LEAVITT PATH ALGEBRAS: GRADED SIMPLE MODULES AND GRADED TYPE THEORY

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Abstract. In this paper, we give a complete characterization of Leavitt path algebras which are graded Σ-V rings, that is, rings over which direct sum of arbitrary copies of any graded simple module is graded injective. Specifically, we show that a Leavitt path algebra L over an arbitrary graph E is a graded Σ-V ring if and only if it is a subdirect product of matrix rings of arbitrary size but with finitely many non-zero entries over K or $K[x, x^{-1}]$ with appropriate matrix gradings. When the graph E is finite, we show that L is a graded Σ-V ring $\iff$ L is graded directly-finite $\iff$ L has bounded index of nilpotence $\iff$ L is graded semi-simple. Examples are constructed showing that the equivalence of these properties in the preceding statement no longer holds when the graph E is infinite. We then characterize the Leavitt path algebras of arbitrary graphs which have bounded index of nilpotence. We also provide an alternative proof of a theorem of Văs [31] describing Leavitt path algebras which are directly-finite. We conclude the paper by developing structure theory for graded regular graded self-injective rings and apply it in the context of Leavitt path algebras. We show that graded self-injective Leavitt path algebras are of graded type I and if the graphs are finite, these are precisely graded Σ-V Leavitt path algebras.

Leavitt path algebras are algebraic analogues of graph C^*-algebras and are also natural generalizations of Leavitt algebras of type $(1, n)$ constructed in [23]. The first objective of this paper is to characterize Σ-V Leavitt path algebras, namely, those over which direct sum of arbitrary copies of any graded simple module is graded injective.

The initial organized attempt to study the module theory over Leavitt path algebras was done in [6] where the simply presented modules over a Leavitt path algebra $L_K(E)$ of a finite graph E with coefficient in the field K, were described. As an important step in the study of the modules over $L_K(E)$ for an arbitrary graph E, the simple $L_K(E)$-modules and also Leavitt path algebras with simple modules of special types, have recently been investigated in a series of papers (see e.g. [5], [11], [28], [20]). In this paper we focus on the question when graded simple modules over Leavitt path algebras are not only graded injective but they are graded Σ-injective and we link this question to the classical ring theoretic properties such as bounded index of nilpotence and direct finiteness.

The study of V-rings and Σ-V rings is a continuation of a long tradition of characterizing rings in terms of certain properties of their cyclic or finitely generated modules (see e.g. [21]). The origin of this tradition can be drawn back to the celebrated characterization due to Osofsky [26, 27] of semisimple rings as those rings for which cyclic right modules have zero injective dimension. Thus obtaining for rings with zero global dimension a dual of Auslander's characterization of global dimension in terms of the projective dimension of cyclic modules. Damiano studied rings over which each proper cyclic right module is injective and showed that such rings are right noetherian [13]. Villamayor was first to study rings over which not necessarily all cyclic but all simple right modules are injective. A ring R is called a right V-ring if each simple right R-module is injective [24]. It is a well-known result due to Kaplansky that a commutative ring is von Neumann regular if and only if it is a V-ring. However, in the case of noncommutative setting, the classes

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of von Neumann regular rings and V-rings are quite independent. It is known due to Cartan, Eilenberg, and Bass that a ring R is right noetherian if and only if every direct sum of injective right R-modules is injective [9]. From this it follows that a ring R is right noetherian if and only if each injective right R-module is Σ-injective. So this observation lead to study of rings over which certain classes of modules are Σ-injective. Faith intitiated the study of rings over which for each cyclic module C, the injective envelope $E(C)$ is Σ-injective and conjectured that such rings are right noetherian [14]. This conjecture is still open. A ring R is called a right Σ-V ring if each simple right R-module is Σ-injective. These rings were introduced in [16] and studied subsequently in [8] and [29]. Baccella proved that a von Neumann regular ring with primitive factors artinian is a right Σ-V ring [8]. He also "proved" that a right Σ-V ring with all primtive factors artinian must be von Neumann regular but it has recently been found that his "proof" has a mistake. In [29] it is proved that a right Σ-V ring is directly-finite and a right non-singular right Σ-V ring has bounded index of nilpotence.

In this paper, we characterize, both graphically and algebraically, the Leavitt path algebra $L := L_K(E)$ of an arbitrary graph E which is a graded Σ-V ring. Specifically, we show that L is a graded Σ-V ring if and only if it is a subdirect product of matrix rings of arbitrary size but with finitely many non-zero entries over K or $K[x, x^{-1}]$ with appropriate matrix gradings. Furthermore, we show that if E is a finite graph, then L is a graded Σ-V ring if and only if L has bounded index of nilpotence if and only if L is graded directly-finite. Recall that a ring R is said to have *bounded index of nilpotence* if there is a positive integer n such that $x^n = 0$ for all nilpotent elements x in R. If n is the least such integer then R is said to have *index of nilpotence* n. We show that a Leavitt path algebra $L_K(E)$ of an arbitrary graph E has index of nilpotence less than or equal to n if and only if no cycle in E has an exit and there is a fixed positive integer n such that the number of distinct paths that end at any given vertex v (and do not include the cycle c in case v lies on c) is $\leq n$. In this case, L is a subdirect product of graded rings $\{A_i : i \in I\}$ where, for each i, $A_i \cong_{\text{gr}} \mathbb{M}_{t_i}(K)$ or $\mathbb{M}_{t_i}(K[x, x^{-1}])$ with appropriate matrix gradings where, for each i, $t_i \leq n$, a fixed positive integer. Văs [31] proved that the Leavitt path algebra over an arbitrary graph E is directly-finite if and only if no cycle in E has an exit. We give an alternative proof for this which is much shorter and establish that graded directly-finite Leavitt path algebras coincide with directly finite ones. We provide an example to show that the two notions are not equivalent in general. Our example is the Leavitt's algebras of ranks different than $(1, n)$, namely $\mathbb{M}_2(L(2, 3))$. In the last section of the paper we focus on when $L_K(E)$ is a graded injective $L_K(E)$-module.

Since Leavitt path algebras are graded von Neumann regular [18], the second objective of the paper is to study those Leavitt path algebras which are graded self-injective. This leads us to develop structure theory for graded regular graded self-injective rings in general and apply it in the context of Leavitt path algebras. We will see that from the three graded types, the self injective Leavitt path algebras are of graded type I. For finite graphs, these turn out to be precisely graded Σ-V Leavitt path algebras.

1. Preliminaries

For the general notation, terminology and results in Leavitt path algebras, we refer to [1], [28] and [30]. We will be using some of the needed results in associative rings, von Neumann regular rings and modules from [15] and [22]. We give below an outline of some of the needed basic concepts and results.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*.

A vertex v is called a *sink* if it emits no edges and a vertex v is called a *regular vertex* if it emits a non-empty finite set of edges. An *infinite emitter* is a vertex which emits infinitely many edges. For each $e \in E^1$, we call e^* a *ghost edge*. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. A *path* μ of length $n > 0$ is a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \cdots, n-1$. In this case $\mu^* = e_n^* \cdots e_2^* e_1^*$ is the corresponding *ghost path*. A vertex is considered a path of length 0.

A path $\mu = e_1 \dots e_n$ in E is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be *based at the vertex* $s(e_1)$. A closed path μ as above is called *simple* provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, ..., n$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

A graph E is said to satisfy *Condition (K)*, if any vertex v on a closed path c is also the base of a another closed path c' different from c.

An *exit* for a path $\mu = e_1 \dots e_n$ is an edge e such that $s(e) = s(e_i)$ for some i and $e \neq e_i$.

If there is a path from vertex u to a vertex v, we write $u \geq v$. A non-empty subset D of vertices is said to be *downward directed* if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and $v \geq w$. A subset H of E^0 is called *hereditary* if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set is *saturated* if, for any regular vertex v, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$. For any vertex $v \in E^0$, the set $T_E(v) = \{w \in E^0 : v \geq w\}$ is called the *tree* of vertex v and it is the smallest hereditary subset of E^0 containing v.

Given an arbitrary graph E and a field K, the *Leavitt path algebra* $L_K(E)$ is defined to be the K-algebra generated by a set $\{v : v \in E^0\}$ of pair-wise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$;

(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$;

(3) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$;

(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

$$v = \sum_{e \in E^1, s(e)=v} ee^*.$$

Every Leavitt path algebra $L_K(E)$ is a $\mathbb{Z}$-*graded algebra*, namely, $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_n$ induced by defining, for all $v \in E^0$ and $e \in E^1$, $\deg(v) = 0$, $\deg(e) = 1$, $\deg(e^*) = -1$. Here the L_n are abelian subgroups satisfying $L_m L_n \subseteq L_{m+n}$ for all $m, n \in \mathbb{Z}$. Further, for each $n \in \mathbb{Z}$, the *homogeneous component* L_n is given by

$$L_n = \Big\{ \sum k_i \alpha_i \beta_i^* \in L_K(E) : \ |\alpha_i| - |\beta_i| = n \Big\}.$$

Elements of L_n are called *homogeneous elements*. An ideal I of $L_K(E)$ is said to be a *graded ideal* if $I = \bigoplus_{n \in \mathbb{Z}} (I \cap L_n)$. If A, B are graded modules over a graded ring R, we write $A \cong_{\mathrm{gr}} B$ if A and B are graded isomorphic. We will also be using the usual grading of a matrix of finite order. For this and for the various properties of graded rings and graded modules, we refer to [17], [19] and [25].

A *breaking vertex* of a hereditary saturated subset H is an infinite emitter $w \in E^0 \backslash H$ with the property that $0 < |s^{-1}(w) \cap r^{-1}(E^0 \backslash H)| < \infty$. The set of all breaking vertices of H is denoted by B_H. For any $v \in B_H$, v^H denotes the element $v - \sum_{s(e)=v, r(e) \notin H} ee^*$. Given a hereditary saturated subset H and a subset $S \subseteq B_H$, (H, S) is called an *admissible pair.* Given an admissible pair (H, S), the ideal generated by $H \cup \{v^H : v \in S\}$ is denoted by $I(H, S)$. It was shown in [30] that the graded ideals of $L_K(E)$ are precisely the ideals of the form $I(H, S)$ for some admissible pair (H, S). Moreover, $L_K(E)/I(H, S) \cong L_K(E \backslash (H, S))$. Here $E \backslash (H, S)$ is a *quotient graph of* E, where $(E \backslash (H, S))^0 = (E^0 \backslash H) \cup \{v' : v \in B_H \backslash S\}$ and $(E \backslash (H, S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1 \text{ with } r(e) \in B_H \backslash S\}$ and r, s are extended to $(E \backslash (H, S))^0$ by setting $s(e') = s(e)$ and $r(e') = r(e)'$.

We will also be using the fact that the Jacobson radical (and in particular, the prime/Baer radical) of $L_K(E)$ is always zero (see [1]). It is known (see [28]) that if P is a prime ideal of L with $P \cap E^0 = H$, then $E^0 \backslash H$ is downward directed.

Let Λ be an infinite set and R a ring. Then $\mathbb{M}_\Lambda(R)$ denotes the ring of $\Lambda \times \Lambda$ matrices in which all except at most finitely many entries are non-zero.

