
(2) *If E is an injective torsion indecomposable nonzero A-module, then either there exist two epimorphisms $M \to E$ and $E \to M$ with nonzero kernels or $0 = \operatorname{Hom}(M', E) = \operatorname{Hom}(E', M)$ for any two submodules $M' \subseteq M$ and $E' \subseteq E$.*
(3) *Let E be an injective indecomposable nonzero P-primary A-module. Then there exist two epimorphisms $M \to E$ and $E \to M$ with nonzero kernels, and the module $M \oplus E$ is not π-projective.*
(4) *Let X be a cyclic nonzero submodule of M of length k, and let Z be an arbitrary cyclic uniserial P-primary module of finite length $d > k+n$, where the integer n is taken from (1). Then there exists an epimorphism $Y \to X$ with nonzero kernel, where Y is a submodule of the module Z. If N is a A-module such that the module Z is isomorphic to a subfactor of the module N, then the module $X \oplus N$ is not π-projective.*
(5) *Let X be a nonzero submodule of M, and let E be an injective indecomposable nonzero P-primary A-module. Then the module $X \oplus E$ is not π-projective.*
(6) *For every nonzero P-primary A-module N, the module $M \oplus N$ is not π-projective.*
(7) *M is a skew-projective module which does not have nonzero finitely generated factor modules.*

Proof. The proof of (1) follows from in [10].
The proof of (2) follows from (1) and Lemma 2.1 (3).

(3) The existence of epimorphisms $f\colon M \to E$ and $g\colon E \to M$ with nonzero kernels follows from (1) and (2). Assume that the module $M \oplus E$ is π-projective. By Lemmas 1.10 (5) and 1.4 (2) f is an isomorphism; this is a contradiction.

(4) Let E be the injective hull of the module Z. It follows from (2) that there exist two epimorphisms $M \to E$ and $E \to M$. It follows from (1) that there exists an epimorphism $f\colon Y \to X_k$ with nonzero kernel, where Y is a submodule of the module Z. Assume that $X_k \oplus N$ is a π-projective module. Since Y is isomorphic to a subfactor of the module N, f is an isomorphism by Lemmas 1.10 (5) and 1.4 (2). This contradicts the fact that $\operatorname{Ker}(f) \neq 0$.

(5) By (3), the module $M \oplus E$ is not π-projective. Therefore, we can assume that $X \neq M$. By (1), X is a cyclic module of length k. It follows from (1) that the module E contains a cyclic uniserial P-primary module Z of finite length $d > k + n$, where the integer n is taken from (1). By (4) there exists an epimorphism $f\colon Y \to X$ with nonzero kernel, where Y is a submodule of the module $Z \subset E$. Assume that the module $X \oplus E$ is π-projective. By Lemmas 1.10 (5) and 1.4 (2), f is an isomorphism. This is a contradiction.

(6) By Lemmas 1.7 (2) and 2.1 (1), the module N has a nonzero uniserial direct summand X. Assume that the module $M \oplus N$ is π-projective. Then the P-primary module $M \oplus X$ is π-projective. In addition, the injective hull of the module X is indecomposable. By (5), the module $M \oplus X$ is not π-projective. This is a contradiction.

(7) By (1), M is a uniserial noncyclic module, and all the proper submodules of M are cyclic and form a countable chain $0 = X_0 \subset X_1 \subset \dots \subset X_k \subset \cdots$, where X_k/X_{k-1} is a simple module for every k, and the module X_k has finite length k for every k. Then M does not have a nonzero finitely generated factor module. Let $\overline{f}$ be an endomorphism of an arbitrary nonzero factor module $M/X_s \equiv \overline{M}$ of the module M, and let $h\colon M \to \overline{M}$ be a natural epimorphism inducing natural epimorphisms $h \mid X_k \equiv h_k\colon X_k \to h(X_k) \equiv \overline{X}_k$. It can be verified directly that $\overline{f}(\overline{X}_k) \subseteq \overline{X}_k$ for all k. Therefore, $\overline{f}$ induces endomorphisms $\overline{f}_k$ of the modules $\overline{X}_k$. By Lemma 2.1 (5), all the X_k are projective $A/r(X_k)$-modules. Then all the X_k are skew-projective A-modules. Therefore, there exist endomorphisms f_k of modules X_k such that $h_k f_k = \overline{f}_k h_k$ for all k. We consider f_i and f_k with $i > k > s$. Then $X_k \subseteq X_i$, $\overline{X}_k \subseteq \overline{X}_i$, $(\overline{f}_i \mid \overline{X}_k - \overline{f}_k)(\overline{X}_k) = 0$. Therefore, $(f_i \mid X_k - f_k)(X_k) \subseteq X_s$ for $i > k$. Let $N_{ik} \equiv \operatorname{Ker}(f_i \mid X_k - f_k) \subseteq X_k$, and let d_{ik} be the length of the module N_{ik}. Since $(f_i \mid X_k - f_k)(X_k) \cong X_k/N_{ik}$ and the length of the module $(f_i \mid X_k - f_k)(X_k)$ does not exceed s, we see that $d_{ik} \geq k - s$. Therefore, $X_{k-s} \subseteq N_{ik}$ for $i > k > s$. For $k > s$, we set $Y_k \equiv X_{k-s}$. Then $f_i \mid Y_k = f_k \mid Y_k$ for $i > k$. In addition, $M = \bigcup_{k>s} Y_k$. This allows to define an endomorphism f of M such that $f \mid Y_k = f_k$. Then $f \mid X_k = f_{k+s}$. If $m \in X_{k+s}$, then $hf(m) = hf_{k+s}(m) = h_{k+s}f_{k+s}(m) = \overline{f}_{k+s}h_{k+s}(m) = \overline{f}_{k+s}h(m) = \overline{f}h(m)$. Therefore, $hf = \overline{f}h$ and the module M is skew-projective. □