We will denote by mod-R (R-mod), the category of all right (left) R-modules. In case of a ring R with local units, we consider the full subcategory Mod-R (R-Mod) of the category mod-R (R-mod) consisting of right (left) modules M which are *unital* in the sense that $MR = R$ ($RM = M$) and *non-degerate* in the sense that $mR = 0$ ($Rm = 0$) with $m \in M$, implies $m = 0$.

We shall be using some of the results from [2] and [30] where the graphs that the authors consider are assumed to be countable. We wish to point out that the results in these two papers hold for arbitrary graphs with no restriction on the number of vertices or the number of edges emitted by any vertex. In fact, these results without any restriction on the size of the graph are stated and proved in [1].

2. Grading of infinite matrices and Leavitt path algebras

We begin with some basics of graded rings and graded modules that we will need throughout this paper. Let Γ be an additive abelian group, R a Γ-graded ring and M and N graded right R-modules. Consider the hom-group $\mathrm{Hom}_R(M,N)$. One can show that if M is finitely generated or Γ is a finite group, then

$$\mathrm{Hom}_R(M,N) = \bigoplus_{\gamma\in\Gamma} \mathrm{Hom}(M,N)_\gamma,$$

where $\mathrm{Hom}(M,N)_\gamma = \{f : M \to N \mid f(\mathbb{M}_\alpha) \subseteq N_{\alpha+\gamma}, \alpha \in \Gamma\}$ (see [19, § 1.2.3]). However if M is not finitely generated, then throughout we will work with the group

$$\mathrm{HOM}_R(M,N) := \bigoplus_{\gamma\in\Gamma} \mathrm{Hom}(M,N)_\gamma,$$

and consequently the Γ-graded ring

$$\mathrm{END}_R(M) := \mathrm{HOM}_R(M,M).$$

If $f \in \mathrm{END}_R(M)$ then $f = \sum_{\gamma\in\Gamma} f_\gamma$, where $f_\gamma \in \mathrm{Hom}(M,M)_\gamma$ and we write $\mathrm{supp}(f) = \{\gamma \in \Gamma \mid f_\gamma \neq 0\}$.

Let us recall the grading of matrices of finite order and then we will indicate how to extend this to the case of infinite matrices in which at most finitely many entries are non-zero (see [17] , [20] and [25]).

Let R be a Γ-graded ring and $(\delta_1, \cdots, \delta_n)$ an n-tuple where $\delta_i \in \Gamma$. Then $\mathbb{M}_n(R)$ is a Γ-graded ring and, for each $\lambda \in \Gamma$, its λ-homogeneous component consists of $n \times n$ matrices

$$\mathbb{M}_n(R)(\delta_1,\cdots,\delta_n)_\lambda = \begin{pmatrix} R_{\lambda+\delta_1-\delta_1} & R_{\lambda+\delta_2-\delta_1} & \cdots & R_{\lambda+\delta_n-\delta_1} \\ R_{\lambda+\delta_1-\delta_2} & R_{\lambda+\delta_2-\delta_2} & & R_{\lambda+\delta_n-\delta_2} \\ & & & \\ & & & \\ R_{\lambda+\delta_1-\delta_n} & R_{\lambda+\delta_2-\delta_n} & & R_{\lambda+\delta_n-\delta_n} \end{pmatrix}. \tag{1}$$

This shows that for each homogeneous element $x \in R$,

$$\deg(e_{ij}(x)) = \deg(x) + \delta_i - \delta_j \tag{2}$$

where $e_{ij}(x)$ is a matrix with x in the ij-position and with every other entry 0.

Now let I be an arbitrary infinite index set and R be a Γ-graded ring. Denote by $\mathbb{M}_I(R)$ the matrix with entries indexed by $I \times I$ having all except finitely many entries non-zero and for each $(i,j) \in I \times I$, the ij-position is denoted by $e_{ij}(a)$, where $a \in R$. Considering a "vector" $\bar{\delta} := (\delta_i)_{i\in I}$, where $\delta_i \in \Gamma$ and following the usual grading on the matrix ring (see equations (1),(2)), define, for each homogeneous element a,

$$\deg(e_{ij}(a)) = \deg(a) + \delta_i - \delta_j. \tag{3}$$

This makes $\mathbb{M}_I(R)$ a Γ-graded ring, which we denote by $\mathbb{M}_I(A)(\bar{\delta})$. Clearly, if I is finite with $|I| = n$, then the usual graded matrix ring $\mathbb{M}_n(R)$ coincides (after a suitable permutation) with $\mathbb{M}_n(R)(\delta_1, \cdots, \delta_n)$.

Now, let us translate it in the context of Leavitt path algebras. Suppose E is a finite acyclic graph consisting of exactly one sink v. Let $\{p_i : 1 \leq i \leq n\}$ be the set of all paths ending at v. Then it was shown in ([1, Lemma 3.4])

$$L_K(E) \cong \mathbb{M}_n(K) \tag{4}$$

under the map $p_i p_j^* \longmapsto e_{ij}$. Now taking into account the grading of $\mathbb{M}_n(K)$, it was further shown in ([17, Theorem 4.14]) that the map (4) induces a graded isomorphism

$$\begin{aligned} L_K(E) &\longrightarrow \mathbb{M}_n(K)(|p_1|, \cdots, |p_n|), \\ p_i p_j^* &\longmapsto e_{ij}. \end{aligned} \tag{5}$$

In the case of a comet graph E (that is, a finite graph E in which every path eventually ends at a vertex on a cycle c without exits), it was shown in [1] that the map

$$\begin{aligned} L_K(E) &\longrightarrow \mathbb{M}_n(K[x, x^{-1}]), \\ p_i c^k p_j^* &\longmapsto e_{ij}(x^k) \end{aligned} \tag{6}$$

induces an isomorphism. Again taking into account the grading, it was shown in ([17, Theorem 4.20]) that the map (6) induces a graded isomorphism

$$\begin{aligned} L_K(E) &\longrightarrow \mathbb{M}_n(K[x^{|c|}, x^{-|c|}])(|p_1|, \cdots, |p_n|), \\ p_i c^k p_j^* &\longmapsto e_{ij}(x^{k|c|}). \end{aligned} \tag{7}$$

Later in [2, Proposition 3.6], the isomorphisms (4) and (6) were extended to infinite acyclic and infinite comet graphs respectively. The same isomorphisms with the grading adjustments will induce graded isomorphisms for Leavitt path algebras of such graphs. We now describe this extension below.

Let E be a graph such that no cycle in E has an exit and such that every infinite path contains a line point or is tail-equivalent to a rational path $ccc\cdots$ where c is a cycle (without exits). Define an equivalence relation in the set of all line points in E by setting $u \sim v$ if $T_E(u) \cap T_E(v) \neq \emptyset$. Let X be the set of representatives of distinct equivalence classes of line points in E, so that for any two line points $u, v \in X$ with $u \neq v$, $T_E(u) \cap T_E(v) = \emptyset$. For each vertex $v_i \in X$, let $\overline{p^{v_i}} := \{p_s^{v_i} : 1 \leq s \leq n_i\}$ be the set of all paths that end at v_i, where n_i could possibly be infinite. Denote by $|\overline{p^{v_i}}| = \{|p_s^{v_i}| : 1 \leq s \leq n_i\}$. Let Y be the set of all distinct cycles in E. As before, for each cycle $c_j \in Y$ based at a vertex w_j, let $\overline{q^{w_j}} := \{q_r^{w_j} : 1 \leq r \leq m_j\}$ be the set of all paths that end at w_j that do not include all the edges of c_j where m_j is could possibly be infinite. Let $|\overline{q^{w_j}}| := \{|q_r^{w_j}| : 1 \leq r \leq m_j\}$. Then the isomorphisms (5) and (7) extend to a $\mathbb{Z}$-graded isomorphism

$$L_K(E) \cong_{\text{gr}} \bigoplus_{v_i \in X} \mathbb{M}_{n_i}(K)(|\overline{p^{v_i}}|) \oplus \bigoplus_{w_j \in Y} \mathbb{M}_{m_j}(K[x^{|c_j|}, x^{-|c_j|}])(|\overline{q^{w_j}}|) \tag{8}$$

where the grading is as in (3).

3. Graded directly-finite Leavitt path algebras

A ring R with identity 1 is said to be *directly-finite* or *Dedekind finite* if for any two elements $x, y \in R$, $xy = 1$ implies $yx = 1$. A ring R with local units is said to be directly-finite if for every $a, b \in R$ and an idempotent $u \in R$ such that $ua = au = a$ and $ub = bu = b$, we have that $ab = u$ implies $ba = u$. Equivalently, for every local unit u, uR is not isomorphic to any proper direct summand. This is same as saying that the corner ring uRu is a directly-finite ring with identity.

A Γ-graded ring R with local units is called *graded directly-finite* if for any two homogeneous elements $x, y \in R^h$ with a homogeneous idempotent $u \in R^h$ such that $xu = x = ux$ and $yu = y = uy$, we have that $xy = u$ implies $yx = u$. Clearly here $u \in R_0$ and if $x \in R_\alpha$ then $y \in R_{-\alpha}$.

Clearly, a directly-finite ring is graded directly-finite but a graded directly-finite ring need not be directly-finite (see Example 3.4). However in this section we show that in the case of Leavitt path algebras these two concepts coincides (Theorem 3.7).

Let R be a Γ-graded ring and A a graded right R-module. Recall that the α-*suspension* of the module A is defined as $A(\alpha) = \bigoplus_{\gamma\in\Gamma} A(\alpha)_\gamma$, where $A(\alpha)_\gamma = A_{\alpha+\gamma}$.

Definition 3.1. Let R be a Γ-graded ring and A a graded right R-module. We say A is *graded directly-finite* if $A \cong_{\mathrm{gr}} A(\alpha) \oplus C$, for some $\alpha \in \Gamma$, then $C = 0$.

The following proposition showcases how the suspensions of a module would come into play when carrying the results from nongraded case to the graded setting. Example 3.3 will then show the delicate nature of graded setting and how things might differ when carrying concepts from nongraded to graded rings.