Lemma 2.3. *Let A be a bounded hereditary Noetherian prime ring, $P \in \mathcal{P}(A)$, and let M be a nonzero P-primary A-module. Then the following conditions are equivalent*:

(1) *the module M is π-projective*;
(2) *the module M is skew-projective*;
(3) *M is either an indecomposable injective A-module or a projective $A/r(M)$-module*;
(4) *M is either an indecomposable A-module or a projective $A/r(M)$-module.*

Proof. Since every simple Artinian ring does not have nonzero torsion modules, we can assume that A is not a simple Artinian ring.

(1) $\Longrightarrow$ (4) Assume that the module M is π-projective module and is not indecomposable. By Lemmas 1.7 (2) and 2.1 (1), $M = X \oplus N$, where N is a nonzero module, X is a nonzero uniserial module, and X is either a cyclic module of finite length k or an injective module. By Lemma 2.2 (6), the module X is not injective. Therefore, X is a cyclic module of length k. By Lemma 2.1 (4), $XP^k = 0$. Let Z be a finitely generated submodule in N. By Lemmas 1.7 (1) and 2.1 (1), T is a finite direct sum of cyclic uniserial modules $Z_1, \dots, Z_n$. By Lemma 2.2 (4), there exists a positive integer d such that the lengths of all of the modules Z_i do not exceed d. By Lemma 2.1 (4), $Z_i P^d = 0$ for all i. Therefore, $ZP^d = 0$. Then $NP^d = 0$. Therefore, $MP^{k+d} = (X \oplus N)P^{k+d} = 0$ and $r(M) \neq 0$. By Lemma 1.5 (2), M is a projective $A/r(M)$-module.

(4) $\Longrightarrow$ (3) Without loss of generality, we can assume that M is an indecomposable A-module which is not a projective $A/r(M)$-module. By Lemma 2.1 (5), the module M is not cyclic. By Lemmas 1.7 (2) and 2.1 (1), the module M is injective.

(3) $\Longrightarrow$ (2) If M is an indecomposable injective module, then M is skew-projective by Lemma 2.2 (7). If M is a projective $A/r(M)$-module, then M is skew-projective by Lemmas 1.1 (1) and (2).

(2) $\Longrightarrow$ (1) The proof follows from Lemma 1.1 (4). $\square$

Theorem 2.1. *Let M be a torsion module over a bounded hereditary Noetherian prime ring A. Then the module M is π-projective $\iff$ M is skew-projective $\iff$ every primary component of the module M is either an indecomposable injective module or a projective module over the factor ring of A with respect to the annihilator of this primary component.*

Theorem 2.4 follows from Lemmas 2.1 (7) and 2.3.