Proposition 3.2. *Let R be a Γ-graded ring and A a graded right R-module. Then A is a graded directly-finite R-module if and only if $\mathrm{END}_R(A)$ is a graded directly-finite ring.*

Proof. First suppose $\mathrm{END}_R(A)$ is not graded directly-finite. Thus there are $x \in \mathrm{Hom}_R(A,A)_\alpha$ and $y \in \mathrm{Hom}_R(A,A)_{-\alpha}$ such that $xy = 1_A$ but $yx \neq 1_A$. Note that yx is a homogeneous idempotent. Now $yA = \bigoplus_{\gamma\in\Gamma}(yA)_\gamma$, where $(yA)_\gamma = yA_\gamma$, is a graded submodule of $A(-\alpha)$. Since $xy = 1$, we have a natural graded isomorphism $A \to yA, a \mapsto ya$. On the other hand, we have

$$A = yA(\alpha) \bigoplus (1-yx)A$$
$$a = yxa + (1-yx)a$$

It follows that $A \cong_{\mathrm{gr}} A(\alpha) \bigoplus (1-yx)A$ so A is *graded directly-infinite.*

Conversely, if $\theta : A \cong_{\mathrm{gr}} A(\alpha) \oplus B$ for some $\alpha \in \Gamma$ and $B \neq 0$, then consider the induced graded projection $x = p\theta : A \to A(\alpha) \oplus B \to A(\alpha)$. So $x \in \mathrm{END}(A)_\alpha$. Now define $y = \theta^{-1} i : A(\alpha) \to A(\alpha) \oplus B \to A$ which is in $\mathrm{END}_R(A)_{-\alpha}$. It is easy to see that $xy = 1_A$ whereas $yx \neq 1_A$ in $\mathrm{END}_R(A)$. ☐

Example 3.3. The following example shows that we do need to take into account the suspensions of the module A in the Definition 3.1. We construct a $\mathbb{Z}$-graded R-module A such that $A \cong_{\mathrm{gr}} A(k) \oplus B$, where $k \neq 0$ and $B \neq 0$, $A(n) \ncong_{\mathrm{gr}} A$ for any $0 \neq n \in \mathbb{Z}$. Further we show that if $A \cong_{\mathrm{gr}} A \oplus C$ then $C = 0$. Indeed we show that there is a graded decomposition $R \cong_{\mathrm{gr}} R(-1) \oplus R(-1)$, as graded right R-module, however $R \ncong R(n)$ for any $0 \neq n \in \mathbb{Z}$. For this example, we consider Leavitt's algebra studied in [23]. Consider the free associative K-algebra R generated by symbols $\{x_i, y_i \mid 1 \leq i \leq 2\}$ subject to relations

$$x_i y_j = \delta_{ij}, \text{ for all } 1 \leq i,j \leq 2, \text{ and } \sum_{i=1}^{2} y_i x_i = 1, \tag{9}$$

where K is a field and δ_{ij} is the Kronecker delta. (This is the Leavitt path algebra associated to the graph with one vertex and two loops). The relations guarantee the right R-module homomorphism

$$\begin{aligned} \phi : R &\longrightarrow R^2 \\ a &\longmapsto (x_1 a, x_2 a) \end{aligned} \tag{10}$$

has an inverse

$$\begin{aligned} \psi : R^2 &\longrightarrow R \\ (a_1, a_2) &\mapsto y_1 a_1 + y_2 a_2, \end{aligned} \tag{11}$$

so $R \cong R \oplus R$ as a right R-module. Thus R is not directly-finite.

Assigning 1 to y_i and -1 to x_i, $1 \leq i \leq 2$, since the relations (9) are homogeneous (of degree zero), the algebra R is a $\mathbb{Z}$-graded algebra. The isomorphism (10) induces a graded isomorphism

$$\begin{aligned} \phi : R &\longrightarrow R(-1) \oplus R(-1) \\ a &\mapsto (x_1 a, x_2 a), \end{aligned} \tag{12}$$

where $R(-1)$ is the suspension of R by -1. However we observe that R is not graded isomorphic to any nontrivial suspension of itself. For this we need to use the graded Grothendieck group of R calculated in [19, §3.9.3]. We have

$$\begin{aligned} K_0^{\mathrm{gr}}(R) &\longrightarrow \mathbb{Z}[1/2] \\ [R(n)] &\longmapsto 2^n. \end{aligned}$$

This shows that R is graded directly-infinite module. Finally if $R \cong_{\mathrm{gr}} R \oplus C$ then C has to be zero as the monoid $V^{\mathrm{gr}}(R)$ is cancellative.

We will prove that for the case of Leavitt path algebras, graded directly-finite rings are indeed directly finite. However this is not always the case (in fact the counterexample can be chosen from Leavitt's algebras of rank different from $(1,n)$). We thank George Bergman for discussion on the construction of a graded directly-finite ring that is not directly-finite which led to the following example.

Example 3.4. Let R be the Leavitt algebra $L(2,3)$ which is generated by symbols $\{x_{i,j}, y_{j,i} : 1 \leq i \leq 3, 1 \leq j \leq 2\}$ subject to the relations $XY = I_3$ and $YX = I_2$, where

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{pmatrix}, \qquad X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}. \tag{13}$$

For an introduction to such rings see [10] for the universal construction approach and [12, §5] for the structure of $L(n,m)$ (which is denoted by $V_{n,m}$). The relations (13) imply that $R \oplus R \cong R \oplus R \oplus R$. However we show that R is not isomorphic to a proper direct summand of itself. The monoid of isomorphism classes of finitely generated projective R-modules is $\langle u : 2u = 3u \rangle$, where u represents the class associated to R (see [17, Theorem 5.21]). This monoid is isomorphic to the monoid described in the following table. One

	0	u	v
0	0	u	v
u	u	v	v
v	v	v	v

can easily see that if R is a proper direct summand of itself, then $u = u + x$ for some $x \neq 0$ in this monoid, which is not possible.

Since R is not a proper direct summand of itself but $R \oplus R$ is, by Proposition 3.2 (with Γ a trivial group), the matrix ring $\mathbb{M}_2(R)$ is not directly-finite, but R is directly-finite. Now consider the ring R as a $\mathbb{Z}$-graded ring concentrated in degree zero (this is not the canonical grading on Leavitt algebras). We have (see Section 2)

$$\mathbb{M}_n(R)(1,2) \cong \mathrm{End}_R\big(R(1) \oplus R(2)\big).$$

We show that the graded R-module $R(1) \oplus R(2)$ is graded directly finite (Definition 3.1). Let

$$R(1) \oplus R(2) \cong_{\mathrm{gr}} R(1+n) \oplus R(2+n) \oplus C, \tag{14}$$

for some $n \in \mathbb{Z}$. For any $j \in \mathbb{Z}$, $(R(1) \oplus R(2))_j = R_{1+j} \oplus R_{2+j}$ is either zero or is one copy of R. Since R does not have any non-trivial direct summand, from Equation 14 it follows that $C_j = 0$, for any $j \in \mathbb{Z}$, i.e., $C = 0$. Now from Proposition 3.2 (with $\Gamma = \mathbb{Z}$) it follows that the matrix ring $\mathbb{M}_2(R)(1,2)$ is graded directly-finite. So $\mathbb{M}_2(R)(1,2)$ is a graded directly-finite ring that is not directly-finite.

Lemma 3.5. *Let E be an arbitrary graph. If $L := L_K(E)$ is graded directly-finite, then no cycle in E has an exit.*

Proof. Assume L is graded directly-finite. We wish to show that no cycle in E has an exit. Suppose, on the contrary, there is a cycle c based at a vertex v and has an exit f at v. Now both c and c^* are homogeneous elements and, by (CK-1) condition, $c^*c = v$. Then, by hypothesis, $cc^* = v$. Left multiplying the last equation by f^*, we obtain $0 = f^*cc^* = f^*v = f^*$, a contradiction. Hence no cycle in E has an exit. □

Next we give an algebraic characterization of graded directly-finite Leavitt path algebras and, at the same time, give an alternative proof of a theorem of L. Vǎs. The main tool is the subalgebra construction given in [4]. This subalgebra construction has turned out to be a very useful tool in making the "local-to-global jump" while proving a ring theoretic property of finite character for a Leavitt path algebra L of an arbitrary graph E This approach has been demonstrated in proving a number theorems such as the following:

(1) L is von Neumann regular $\Longleftrightarrow$ E is acyclic ([4]);

(2) Every simple left/right L-module is graded $\Longleftrightarrow$ L is von Neumann regular ([20]);

(3) Every Leavitt path algebra L is a graded von Neumann regular ring ([18], [20]);

(4) Every Leavitt path algebra L is a right/left Bezout ring [3].

Proposition 2 in [4], as stated, does not include the additional properties implied by the subalgebra construction. Indeed, a careful inspection of the construction in [4] shows that the morphism θ in the construction is actually a graded morphism whose image is a graded submodule of L and it also reveals some properties of cycles.

We include these facts in the following stronger formulation of Proposition 2 of [4] which we shall be using.

Theorem 3.6. *Let E be an arbitrary graph. Then the Leavitt path algebra $L := L_K(E)$ is a directed union of graded subalgebras $B = A \oplus K\varepsilon_1 \oplus \cdots \oplus K\varepsilon_n$ where A is the image of a graded homomorphism θ from a Leavitt path algebra $L_K(F_B)$ to L with F_B a finite graph (depending on B), the elements ε_i are homogeneous mutually orthogonal idempotents and $\oplus$ denotes a ring direct sum. Moreover, any cycle c in the graph F_B gives rise to a cycle c' in E such that if c has an exit in F_B then c' has an exit in E.*

We now give an algebraic description of graded directly-finite Leavitt path algebras and an alternative and a shorter proof of a theorem of L.Vǎs [31].

Theorem 3.7. *Let E be an arbitrary graph. Then the following properties are equivalent for $L := L_K(E)$:*

(a) *L is graded directly-finite;*

(b) *No cycle in E has an exit;*

(c) *L is a directed union of graded semisimple Leavitt path algebras; specifically, L is a directed union of graded direct sums of matrices of finite order over K or $K[x, x^{-1}]$ with appropriate matrix gradings.*

(d) *L is directly-finite.*

Proof. We have already proved (a) $\Longrightarrow$ (b) in Lemma 3.5.

Assume (b). By Theorem 3.6, L is a directed union of graded subalgebras $B = A \oplus K\varepsilon_1 \oplus \cdots \oplus K\varepsilon_n$, where A is the image of a graded homomorphism θ from a Leavitt path algebra $L_K(F_B)$ to L with F_B a finite graph depending on B. Moreover, any cycle with an exit in F_B gives rise to a cycle with an exit in E. Since no cycle in E has an exit, no cycle in the finite graph F_B has an exit and so (see Section 2 and (8)) $L_K(F_B) \cong_{\mathrm{gr}} \bigoplus_{v_i \in X} \mathbb{M}_{n_i}(K)(|\overline{p^{v_i}}|) \oplus \bigoplus_{w_j \in Y} \mathbb{M}_{m_j}(K[x^{|c_j|}, x^{-|c_j|}])(|\overline{q^{w_j}}|)$, where n_i and m_j are positive integers and X, Y are finite sets. Since the matrix rings $\mathbb{M}_{n_i}(K)$ and $\mathbb{M}_{m_j}(K[x, x^{-1}])$ with any matrix grading are graded simple rings, A and hence B is a direct sum of finitely many matrix rings of finite order with appropriate matrix gradings over K and/or $K[x, x^{-1}]$. This proves (c).

Now (c) $\Longrightarrow$ (d) follows from the known fact that matrix rings $\mathbb{M}_{n_i}(K)$ and $\mathbb{M}_{m_j}(K[x, x^{-1}])$ are directly-finite and finite ring direct sums of such matrix rings are directly-finite and, by condition (c), any finite set of elements of L belongs to graded direct sum of finitely many matrix rings of finite order with appropriate matrix gradings over K or $K[x, x^{-1}]$. This shows L is directly-finite.

Obviously (d) implies (a) and this completes the proof. □

Remark 3.8. In passing, observe that an arbitrary von Neumann regular ring R need not be directly-finite, as is clear when R is the ring of linear transformations on an infinite dimensional vector space V over a field K. But if a Leavitt path algebra $L_K(E)$ of a graph E is von Neumann regular then it is always directly-finite due to the fact that E becomes acyclic ([4]) and so vacuously satisfies the condition that no cycle in E has an exit. This also indicates that the ring R of all linear transformations on an infinite dimensional vector space cannot be realized as the Leavitt path algebra of a directed graph.