Lemma 2.4. *Let T be an injective module, F be a hereditary module, and let $M \equiv T \oplus F$. Then for any submodule N of M, there exists a direct decomposition $M = T \oplus F_1$ such that $N = N \cap T \oplus N \cap F_1$.*

Proof. Let $h\colon N \to F$ be the homomorphism with the kernel $N \cap T$, induced by a natural projection $T \oplus F \to F$ with kernel T. Since $h(N)$ is a submodule of the hereditary module F, the module $h(N)$ is projective. Therefore, there exists a direct decomposition $N = N \cap T \oplus N_1$, where $N_1 \cap T = 0$. Let E be the injective hull of M. Since T is an injective submodule of the injective module E and $T \cap N_1 = 0$, there exists a direct decomposition $E = T \oplus E_1$ with $N_1 \subseteq E_1$. Since $T \subseteq M$, we see that $M = M \cap (T \oplus E_1) = T \oplus (M \cap E_1) = T \oplus F_1$, where $F \equiv M \cap E_1$ and $N_1 \subseteq M \cap E_1 = F$. $\square$

Lemma 2.5. *Let T be an injective module without nonzero Noetherian factor modules, F be a hereditary Noetherian module, and let $M \equiv T \oplus F$.*

(1) *If the module T is skew-projective, then M is a skew-projective module.*
(2) *If all idempotent endomorphisms of all of the factor modules of the module T can be lifted to endomorphisms of the module T, then all idempotent endomorphisms of all of the factor modules of M can be lifted to endomorphisms of M.*
(3) *If the module T is quasi-projective, then M is a quasi-projective module.*
(4) *If T is a uniserial module, then the module M is π-projective, and all idempotent endomorphisms of all of the factor modules of M can be lifted to endomorphisms of M.*

Proof. (1) Let $\overline{f}$ be an endomorphism of a factor module M/N of M, and let h be a natural epimorphism. By Lemma 2.4, there exists a direct decomposition $M = F_1 \oplus T$ with $N = N \cap F_1 \oplus N \cap T$. Then $M/N = h(F_1) \oplus h(T)$. Let $h_1 \equiv h \mid F_1$, $h_2 \equiv h \mid T$, $\overline{f}_1 \equiv \overline{f} \mid h(F_1)$, and let $\overline{f}_2 \equiv \overline{f} \mid h(T)$. Since $F_1 \cong M/T \cong F$, the module F_1 is Noetherian. We have the homomorphism $\overline{f}_1 h_1$ from the module F_1 into the module $h(M)$. Since the module F_1 is projective, there exists a homomorphism $f_1\colon F_1 \to M$ with $h_1 f_1 = \overline{f}_1 h_1$. Since the module $h(T)$ does not have a nonzero Noetherian homomorphic image and the module $h(F_1)$ is Noetherian, $\operatorname{Hom}\big(h(T), h(F_1)\big) = 0$. Therefore, $\overline{f}_2\big(h(T)\big) \subseteq h(T)$ and $\overline{f}_2$ is an endomorphism of the factor module $h(T)$ of the module T. Since the module T is skew-projective, there exists an endomorphism f_2 of the module T such that $h_2 f_2 = \overline{f}_2 h_2$. By the rule $f(x+y) = f_1(x) + f_2(y)$ ($x \in F_1$, $y \in T$), the endomorphism f of the module $M = F_1 \oplus T$ is defined. Then $hf = \overline{f}h$.

(2) and (3) The proof of (2) and (3) is analogous to the proof of (1).

(4) The proof follows from (2). $\square$

Lemma 2.6. *Let A be a right hereditary right Noetherian prime ring, Q be the injective hull of the module A_A, and let E be an injective nonsingular indecomposable nonzero right A-module.*

(1) *Every injective indecomposable right A-module is a homomorphic image of the module Q.*
(2) *For every injective torsion indecomposable right A-module X, there exists an epimorphism (with nonzero kernel) $f\colon E \to X$.*
(3) *For every injective torsion indecomposable right A-module X, the module $X \oplus E$ is not π-projective.*
(4) *Every injective π-projective right A-module M is either a torsion-free module or a torsion module.*

Proof. (1) The proof follows from [2, Lemma 1.14 (2)].

(2) We can assume that $X \neq 0$. By (1) and Lemma 1.11 (7), there exists an epimorphism $h\colon E^n \to X$ which induces a nonzero homomorphism $f\colon E \to X$. Since $f(E)$ is a homomorphic image of the injective right module E over the right hereditary ring A, the module $f(E)$ is injective [1, 39.16]. Therefore, $f(E)$ is a nonzero direct summand of the indecomposable module X. Then $f(E) = X$. By Lemma 1.6 (2), the module E is torsion-free. Therefore, $\operatorname{Ker}(f) \neq 0$.

(3) Assume that the module $X \oplus E$ is π-projective. By Lemma 1.4 (2), the module X is E-projective. By Lemma 1.10 (5), $\operatorname{Ker}(f) = 0$; this is a contradiction.

(4) Since A is a right Noetherian ring and M is an injective right A-module, $M = \oplus M_i$, where all the M_i are injective uniform modules (see [1, 27.5]). By Lemma 1.6 (6), every module M_i is either a torsion-free module or a torsion module. By (3), either all the M_i are torsion-free modules or all the M_i are torsion modules. □

Theorem 2.2. *Let M be a mixed module over a bounded hereditary Noetherian prime ring A. Then the following conditions are equivalent*:

1. *the module M is π-projective*;
2. *the module M is skew-projective*;
3. *$M = T \oplus F$, where F is a finitely generated projective nonzero module, T is an injective torsion nonzero module, and every primary component of the module T is an indecomposable module.*

Proof. The implication (2) $\Longrightarrow$ (1) follows from Lemma 1.1 (4).

(1) $\Longrightarrow$ (3) By Lemma 1.14 (4), $M = T \oplus F = X \oplus N$, where T is an injective torsion π-projective nonzero module, F is a π-projective torsion-free nonzero module, X is a nonzero uniserial injective countably generated torsion module, and N is a nontorsion module. By Lemma 2.2 (5), every primary component of the module T is an indecomposable module. We prove that F is a finitely generated projective nonzero module. By Theorem A, it is sufficient to prove that F is a reduced finite-dimensional module. By Lemma 1.11 (6), the module F is finite-dimensional. Assume that F has a nonzero injective direct summand E. Then $X \oplus E$ is a mixed injective π-projective module; this is a contradiction to Lemma 2.6 (5).

(3) $\Longrightarrow$ (2) By Theorem 2.1, the module T is skew-projective. It follows from Lemmas 2.2 (7) and 1.7 (2) that T does not have a nonzero Noetherian factor module. The projective right module F over the right hereditary ring A is a hereditary module [1, 39.16]. The finitely generated right module F over the right Noetherian ring A is a Noetherian module. By Lemma 2.5 (1), the module M is skew-projective. □

End of the proof of Theorem 2. The following two cases are the only possible cases: 1) M is a torsion module; 2) M is a mixed module. In the case 1), Theorem 2 follows from Theorem 2.1 and Lemma 2.5 (2). In the case 2), Theorem 2 follows from Theorem 2.2 and Lemma 2.5 (2). □

Lemma 2.7. *Let A be a serial hereditary Noetherian prime ring which is not a simple Artinian ring.*

1. *Every indecomposable nonsingular injective nonzero right A-module E is a uniserial countably generated module without nonzero Noetherian factor modules.*
2. *Every finite-dimensional nonsingular injective right A-module T does not have nonzero Noetherian factor modules.*

Proof. (1) Since E is an indecomposable injective module over the serial Noetherian ring A, the module E is uniserial [6, 11.55 (2)]. Since every semiprimitive serial ring is Artinian, the ring A is not right primitive. Let $0 \neq m \in E$. Then E/mA is a uniserial torsion module. By Lemma 1.7 (2), the module E/mA is countably generated. Therefore, the module E is countably generated. Assume that E has a nonzero Noetherian factor module E/N. Since the nonzero uniserial finitely generated module E/N is cyclic, there exists an element $x \in E \setminus N$ such that $E/N = (x+N)A$. Then $E = N + xA$. Since E is a uniserial module and $xA \not\subseteq N$, the module $E = xA$ is Noetherian. By Lemma 1.11 (8), A is a simple Artinian ring. This is a contradiction.

(2) Assume that there exists an epimorphism $h\colon T \to \overline{T}$ such that $\overline{T}$ is a nonzero Noetherian module. Since T is a finite direct sum of indecomposable injective modules, $T = T_1 \oplus \dots \oplus T_n$ by (1), where all

the modules T_i do not have a nonzero Noetherian factor module. Therefore, $h(T_i) = 0$ for all i. Then $\overline{T} = \sum_{i=1}^{n} h(T_i) = 0$; this is a contradiction. $\square$

Lemma 2.8. *Let A be a hereditary Noetherian prime ring. Then the following conditions are equivalent*:

(1) *A is a serial ring*;
(2) *there exists a π-projective injective nonzero right A-module which is not singular*;
(3) *for every indecomposable injective nonsingular module T and every finitely generated projective module F, the module $T \oplus F$ is π-projective.*

Proof. The equivalence of conditions (1) and (2) is proved in [2,Theorem 1].

The implication (3) $\Longrightarrow$ (2) is obvious.