4. Leavitt path algebras which are graded Σ-V rings

A ring R over which every simple left/right module is injective is called a left/right *V-ring*. Here V comes from the name of Orlando Vallamayor. If for every simple left/right R-module S, direct sums of arbitrary copies of S are injective, then R is called a left/right *Σ-V ring*. We need to define a graded version of this notion.

Definition 4.1. Let R be a Γ-graded ring. A graded left/right R-module M is said to be *graded Σ-injective* if every direct sum $\bigoplus_{i\in I} M(\alpha_i)$, where I is an arbitrary index set and $\alpha_i \in \Gamma$, is graded injective. The Γ-graded ring R is called a *graded left/right Σ-V ring* if every graded simple left/right R-module is graded Σ-injective.

In this section, we obtain a complete description of Leavitt path algebras of arbitrary graphs which are graded Σ-V rings. These turn out be the subdirect products of matrix rings of arbitrary size, with finitely many non-zero entries, over K or $K[x, x^{-1}]$ equipped with appropriate matrix gradings (Theorem 4.10 and Corollary 4.11).

We begin with the following useful observation.

Proposition 4.2. *Let E be an arbitrary graph. If $L := L_K(E)$ is a graded Σ-V ring, then L is graded directly-finite and consequently, no cycle in E has an exit.*

Proof. Suppose, on the contrary, L is not graded directly-finite. Then there exists homogeneous elements x, y and a local unit v such that $xy = v$ but $yx \neq v$ where $v = v^2$ satisfies $vx = x = xv$, $vy = y = yv$. Since yx is a homogeneous idempotent, we have a decomposition

$$vL = yxL \oplus (v - yx)L = yL \oplus (v - yx)L, \tag{15}$$

where $yxL = yL$ due to the fact that $yL = yvL = yxyL \subseteq yxL$. Now the map $va \longmapsto yva = ya$ gives rise to a graded isomorphism $vL \longrightarrow yL(|y|)$. It now follows from (15) that $vL \cong_{\text{gr}} A_1 \oplus_{\text{gr}} B_1$, where $A_1 = yL(|y|)$ and $B_1 = (v - yx)L$. As $A_1 \cong_{\text{gr}} vL$, we have a graded decomposition $A_1 = A_2 \oplus_{\text{gr}} B_2$ such that $A_2 \cong_{\text{gr}} A_1$ and $B_2 \cong_{\text{gr}} B_1$. Apply the same arguments to A_2 to get a graded decomposition $A_2 = A_3 \oplus_{gr} B_3$ where $A_3 \cong_{\text{gr}} A_2$ and $B_3 \cong_{\text{gr}} B_2$. Proceeding like this, we obtain infinitely many independent graded direct summands of vL, $B_1 \cong_{\text{gr}} B_2 \cong_{\text{gr}} \cdots \cong_{\text{gr}} B_n \cong_{\text{gr}} \cdots$. For each i, choose a maximal graded L-submodule M_i in B_i in such a way that $B_i/M_i \cong_{\text{gr}} B_j/M_j$ for all i, j. Now

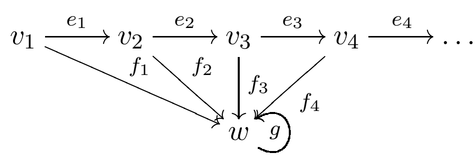
$$(\bigoplus_{n=1}^{\infty} B_n)/(\bigoplus_{n=1}^{\infty} M_n) \cong_{\text{gr}} \bigoplus_{n=1}^{\infty} (B_n/M_n)$$

is a graded direct sum of isomorphic graded simple L-modules, so it is graded injective, as L is a graded Σ-V ring. Consequently, $(\bigoplus_{n=1}^{\infty} B_n)/(\bigoplus_{n=1}^{\infty} M_n)$ is a graded direct summand of the graded cyclic module $vL/(\bigoplus_{n=1}^{\infty} M_n)$. This means that $\bigoplus_{n=1}^{\infty}(B_n/M_n)$ is a cyclic module, a contradiction. Hence L is graded directly-finite. Now Lemma 3.5 implies that no cycle in E has an exit. □

Before we proceed further, the following observations about prime and primitive ideals of graded rings may be worth noting:

Recall that a graded ideal P in a Γ-graded ring R is *graded prime* if for any two homogeneous elements $x, y \in R$, $xRy \subseteq P$ implies $x \in P$ or $y \in P$. A graded ideal P of R is said to be *graded left/right primitive* if P is the annihilator of a graded simple left/right R-module. If Γ is an ordered group, then it is known (see Proposition 1.4, Chapter II, [25]) that a graded ideal P of a Γ-graded ring R is graded prime if and only if P is prime. A similar statement for graded primitive ideals is no longer true. It may happen that a graded ideal of such a Γ-graded ring may be graded primitive, but not primitive as the following example shows:

Example 4.3. Consider the following graph E, where $E^0 = \{w, v_1, v_2, v_3, \cdots\}$ and $E^1 = \{e_1, e_2, e_3, \cdots; f_1, f_2, f_3, \cdots; g\}$. Further, for all $n \geq 1$, $s(e_n) = v_n = s(f_n)$; $r(e_n) = v_{n+1}$; $r(f_n) = w$ and $s(g) = w = r(g)$. Thus the graph looks like



(16)

If I is the ideal generated by w, then by [2, Proposition 3.7], $I \cong_{\text{gr}} \mathbb{M}_\infty(K[x, x^{-1}])$. Now E does not satisfy Condition (L), as it contains the loop g without exits and E^0 is downward directed. Hence, by [28, Theorem 4.3], $\{0\}$ is a prime but not a primitive ideal and $L_K(E)$ is not a primitive ring. But the graded ideal $\{0\}$ is a graded primitive ideal and $L_K(E)$ is a graded primitive ring. To see this, note that the graded ideal I is graded isomorphic to a direct sum of graded simple right ideals S of I, all of which is isomorphic to $K[x, x^{-1}]$ and such an S is also a graded simple right $L_K(E)$ ideal. Moreover, the annihilator ideal of S in $L_K(E)$ is $\{0\}$ due to the fact that I is the only non-zero proper (graded) ideal of $L_K(E)$ and $SI = S \neq 0$.

We first give a graded version of a definition given in [2].

Definition 4.4. A graded ring R is said to be categorically graded left/right artinian if every finitely generated graded left/right R-module is graded artinian.

Lemma 4.5. *A Leavitt path algebra $L := L_K(E)$ of an arbitrary graph E is categorically graded left (right) artinian if and only, for every vertex v, Lv (vL) is graded left (right) artinian.*

Proof. Suppose for every vertex v, Lv is a graded left artinian L-module. Since $L = \bigoplus_{v \in E^0} Lv$, every finitely generated graded left L-module M will be a graded homomorphic image of $\bigoplus_{i \in I} Lv_i(\alpha_i)$, where I is some finite set and $\alpha_i \in \Gamma$. Since Lv is graded left artinian, so is any shifting of Lv. This implies $\bigoplus_{i \in I} Lv_i(\alpha_i)$ is graded left artinian and so its homomorphic image M. Hence L is categorically graded left artinian. The proof for the right artinian case is similar. $\square$

We next state a graded version of a known property of right ideals in von Neumann regular rings.

Lemma 4.6. *Let R be a graded von Neumann regular ring. If A is a countably generated but not a finitely generated graded right ideal of R, then $A = \bigoplus_{n \geq 1} e_n A$ where the e_n are homogeneous idempotents.*

Proof. We can easily write A as the union of a strictly ascending chain of finitely generated graded right ideals $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$. Since R is graded von Neumann regular, each A_n is a principal ideal generated by a homogeneous idempotent, say f ([18]). We claim that $A_n = fR$ is a graded direct summand of R. Let $B = \{x - fx : x \in R\}$. We claim that B is a graded right ideal of R. Let $0 \neq a \in B$ and suppose $a = a_{n_1} + \cdots + a_{n_k}$ is a homogeneous decomposition a, where each a_{n_i} is a homogeneous element in R_{n_i}, $n_i \in \Gamma$. Then $0 = ea = ea_{n_1} + \cdots + ea_{n_k}$ is a homogeneous sum and so, by the the independence of the subgroups R_{n_i}, $ea_{n_i} = 0$ for all $i = 1, \cdots, k$. Clearly then $a_{n_i} = a_{n_i} - ea_{n_i} \in B$. Thus B is a graded ideal and clearly $fR \oplus B = R$ is a graded decomposition of R.

For each n, let $A_{n+1} = A_n \oplus C_{n+1}$ be a graded decomposition of A_{n+1}. It is then clear that $A = A_1 \oplus \bigoplus_{n\geq 1} C_{n+1}$. Now A_1 and, for $n \geq 1$, C_{n+1} are all finitely generated graded right ideals of R and so $A_1 = e_1 R$ and, for all $n \geq 1$, $C_{n+1} = e_{n+1}R$, where $e_1, e_2, \cdots$ are all homogeneous idempotents. We then conclude that $A = \bigoplus_{n\geq 1} e_n A$, as desired. ☐

Next, we proceed to prove the graded version of Lemma 6.17 from [15] for a ring with local units.

Lemma 4.7. (*Lemma 2.9* [7]) *Suppose R is a Γ-graded ring with local units. For any graded left R module M, the map $\sum_{i=1}^n r_i \otimes m_i \longmapsto \sum_{i=1}^n r_i m_i$, where the r_i and the m_i are homogeneous elements, induces a graded isomorphism $R \otimes_R M \longrightarrow M$.*

Proof. Since $RM = M$, the given map induces an epimorphism. Suppose $\sum_{i=1}^n r_i m_i = 0$. Let ϵ be a homogeneous local unit in R such that $\epsilon r_i = r_i = r_i \epsilon$ for all $i = 1, \cdots, n$. Then $\sum_{i=1}^n r_i \otimes m_i = \sum_{i=1}^n \epsilon r_i \otimes m_i = \sum_{i=1}^n \epsilon \otimes r_i m_i = \epsilon \otimes \sum_{i=1}^n r_i m_i = \epsilon \otimes 0 = 0$. Since tensor product commutes with direct sums (of homogeneous components), this induces a graded isomorphism. ☐

Lemma 4.8. *Suppose R is a Γ-graded ring with local units. Suppose M is a graded flat left R-module. Then, for any graded right ideal J of R, the map $\sum_{i=1}^n j_i \otimes m_i \longmapsto \sum_{i=1}^n j_i m_i$ where the j_i, m_i are homogeneous elements respectively in J and M, induces a graded isomorphism $J \otimes_R M \longrightarrow JM$.*

Proof. Note that JM is the subgroup of M generated by $\{\sum_{i=1}^n j_i m_i \in M : j_i \in J_{\alpha_i}, m_i \in M_{\beta_i}, \alpha_i\beta_i \in \Gamma\}$. Since ${}_R M$ is flat, we get an exact sequence

$$0 \longrightarrow J \otimes_R M \longrightarrow R \otimes_R M \longrightarrow M,$$

where the second map is the graded isomorphism given in Lemma 4.7 above and the first map is a graded monomorphism. Since the image of the composite map is JM, the stated map is a graded isomorphism $J \otimes_R M \longrightarrow JM$. ☐

Proposition 4.9. *Suppose $\phi : R \longrightarrow S$ is a graded epimorphism between two Γ-graded rings with local units and suppose S is graded flat as a graded left R-module. If A is a graded injective right S-module, then A is also graded injective as a right R-module.*

Proof. Let J be a graded right ideal of R and suppose $f : J \longrightarrow A$ is a graded morphism of right R-modules. Then f induces a graded S-module morphism $J \otimes_R S \longrightarrow A \otimes_R S \longrightarrow A$ where the second map is a graded isomorphism given in Lemma 4.7. Since S is flat as a left R-module, $J \otimes_R S \longrightarrow JS$ is a graded isomorphism by Lemma 4.8. Hence we get a graded morphism of right S-modules $g : JS \longrightarrow A$ such that $g(\phi(x)) = f(x)$ for all $x \in J$, noting that $JS = \phi(J)$. Since A is a graded injective right S-module, we get a graded S-morphism $h : S \longrightarrow A$ such that $h|JS = g$, so that $h(g(\phi(x)) = f(x)$ for all $x \in J$. Then $\theta = h\phi : R \longrightarrow A$ is a graded morphism such that, for all $x \in J$, $\theta(x) = h(\phi(x)) = f(x)$. Hence A is graded injective as a right R-module. ☐

We are now ready to prove our main theorem.

Theorem 4.10. *Let E be an arbitrary graph. Then the following properties are equivalent for $L := L_K(E)$:*

(a) *L is a graded right Σ-V ring;*

(b) *For every graded primitive ideal P of L, L/P is categorically graded artinian;*

(c) *For every graded primitive ideal P, L/P is graded isomorphic to $\mathbb{M}_\Lambda(K)(|\overline{p^{v_i}}|)$ or $\mathbb{M}_\Upsilon(K[x^n, x^{-n}])(|\overline{q^w}|)$, where Λ and Υ are arbitrary index sets and n is a suitable positive integer.*

Proof. Assume (a). Let P be a graded primitive ideal of L, say $P = I(H, S)$ for some admissible pair (H, S). Suppose, by way of contradiction, $\bar{L} = L/P \cong L_K(E\backslash(H, S))$ is not categorically graded right artinian. Then by Lemma 4.5 there is a vertex $v \in E\backslash(H, S)$ such that $v\bar{L}$ is not graded right artinian.

This means that $v\bar{L}$ contains at least one graded right ideal N which is not finitely generated, because if every graded right ideal inside $v\bar{L}$ is finitely generated, it will be a graded direct summand of $\bar{L}$ and hence of $v\bar{L}$, as $\bar{L} \cong L_K(E\backslash(H,S))$ is graded von Neumann regular. This means that $v\bar{L}$ is graded semisimple and since $v\bar{L}$ is cyclic, it will be graded right artinian, a contradiction. By Lemma 4.6, N contains a graded right ideal $A = \bigoplus_{n\geq 1} e_n\bar{L}$ where the e_n are homogeneous idempotents. Let S be a graded right simple L-module annihilated by P, so S is a faithful graded simple $\bar{L}$-module. Consider the module $M = \bigoplus_{n\geq 1} S_n$ where $S_n \cong_{\text{gr}} S$ for all n. Now, for each n, the faithful module S_n contains an element x_n such that $x_n e_n \neq 0$. Beacuse, otherwise $S_n e_n = 0$, contradicting the fact that S_n is a faithful module. For each $n \geq 1$, define a graded homomorphism $f_n : e_n\bar{L} \to S_n$ mapping $e_n \mapsto x_n e_n$ which gives rise to a graded homomorphism

$$f = \oplus f_n : \bigoplus_{n\geq 1} e_n\bar{L} \longrightarrow \bigoplus_{n\geq 1} S_n = M.$$

By hypothesis, M is injective and so f extends to a graded homomorphism $g : v\bar{L} \longrightarrow M$. If $g(v) = a$, then, on the one hand $g(e_n) = g(ve_n) = g(v)e_n = ae_n$. On the other hand, $g(e_n) = f(e_n) = x_n e_n \neq 0$. Thus $ae_n = x_n e_n \in S_n$ and non-zero for all $n \geq 1$. This is a contradiction, since $a \in S_1 \oplus \cdots \oplus S_k$ for some $k \geq 1$ and this implies that, for any $n > k$, $ae_n \in (S_1 \oplus \cdots \oplus S_k) \cap S_n = 0$. Thus $\bar{L}$ is categorically graded artinian and a primitive ring. This proves (b).

Assume (b). The proof (b) $\Longrightarrow$ (c) is essentially the graded version of the ideas obtained by combining parts of the proofs of Theorem 2.4 and Theorem 3.7 of [2]. For clarity, we outline the initial steps of the proof. Let $P = I(H,S)$ be a graded primitive ideal of L where $P \cap E^0 = H$. Since $\bar{L} = L/P \cong L_K(E\backslash(H,S))$ is categorically graded artinian, $E\backslash(H,S)$ must be row-finite; because if v is an infinite emitter in $E\backslash(H,S)$ with $\{f_n : n = 1,2,3,\cdots\} \subseteq s^{-1}(v)$, then $v\bar{L}$ contains an infinite descending chain of graded right ideals

$$\bigoplus_{n\geq 1} f_n f_n^* \bar{L} \supsetneqq \bigoplus_{n\geq 2} f_n f_n^* \bar{L} \supsetneqq \bigoplus_{n\geq 3} f_n f_n^* \bar{L} \supsetneqq \cdots,$$

a contradiction. Also no path p in $E\backslash(H,S)$ contains infinitely many bifurcations; because, otherwise, we write p as a concatenation of paths $p = p_1p_2p_3\cdots$ where, for each n, there is a bifurcating edge e_n with $s(e_n) = r(p_n)$. If $v = s(p)$, then one can verify that $v\bar{L}$ contains an infinite descending chain of graded right ideals

$$p_1p_1^*\bar{L} \supsetneqq p_1p_2p_2^*p_1^*\bar{L} \supsetneqq p_1p_2p_3p_3^*p_2^*p_1^*\bar{L} \supsetneqq \cdots$$

contradicting that $v\bar{L}$ is graded right artinian. Likewise, if there is a cycle c with an exit f at a vertex v in $\bar{L}$, then again $v\bar{L}$ will contain the infinite descending chain of graded right ideals

$$cc^*L \supsetneqq c^2(c^*)^2L \supsetneqq c^3(c^*)^3L \supsetneqq \cdots$$

which is a contradiction. Thus no cycle in $E\backslash(H,S)$ has an exit. This means that every path in $E\backslash(H,S)$ eventually ends at a line point or at a cycle without exits. Also $E\backslash(H,S)$ is row-finite. Moreover, since $E^0\backslash H$ is downward directed, either $E\backslash(H,S)$ contains a unique sink w and no cycles or $E\backslash(H,S)$ contains a unique cycle c without exits. Downward directness of $E^0\backslash H$ also implies, either all the paths in $E\backslash(H,S)$ end at the unique sink w or all the paths in $E\backslash(H,S)$ end at the unique cycle c (without exits). Then by Theorem 3.7 of [2],

$$L_K(H,S) \cong \mathbb{M}_\Lambda(K) \text{ or } L_K(E\backslash(H,S)) \cong \mathbb{M}_\Upsilon(K[x^{|c|}, x^{-|c|}])$$

which is a graded isomorphism with the matrix gradings as given in Section 2, where Λ is the set of all paths in $E\backslash(H,S)$ that end at the unique sink w and Υ is the set of all paths that end at c but do not contain the entire cycle c. This proves (c).

Assume (c). Let S be a graded simple right L-module and let P be the right annihilator of S in L. Then P is a graded primitive ideal and so, by hypothesis, either $L/P \cong_{\text{gr}} \mathbb{M}_\Lambda(K)(|\overline{p^{v_i}}|)$ or $\mathbb{M}_\Upsilon(K[x^n, x^{-n}])(|\overline{q^w}|)$.

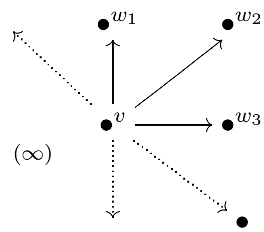
Consequently, L/P is graded semi-simple [20]. In particular, as a right L/P-module, S is graded $\sum$-injective. Since L is graded von Neumann regular [18, 20], L/P is graded flat as a graded right L-module. Then by Proposition 4.9, S is Σ-injective as a graded right L-module. This proves (a). □

Now the intersection of all graded primitive ideals of $L_K(E)$, being the graded Jacobson radical of $L_K(E)$, is zero (see [18]) and so we get the following corollary.

Corollary 4.11. *If for arbitrary graph E, $L := L_K(E)$ is a graded Σ-V ring, then L is a graded subdirect product of matrix rings of arbitrary size but with finitely many non-zero entries over K or $K[x, x^{-1}]$ equipped with appropriate matrix gradings.*

If $L_K(E)$ is a graded Σ-V ring, $L_K(E)$ need not decompose as a graded direct sum of matrix rings, as the following example shows.

Example 4.12. Consider the following "infinite clock" graph E:



(17)

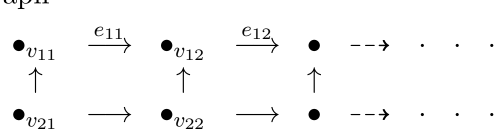
Thus $E^0 = \{v\} \cup \{w_1, w_2, \cdots\}$, where the w_i are all sinks. For each $n \geq 1$, let e_n denote the single edge connecting v to w_n. The graph E is acyclic and so every ideal of L is graded. The number of distinct paths ending at any given sink (including the sink) is ≤ 2. For each $n \geq 1$, $H_n = \{w_i : i \neq n\}$ is a hereditary saturated set, $B_{H_n} = \{v\}$ and $E^0 \backslash H_n = \{v, w_n\}$ is downward directed. Hence the ideal P_n generated by $H_n \cup \{v - e_n e_n^*\}$ is a graded primitive ideal and $L_K(E)/P_n \cong \mathbb{M}_2(K)$. Moreover, every graded (prime =) primitive ideal P of $L_K(E)$ is equal to P_n for some n. By Theorem 4.10, $L_K(E)$ is a graded $\sum$-V-ring.

But $L_K(E)$ cannot decompose as a direct sum of the matrix rings $\mathbb{M}_2(K)$. Because, otherwise, v would lie in a direct sum of finitely many copies of $\mathbb{M}_2(K)$. Since the ideal generated by v is $L_K(E)$, $L_K(E)$ will then be a direct sum of finitely many copies of $\mathbb{M}_2(K)$. This is impossible since $L_K(E)$ contains an infinite set of orthogonal idempotents $\{e_n e_n^* : n \geq 1\}$.

We can also describe the internal structure of this ring $L_K(E)$. The socle S of $L_K(E)$ is the ideal generated by the sinks $\{w_i : i \geq 1\}$, $S \cong \bigoplus_{\aleph_0} \mathbb{M}_2(K)$ and $L_K(E)/S \cong K$.

By Proposition 4.2, if a Leavitt path algebra L is a graded Σ-V-ring, then L is (graded) directly-finite. We construct examples showing that the converse is not true.