(1) $\Longrightarrow$ (3) We can assume that A is not a simple Artinian ring. By Lemma 2.7 (1), T is a uniserial module without nonzero Noetherian factor modules. Since F is a finitely generated projective module over the hereditary Noetherian ring A, the module F is Noetherian and hereditary. By Lemma 2.5 (4), the module $T \oplus F$ is π-projective. $\square$

Lemma 2.9. *Let A be a hereditary Noetherian prime ring. Then the following conditions are equivalent*:

(1) *there exists a positive integer n such that the ring A is isomorphic to the ring of all of the $(n \times n)$ matrices over a complete uniserial Noetherian domain D*;
(2) *there exists a π-projective injective nonsingular finite-dimensional nonuniform right A-module*;
(3) *every injective nonsingular finite-dimensional right A-module is quasi-projective*;
(4) *for every injective nonsingular finite-dimensional module T and every finitely generated projective module F, the module $T \oplus F$ is quasi-projective*;
(5) *for every injective nonsingular finite-dimensional module T and every finitely generated projective module F, the module $T \oplus F$ is π-projective.*

Proof. The equivalence of conditions (1), (2) and (3) is proved in [2, Lemma 1.21].

The implication (4) $\Longrightarrow$ (5) follows from Lemma 1.1 (5).

The implication (5) $\Longrightarrow$ (2) is obvious.

(3) $\Longrightarrow$ (4) We can assume that A is not a simple Artinian ring. By Lemma 2.7 (2), the module T does not have a nonzero Noetherian factor module. Since F is a finitely generated projective module over the hereditary Noetherian ring A, the module F is Noetherian and hereditary. By Lemma 2.5 (3), the module $T \oplus F$ is quasi-projective. $\square$

End of the proof of Theorem 3. Without loss of generality, we can assume that M is a nonsingular finite-dimensional module. The following three cases are the only possible cases: 1) M is a reduced finite-dimensional nonsingular module; 2) $M = T \oplus F$, where T is a indecomposable injective finite-dimensional nonsingular nonzero module, and F is a reduced finite-dimensional nonsingular module; 3) $M = T \oplus F$, where T is a nonuniform nonsingular injective nonzero module, and F is a reduced finite-dimensional nonsingular module. In the case 1), Theorem 3 follows from Theorem A. In the case 2), Theorem 3 follows from Theorem A and Lemma 2.8. In the case 3), Theorem 3 follows from Theorem A and Lemma 2.9. $\square$

Corollary 2.1. *Let M be a module over a bounded hereditary Noetherian prime ring A. Then M is a π-projective nonreduced nonsingular module $\Longleftrightarrow$ one of the following conditions holds*:

(i) *A is a serial ring and $M = T \oplus F$, where T is an injective indecomposable nonsingular nonzero module, and F is a finitely generated projective module*;
(ii) *there exists a positive integer n such that the ring A is isomorphic to the ring of all of the $(n \times n)$ matrices over a complete uniserial Noetherian domain D, $M = T \oplus F$, where T is an injective finite-dimensional nonsingular nonzero module, and F is a finitely generated projective module.*

Proof. By Theorem 3, it is sufficient to prove that every π-projective nonreduced nonsingular module M over the serial ring A is finite-dimensional. Since M has a nonzero injective direct summand and every injective module over a Noetherian ring is a direct sum of indecomposable modules [5, 20.6], we obtain that $M = X \oplus Y$, where X is an indecomposable injective nonsingular nonzero module, and Y is an infinite-dimensional nonsingular submodule. By Lemma 2.7 (1), the module X is countably generated. By Lemma 1.11 (6), the module M is not π-projective. This is a contradiction. $\square$

Corollary 2.2. *Let A be a bounded hereditary prime ring such that all π-projective reduced nonsingular infinite-dimensional A-modules are projective. Then the module M is π-projective $\iff$ one of the following conditions holds*:

(i) *M is a projective module*;

(ii) *$M = T \oplus F$, where T is a torsion injective nonzero module such that every primary component of T is an indecomposable module, and F is a nonzero finitely generated projective module*;

(iii) *M is a torsion module such that every primary component of M is either an indecomposable injective module or a projective module over the factor ring of A with respect to the annihilator of this primary component*;

(iv) *A is a serial ring and $M = E \oplus F$, where E is an injective indecomposable torsion-free module, and F is a finitely generated projective module*;

(v) *there exists a positive integer n such that the ring A is isomorphic to the ring of all of the $(n \times n)$ matrices over a complete uniserial Noetherian domain D, $M = E \oplus F$, where E is an injective finite-dimensional torsion-free module, and F is a finitely generated projective module.*

Corollary 2.2 follows from Theorems A, 2, 3, and Corollary 2.1.

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