Example 4.13. Let E be the graph



If $S = \langle v_{11} \rangle$ is the ideal generated by v_{11}, then S is the graded socle and $S \neq L_K(E)$ as $v_{2n} \notin S$ for all $n \geq 1$. Being graded semi-simple, S is a graded Σ-V ring and $L_K(E)/S$ is also graded semi-simple and hence is a graded Σ-V ring. But $L_K(E)$ is not a graded Σ-V ring. To see this, first note that, since E is acyclic, by Theorem 2.10 in [20], the graded socle $S \cong_{\text{gr}} \mathbb{M}_\Lambda(K)$. So S is a graded direct sum of isomorphic graded simple modules. If $L_K(E)$ is a Σ-V ring, S is graded injective and hence a graded direct summand of $L_K(E)$. Then $L_K(E) \cong_{\text{gr}} S \oplus (L_K(E)/S)$ is a graded direct sum of graded simple modules and so $L_K(E) = \text{soc}^{\text{gr}}(L_K(E)) = S$, a contradiction. Hence $L_K(E)$ is not a graded Σ-V ring. On the other hand, by Theorem 3.7, $L_K(E)$ is (graded) directly-finite, since the graph E is acyclic and so E vacuously satisfies the condition that no cycle in E has an exit.

Example 4.14. By by Theorem 3.7, the example 4.3 is a graded directly-finite Leavitt path algebra L as the single loop does not have an exit. But L is not a graded Σ-V ring since L is a graded primitive ring, but is not categorically graded right artinian as v_1L contains an infinite descending chain of graded right ideals

$$\bigoplus_{n\geq 1} e_ne_n^*\bar{L} \supset \bigoplus_{n\geq 2} e_ne_n^*\bar{L} \supset \bigoplus_{n\geq 3} e_ne_n^*\bar{L} \supset \cdots\cdots$$

thus in contradiction to Theorem 4.10.

The next result shows that, for finite graphs E, we get a stronger conclusion about $L_K(E)$ which are Σ-V rings.

Theorem 4.15. *Let E be a finite graph. Then the following are equivalent for the Leavitt path algebra $L := L_K(E)$:*

(a) *L is a graded Σ-V ring;*

(b) *No cycle in E has an exit;*

(c) *L is (graded) directly-finite;*

(d) *$L \cong_{\mathrm{gr}} \bigoplus_{v_i\in X} \mathbb{M}_{n_i}(K)(|\overline{p^{v_i}}|) \oplus \bigoplus_{w_j\in Y} \mathbb{M}_{m_j}(K[x^{|c_j|}, x^{-|c_j|}])(|\overline{q^{w_j}}|)$, where n_i and m_j are positive integers, X, Y are finite sets being the set of sinks and the set of all cycles in E respectively.*

(e) *L is graded left/right semi-simple, that is, L is a graded direct sum of (finitely many) graded simple left/right L-modules;*

(f) *L has bounded index of nilpotence;*

(g) *L is a graded self-injective ring.*

Proof. Now, by Proposition 4.2 and Theorem 3.7, (a) $\Longrightarrow$ (b) and (b) $\Longleftrightarrow$ (c). The equivalence (d) $\Longleftrightarrow$ (g) was shown in [20].

Assume (b). Since E is a finite graph, every path in E eventually ends at a sink or is tail equivalent to a rational path $ccc\cdots$ for some cycle c without exits. Then, by [1, Theorem 3.7],

$$L \cong \bigoplus_{v_i\in X} \mathbb{M}_{n_i}(K) \oplus \bigoplus_{c_j\in Y} \mathbb{M}_{m_j}(K[x, x^{-1}]), \tag{18}$$

where X and Y are finite sets denoting respectively the set of sinks and the set of distinct cycles in E and for each i, n_i, denotes the number of paths ending at the sink v_i and for each j, m_j denotes the number of paths tail equivalent to the rational path $c_jc_jc_j\cdots$. As established in Section 2, the matrices in (18) can be given appropriate matrix gradings giving rise a graded isomorphism

$$L \cong_{\mathrm{gr}} \bigoplus_{v_i\in X} \mathbb{M}_{n_i}(K)(|\overline{p^{v_i}}|) \oplus \bigoplus_{w_j\in Y} \mathbb{M}_{m_j}(K[x^{|c_j|}, x^{-|c_j|}])(|\overline{q^{w_j}}|). \tag{19}$$

This proves (d).

Assume (d). The matrices $\mathbb{M}_{n_i}(K)(|\overline{p^{v_i}}|)$ and $\mathbb{M}_{n_j}(K[x^{|c_j|}, x^{-|c_j|}])(|\overline{q^{w_j}}|)$ are both direct sums of graded simple left/right modules (see [20]). Consequently, L is graded left/right semi-simple, thus proving (e).

Assume (e). Since L is graded left/right semi-simple, every graded left/right L-module is injective. In particular, L is a graded $\sum$-V ring, thus proving (a).

Assume (d). Now a matrix ring of finite order n over K or $K[x, x^{-1}]$ has bounded index of nilpotence n (see [15]). Since the index sets Λ and Υ in (19) are finite, we then conclude that L has bounded index of nilpotence, thus proving (f).

Assume (f), so L has bounded index of nilpotence , say n. Suppose, by way of contradiction, that E contains a cycle c with an exit f at a vertex v. Consider the set $\{\varepsilon_{ij} = c^iff^*(c^*)^j : 1 \leq i, j \leq n+1\}$.

Clearly the ε_{ij} form a set of matrix units as $(\varepsilon_{ii})^2 = \varepsilon_{ii}$ and $\varepsilon_{ij}\varepsilon_{kl} = \varepsilon_{il}$ or 0 according as $j = k$ or not. Then it is easy to see that the set $S = \{\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} k_{ij}\varepsilon_{ij} : k_{ij} \in K\}$ is a subring of L isomorphic to the matrix ring $\mathbb{M}_{n+1}(K)$. Since S has bounded index of nilpotence $n+1$, this is a contradiction. Hence no cycle in E has an exit, thus proving (b).

This finishes equivalence of all the statements. □

The equivalences among the various properties in Theorem 4.15 no longer seem to hold if the graph E is arbitrary. As we shall see in the next section, Condition (f) of Theorem 4.15 (the bounded index of nilpotence) is, in general, a stronger condition in the sense that it implies Conditions (a), (b), (c) of Theorem 4.15, but the converse does not hold.

5. Leavitt path algebras with bounded index of nilpotence

In this section, we characterize the Leavitt path algebras of arbitrary graphs which have bounded index of nilpotence. Recall that a ring R is said to have *bounded index of nilpotence* if there is a positive integer n such that $x^n = 0$ for all nilpotent elements x in R. If n is the least such integer then R is said to have *index of nilpotence* n.

We begin with the following useful proposition which extends the idea in the proof of (f)⇒(b) in Theorem 4.15 and part of which is implicit in some of the earlier papers on Leavitt path algebras.

Proposition 5.1. *Let E be an arbitrary graph and let $L := L_K(E)$.*

(a) *If there is a cycle with an exit in E, then, for every integer $n \geq 1$, there is a subring S_n of L isomorphic to the matrix ring $\mathbb{M}_n(K)$.*

(b) *Suppose v is a vertex in E which either (i) is the base of a cycle c without exits or (ii) does not lie on a closed path (in particular, v is a sink). If, for some $n \geq 1$, v is the range of n distinct paths $p_1, \cdots, p_n$ each of which, in case (i), does not include the entire cycle c, then the set*

$$T_n = \Big\{\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}p_ip_j^* : k_{ij} \in K\Big\}$$

is a subring of L isomorphic to the matrix ring $\mathbb{M}_n(K)$.

Proof. (a) The statement (a) follows from the proof of (f) $\Longrightarrow$ (b) in Theorem 4.15.

(b) First observe that $p_j^*p_k \neq 0$ if and only if $p_j = p_k$. Because, if $p_j^*p_k \neq 0$, then either $p_j = p_kp'$ or $p_k = p_jq'$ for some paths p', q'. Since $r(p_j) = r(p_k) = v$, $s(p') = v = r(p')$ and $s(q') = v = r(q')$. In case (i), v is the base of a cycle c without exits and so both p', q' must be of the form c^m for some $m \geq 0$. Since both the paths p_j and p_k do not include the entire cycle c, so $p' = v = q'$. In case (ii), v does not lie on a closed path, and so we again conclude that $p' = v = q'$. Thus in both cases $p_j = p_k$. Conversely, if $p_j = p_k$, then clearly $p_j^*p_k = p_j^*p_j = v \neq 0$.

For all i, j, let $\varepsilon_{ij} = p_ip_j^*$. Clearly, $(\varepsilon_{ii})^2 = \varepsilon_{ii}$ and $\varepsilon_{ij}\varepsilon_{kl} = \varepsilon_{il}$ or 0 according as $j = k$ or not. Thus the ε_{ij} form a set of matrix units and it is readily seen that $T_n = \{\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}p_ip_j^* : k_{ij} \in K\}$ is a subring of L isomorphic to the matrix ring $\mathbb{M}_n(K)$. □

We are now ready to describe all the Leavitt path algebras with bounded index of nilpotence.

Theorem 5.2. *Let E be an arbitrary graph. Then the following properties are equivalent for $L := L_K(E)$:*

(a) *L has bounded index of nilpotence $\leq n$;*

(b) *No cycle in E has an exit and there is a fixed positive integer n such that the number of distinct paths that end at any given vertex v (and do not include the cycle c in case v lies on c) is $\leq n$;*

(c) *For any graded prime ideal P of L, $L/P \cong_{\rm gr} \mathbb{M}_t(K)$ or $\mathbb{M}_t(K[x,x^{-1}])$ where $t \leq n$ with appropriate matrix gradings;*

(d) *L is a graded subdirect product of graded rings $\{A_i : i \in I\}$. Here for each i, $A_i \cong_{\rm gr} \mathbb{M}_{t_i}(K)$ or $\mathbb{M}_{t_i}(K[x,x^{-1}])$ with appropriate matrix gradings where, for each i, $t_i \leq n$, a fixed positive integer.*

Proof. Assume (a) so L has the index of nilpotence $\leq n$. Now no cycle in E can have an exit, because, otherwise, by Proposition 5.1(a), L will contain subrings of matrices of arbitrary size giving rise to unbounded nilpotence index. Then any vertex v in E either does not lie on a closed path or is the base of a cycle without exits. If there are are more than n distinct paths ending at v, then again, by Proposition 5.1 (b), L will contain a copy of a matrix ring of order $> n$ over K which will imply that the nilpotence index of L is $> n$, a contradiction. This proves (b).

Assume (b). Let $P = I(H,S)$ be a graded prime ideal of L. Our hypothesis implies that no cycle in $E\backslash(H,S)$ has an exit and that n is also the upper bound for the number of distinct paths ending at any vertex in $E\backslash(H,S)$. So every path in $E\backslash(H,S)$ containing no repeated edges has length $\leq n$. This means that every path ends at a sink or at a cycle without exits. But, by Theorem 3.12 of [28], $(E\backslash(H,S))^0 = E^0\backslash H$ is downward directed. Consequently, $E\backslash(H,S)$ contains either exactly one sink w or exactly one cycle c without exits based at a vertex v. Since there are no more than n distinct paths ending at w or at v without including c, we appeal to Proposition 3.4 and Theorem 3.7 of [2] to conclude that $L/P \cong L_K(E\backslash(H,S)) \cong \mathbb{M}_t(K)$ or $\mathbb{M}_t(K[x,x^{-1}])$ according as $E\backslash(H,S)$ contains a sink or a cycle without exits. Furthermore, as discussed in Section 2, there is a grading isomorphism on these algebras taking into account the length of paths ending at v or w. This proves (c).

Assume (c). Now, for any graded prime ideal P, $L/P \cong_{\rm gr} \mathbb{M}_t(K)$ or $\mathbb{M}_t(K[x,x^{-1}])$ with appropriate matrix gradings, where $t \leq$ a fixed integer n. First note that the intersection of all graded prime ideals of L is zero. To see this, given any vertex v, let M be a graded ideal maximal among graded ideals with respect to $v \notin M$. We claim that M is a graded prime ideal. To show that the graded ideal M is graded prime, we need only show it is a prime ideal [28]. Suppose, on the contrary, M is not prime. Then there exist $a, b \in L$ such that $aLb \subseteqq M$, but $a \notin M, b \notin M$. By the maximality of M, $v \in M + LaL$ and $v \in M + LbL$. Then $v = v^2 \in (M + LaL)(M + LbL) \subseteqq M^2 + M + M + LaLbL \subseteqq M$, as $aLb \subseteqq M$, a contadiction. Hence M is a (graded) prime ideal. Consequently, L is a graded subdirect product of rings graded isomorphic to $\mathbb{M}_t(K)$ or $\mathbb{M}_t(K[x,x^{-1}])$ under appropriate matrix gradings, where $t \leq n$, a fixed positive integer. This proves (d).

Assume (d). Let $t \leq n$. Clearly the matrix ring $\mathbb{M}_t(K)$ has index of nilpotence $\leq n$. Let Q be the quotient field of $K[x,x^{-1}]$. Then $M_n(K[x,x^{-1}])$ is a subring of $M_n(Q)$. Since $M_n(Q)$ has index of nilpotence n, $M_n(K[x,x^{-1}])$ has index of nilpotence $\leq n$. Consequently, a subdirect product of such rings will also have index of nilpotence $\leq n$. This proves (a). ∎

One consequence of Theorem 5.2 is the following.

Proposition 5.3. *Let E be an arbitrary graph. If $L := L_K(E)$ has bounded index of nilpotence n, then L is a graded Σ-V ring, but not conversely.*

Proof. If L has bounded index of nilpotence n, then for any graded primitive ideal P of L, we have, by Theorem 5.2(c), $L/P \cong_{\rm gr} \mathbb{M}_t(K)$ or $\mathbb{M}_t(K[x,x^{-1}])$ with appropriate matrix gradings, where $t \leq n$. By Theorem 4.10 (c), we then conclude that L is a graded Σ-V ring.

To see that the converse does not hold, let E be the graph consisting of a sink w and countably infinite edges $\{e_n : n \geq 1\}$ with distinct sources such that $r(e_n) = w$ for all n (this is the "opposite graph of the infinite clock" (17)). Then

$$L_K(E) \cong_{\rm gr} \mathbb{M}_\infty(K)(\bar{\delta}),$$

the infinite matrices with finitely many non-zero entries, where $\bar{\delta} := (\delta_i)_{i\in\mathbb{Z}}$, where $\delta_i = 1, i \in \mathbb{Z}$. Now this graded ring is graded semi-simple and so all graded left/right L-modules are injective and hence L is a $\sum$-V-ring. But L does not have bounded index of nilpotence as $\mathbb{M}_\infty(K)$ has unbounded nilpotence index. □

Remark 5.4. The Leavitt path algebra in Theorem 5.2 need not decompose as a direct sum of matrix rings, as is clear from Example 4.12 in which the Leavitt path algebra has bounded index of nilpotence 2. But the decomposition is possible if the graph is row-finite, as shown in the following theorem.

Theorem 5.5. *Let E be a row-finite graph. Then the following properties are equivalent for $L := L_K(E)$:*

(a) *L has bounded index of nilpotence $\leq n$;*

(b) *There is a fixed positive integer n and a graded isomorphism*

$$L \cong_{\mathrm{gr}} \bigoplus_{i\in I} \mathbb{M}_{n_i}(K) \oplus \bigoplus_{j\in J} \mathbb{M}_{m_j}(K[x, x^{-1}])$$

where I, J are arbitrary index sets and, for all $i \in I$ and $j \in J$, $n_i, m_j \leq n$. Thus, in particular, L is graded semi-simple (that is, a direct sum of graded simple left/right ideals).

Proof. Assume (a). By Theorem 5.2, no cycle in E has an exit and the number of distinct paths that end at any vertex v is $\leq n$, with the proviso that if v sits on a cycle c, then these paths do not include the entire cycle c. If A is the graded ideal generated by all the sinks in E and all the vertices on cycles without exits, then, by Theorem 3.7 of [2], $A \cong \bigoplus_{i\in I} \mathbb{M}_{n_i}(K) \oplus \bigoplus_{j\in J} \mathbb{M}_{m_j}(K[x, x^{-1}])$. By giving appropriate matrix gradings (see Section 2), this isomorphism becomes a graded isomorphism. We claim that $L = A$. Let $H \subseteq A$ be the set of all sinks or vertices on cycles in E. By hypothesis, every path in E that does not include an entire cycle has length $\leq n$ and eventually ends at a vertex in H. So if u is any vertex in E, using the fact that all the vertices in E are regular and by a simple induction on the length of the longest path from u, we can conclude that u belongs to the saturated closure of H. This implies that $L = A$. This proves (b).

(b) $\Longrightarrow$ (a) follows from the fact that the matrix rings $\mathbb{M}_{n_i}(K)$ and $\mathbb{M}_{m_j}(K[x, x^{-1}])$ with $n_i, m_j \leq n$ have index of nilpotence $\leq n$. □

As another application of Theorem 3.6, we get an analogous description of Leavitt path algebras with bounded index of nilpotence.

Theorem 5.6. *Let E be an arbitrary graph. Then $L := L_K(E)$ has bounded index n if and only if L is a directed union of direct sums of finitely many matrix rings of finite order $\leq n$.*

6. Leavitt path algebras which are Σ-V rings

In this section, we show how the ideas of Section 2 without the use of gradings give rise to the description of the Leavitt path algebra $L_K(E)$ of an arbitrary graph E which is a (not necessarily graded) Σ-V ring. In this case, the graph E is shown to be acyclic so L becomes von Neumann regular and is a subdirect product of semi-simple rings.

Recall that a ring R is said to be a left/right V-ring if every simple left/right R-module is injective. We begin with establishing a necessary condition for a Leavitt path algebra to be a V-ring.

Lemma 6.1. *Suppose R is a ring with local units. If R is a right V-ring, then R is right weakly regular, that is, $I^2 = I$ for any right ideal I of R.*

Proof. It is easy to check that a theorem of Villamayor on V-rings (see [22]) holds for a V-ring R with local units, namely, every right ideal is an intersection of maximal right ideals of R. Let I be a non-zero right ideal of R. If $I \neq I^2$, let $a \in I \backslash I^2$. Since I^2 is an intersection of maximal right ideals, there is a maximal right ideal M containing I^2 such that $a \notin M$. Then $R = aR + M$. Let u be a local unit satisfying $au = ua = a$. Write $u = ax + m$ where $x \in R$ and $m \in M$. Then $a = ua = axa + ma \in I^2 + M = M$, a contradiction. Hence $I^2 = I$ for every right ideal I of R. □

We need the following result from [7].

Lemma 6.2 (Theorem 3.1, [7])**.** *Let $L := L_K(E)$ be a Leavitt path algebra of an arbitrary graph E. If $I^2 = I$ for every right ideal I of L, then the graph E satisfies Condition (K), that is, every vertex v on a simple closed path c is also part of another simple closed path c' different from c.*

Proposition 6.3. *If $L := L_K(E)$ is a Σ-V ring, then the graph E contains no cycles and L is von Neumann regular.*

Proof. Now the same proof of Proposition 4.2 without the grading shows that L is directly-finite and so, by Lemma 3.5, no closed path in the graph E has an exit. On the other hand, since L is also a right V-ring, Lemma 6.1 and Lemma 6.2 imply that the graph E satisfies Condition (K) which in particular implies that every cycle in E has an exit. In view of these contradicting statements we conclude that the graph E contains no cycles. By [4, Theorem 1] L is von Neumann regular. □

Now using Proposition 6.3 and repeating the ungraded version of the proof of Theorem 4.10, we obtain the following description of Leavitt path algebras which are Σ-V rings.

Theorem 6.4. *Let E be an arbitrary graph. Then the following properties are equivalent for $L := L_K(E)$:*

(a) *L is a Σ-V ring;*

(b) *L is von Neumann regular and, for each primitive ideal P of L, $L/P \cong \mathbb{M}_n(K)$, where n could be possibly infinite.*

Remark 6.5. In view of Theorems 4.10 and 6.4, it is clear that the well-known Leavitt ring $L(1, n)$ is not a Σ-V ring and also not a graded Σ-V ring.

7. The type of graded regular self-injective Leavitt path algebras

The class of von Neumann regular rings (regular for short) constitutes a large class and from the categorical view point they are a natural generalisation of the basic building blocks of ring theory, i.e., simple rings. For an associative ring with identity if all modules are free or projective, then the ring is a division ring or a semi-simple ring, respectively. However if all modules are flat then the ring is regular. There is an element-wise definition for such rings; any element has an "inner inverse", i.e., for an element a, there is b such that $aba = a$. Such rings have very rich structures, and Ken Goodearl has devoted an entire book on them [15]. In the case of regular rings which are self injective, there is a structure theory developed which shows that the ring can be decomposed uniquely into three types based on the behaviour of idempotent elements [15, §10].

For the class of graded rings, one can define the notion of graded von Neumann regular rings (graded regular for short) as rings in which any homogeneous element has an inner inverse. There are many interesting class of rings which are not regular but are graded regular. One such class is the Leavitt path algebras [18].

The aim of this section is to develop sufficient structure theory for graded von Neumann regular graded self-injective rings parallel to the structure theory for the injective von Neumann regular rings and use it to show that Leavitt path algebras are of graded type I. In fact we will prove that for finite graphs, self-injective Leavitt path algebras are of graded type I and are precisely Σ-V Leavitt path algebras (Theorem 7.11).

In order to carry over the structure theory of regular self injective rings from nongraded setting to the graded case one needs to carefully analyse the behaviour of the homogeneous components throughout the proofs, as in the graded setting, suspensions of the modules would come into consideration. A prototype example of regular self injective rings is the ring of column finite matrices over a division ring. This would not readily generalize to the graded setting; the column finite matrices over a graded division rings is not necessarily graded. For this we need to consider a "closed" subring of such matrices, i.e., the $\operatorname{END}(M)$ as detailed in Section 2.

Definition 7.1. Let $R=\bigoplus_{\gamma\in\Gamma}R_\gamma$ be a Γ-graded ring. Denote by $I^{\mathrm{gr}}(R)$ the set of homogeneous idempotents and by $B^{\mathrm{gr}}(R)$ the set of central homogeneous idempotents.

(1) We say $e\in I^{\mathrm{gr}}(R)$ is *graded abelian idempotent* if $I^{\mathrm{gr}}(eRe)=B^{\mathrm{gr}}(eRe)$. We say R is *graded abelian* if 1 is a graded abelian idempotent, i.e., $I^{\mathrm{gr}}(R)=B^{\mathrm{gr}}(R)$.
(2) We say $e\in I^{\mathrm{gr}}(R)$ is *graded directly-finite* if eRe is a graded directly-finite ring, i.e., if $x,y\in(eRe)^h$ such that $xy=1$ then $yx=1$.
(3) We say $e\in I^{\mathrm{gr}}(R)$ is *graded faithful* if $y\in B^{\mathrm{gr}}(R)$ such that $ey=0$, then $y=0$.

Next we define different types of graded regular graded self-injective rings.

Definition 7.2. Let R be a Γ-graded von Neumann regular, graded right self-injective ring.

(1) We say R is of *graded type I*, if R contains a graded faithful abelian idempotent.
(2) We say R is of *graded type II*, if R contains a graded faithful directly-finite idempotent but contains no nonzero graded abelian idempotent.
(3) We say R is of *graded type III*, if R contains no nonzero graded directly-finite idempotent.

We start with exploring the relations between the ring R and its zero homogeneous ring R_0.

Lemma 7.3. *If R is a Γ-graded regular graded self-injective ring then R_0 is a regular self-injective ring. Furthermore if R is strongly graded, the converse also holds.*

Proof. If R is graded regular, clearly R_0 is regular and consequently R is flat over R_0. Let I be a right ideal of R_0. Tensoring any map $f:I\to R_0$ with R, thanks to R being graded self-injective, $f\otimes 1$ lifts to a graded homomorphism $\overline{f}:R\to R$ which its zero components $\overline{f}_0\in\operatorname{Hom}(R_0,R_0)$ extends f. Thus R_0 is self-injective as well.

If R is strongly graded, by Dade's theorem ([19, §1.5]) Gr-R is Morita equivalent to Mod-R_0. Thus any diagram can be completed via going to Mod-R_0. □

One can further establish that if a graded regular self-injective ring R has graded type I (type II), then R_0 has type I (type II) (see Remark 7.12).

Theorem 7.4. *Let R be a Γ-graded regular ring and A be a graded projective right R-module. Then any graded finitely generated submodule B of A is a direct summand of A. In particular B is graded projective.*

Proof. Since A is graded projective, there is a graded module C such that $A\oplus C\cong_{\mathrm{gr}}F$, where $F\cong_{\mathrm{gr}}\bigoplus_{i\in I}R(\alpha_i)$, is a graded free R-module. Consider A as a graded submodule of F. Since $B\leq^{\mathrm{gr}}A$ is finitely generated, there is a graded finitely generated free R-module G which is a direct summand of F and contains B.

Suppose B is generated by n elements b_i of degree α_i, $1\leq i\leq n$, respectively, and consider a graded free module H containing G which has a graded basis of at least n elements h_i of degrees α_i's. Sending $h_i\mapsto b_i$, $1\leq i\leq n$, and the rest of the basis to zero, will define a graded homomorphism of degree zero $f:H\to H$ such that $f(H)=B$. Since $\operatorname{End}_R(H)$ is graded von Neumann regular, there is $g\in\operatorname{End}_R(H)_0$ such that $fgf=f$. Thus fg is a homogeneous idempotent and $fg(H)=f(H)=B$. Thus

$$H=fg(H)\bigoplus(1-fg)(H)=B\bigoplus(1-fg)(H).$$

and so B is a direct summand of H. (Note that no suspension of B is required as f and g are of degree zero.) To finish the proof, we use the following fact twice: if $N\leq M\leq F$ and N and M are direct summand of F, then N is a direct summand of M. □

We need to adapt Theorem 1.22 in [15] in the graded setting. Recall that a graded right/left ideal A of the graded ring R is called *graded right/left essential* and denoted by $A \leq_e^{\mathrm{gr}} R$, if for any non-zero graded right/left ideal B of R, we have $A \cap B \neq 0$.

Theorem 7.5. *Let R be a Γ-graded ring, and A be a graded quasi-injective right R-module. Let $Q = \mathrm{END}_R(A)$. Then*

(a) $J^{\mathrm{gr}}(Q) = \big\{ f \in \mathrm{END}_R(A) \mid \ker f_\alpha \leq_e^{\mathrm{gr}} A,$ *for all* $\alpha \in \mathrm{supp}(f) \big\}$.

(b) $Q/J^{\mathrm{gr}}(Q)$ *is graded von Neumann regular.*

(c) *If $J^{\mathrm{gr}}(Q) = 0$ then Q is graded right self-injective.*

Proof. Let $K = \big\{ f \in \mathrm{END}_R(A) \mid \ker f_\alpha \leq_e^{\mathrm{gr}} A, \text{ for all } \alpha \in \mathrm{supp}(f) \big\}$. One checks that J is a graded two sided ideals of A. We first show that $K \subseteq J^{\mathrm{gr}}(Q)$. It is enough to show that $K^h \subseteq J^{\mathrm{gr}}(Q)$. Let $f \in K^h$, i.e., $f \in \mathrm{Hom}(A,A)_\alpha$, for some $\alpha \in \Gamma$ and $\ker f \leq_e^{\mathrm{gr}} A$. We show that for any $r \in \mathrm{Hom}(A,A)_{-\alpha}$, $1 - rf$ is invertible. We have $\ker(1-rf) \cap \ker f = 0$. Since $\ker f$ is essential, and $\ker(1-rf)$ is a graded right ideal of A, it follows that $\ker(1-rf) = 0$. Thus $\theta := 1 - rf : A \to (1-rf)A$ is graded isomorphism. Since A is graded quasi-injective, there is a graded homomorphism g which completes the following diagram.

$$\begin{array}{ccc} (1-rf)A & \hookrightarrow & A \\ \downarrow{\scriptstyle \theta^{-1}} & \swarrow{\scriptstyle g} & \\ A & & \end{array}$$

It follows that $g(1-rf) = 1$. So $f \in J^{\mathrm{gr}}(Q)$ and consequently $K \subseteq J^{\mathrm{gr}}(Q)$. Next we show that Q/K is a graded regular ring. Let $f \in Q^h$, i.e., $f \in \mathrm{Hom}(A,A)_\alpha$ for some $\alpha \in \Gamma$. Set $S = \{ N \leq^{\mathrm{gr}} A \mid N \cap \ker f = 0 \}$. This is a poset in which each chain has an upper bound. Thus by Zorn's lemma S has a maximal element B. It follows that $B \oplus \ker f \leq_e^{\mathrm{gr}} A$. Since $\theta = f : B \to f(B)$ is graded isomorphism, and A is graded quasi-injective, there is graded homomorphism g such that

$$\begin{array}{ccc} f(B) & \hookrightarrow & A \\ \downarrow{\scriptstyle \theta^{-1}} & \swarrow{\scriptstyle g} & \\ A & & \end{array}$$

It follows that $gf = 1$ on B. Thus $(fgf - f)B = 0$. So $B \oplus \ker f \leq \ker(fgf - f)$ and therefore $fgf - f \in \mathrm{Hom}(A,A)_\alpha$ and $\ker(fgf - f) \leq_e^{\mathrm{gr}} A$. So $fgf - f \in K$. Thus in Q/K we have $\overline{fgf} = \overline{f}$, i.e., Q/K is graded regular. So $J^{\mathrm{gr}}(Q/R) = 0$, which gives that $J^{\mathrm{gr}}(Q) = K$. So $Q/J^{\mathrm{gr}}(Q)$ is graded regular.

(c) Suppose $J^{\mathrm{gr}}(Q) = 0$. Then Q is graded regular and A is the graded left Q-module. So A is (graded) flat over Q. Suppose J is a graded right ideal of Q and $f : J \to Q$ a graded Q-module homomorphism. We will show that there is a $h \in \mathrm{Hom}(A,A)_0$ such that one can complete the following diagram and thus Q is self-injective.

$$\begin{array}{ccc} J & \hookrightarrow & Q \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle h} & \\ Q & & \end{array} \qquad (20)$$

We have the following commutative diagram of graded maps.

$$\begin{array}{ccc} J \otimes_Q A & \xrightarrow{f \otimes 1} & Q \otimes_Q A \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ JA & \xrightarrow[g]{} & A \end{array}$$

Since A is quasi-injective the graded homomorphism g extends to a $h \in \mathrm{Hom}(A, A)_0$ such that $h(xy) = g(x)y = f(x)y$ for any $x \in J$ and $y \in A$. Thus we have $f(x) = h(x)$ for any $x \in J$. This completes Diagram 20 and thus the proof. □

Corollary 7.6. *Let R be a graded ring, A a graded right R-module and $Q = \mathrm{END}_R(A)$. If A is graded semi-simple or graded non-singular quasi-injective, then Q is graded regular self-injective ring.*

Proof. We show that $J^{\mathrm{gr}}(Q) = 0$ and the corollary follows from Theorem 7.5. If $f \in J^{\mathrm{gr}}(Q)$ then $\ker f_\alpha \leq_e^{\mathrm{gr}} A$ for all $\alpha \in \mathrm{supp}(f)$. If A is graded semi-simple, then f is homogeneous and $\ker f = A$ and so $f = 0$. If A is graded non-singular, then $f_\alpha : A \to A(-\alpha)$ is a graded homomorphism. Thus $A/\ker f \cong A(-\alpha)$. If A is non-singlaur then $A(-\alpha)$ is non-singular as well and so $f = 0$. Thus $J^{\mathrm{gr}}(Q) = 0$. □

The corollary below follows immediately from Corollary 7.6 and it gives a prototype example of graded regular self-injective rings. Recall that a Γ-graded ring R is called a *graded division ring* if any non-zero homogeneous element is invertible.

Corollary 7.7. *Let R be a graded division ring and A be a right graded R-module. Then $\mathrm{END}_R(A)$ is a graded regular self-injective ring.*

Corollary 7.8. *Let R be a graded ring such that R is graded right nonsingular. Then its maximal graded right quotient ring $Q^{\mathrm{gr}}(R)$ is graded regular and graded right self-injective.*

Since the Leavitt path algebra $L_K(E)$ associated to an arbitrary graph E is graded regular [18, 20], it is graded nonsingular. Consequently, its maximal graded right quotient ring is graded regular and graded right self-injective.

Since for any central homogeneous idempotent e, $eR(1-e)R = 0$, if R is graded prime, it follows that $B^{\mathrm{gr}}(R) = \{0, 1\}$. If R is, in addition, graded regular and abelian, i.e., $I^{\mathrm{gr}}(R) = B^{\mathrm{gr}}(R)$, then for any $0 \neq x \in R^h$, there is $y \in R^h$ such that $xyx = x$, so $xy \in B^{\mathrm{gr}}(R)$, and thus $xy = 1$. It follows that R is a graded division ring. The following proposition shows that if a corner of R is abelian, then R is graded isomorphic to matrices over a graded division ring.

Theorem 7.9. *Let R be a Γ-graded ring. Then $R \cong_{\mathrm{gr}} \mathrm{END}_D(A)$, where D is a graded division ring and A is a graded D-module if and only if R is graded prime, graded regular, self-injective ring with* $\mathrm{soc}^{\mathrm{gr}}(R_R) \neq 0$.

Proof. Suppose $R \cong_{\mathrm{gr}} \mathrm{END}_D(A)$, where D is a graded division ring and A is a graded D-module. Then by Corollary 7.6, R is graded regular, self-injective. For $0 \neq x, y \in \mathrm{END}_D(A)^h$, there are homogeneous elements $a, b \in A$ such that $x(a) \neq 0$ and $y(b) \neq 0$. Since $y(b)$ is homogeneous, one can extend it to a graded basis for D [19, §1.4]. Thus there is a $z \in \mathrm{END}_D(A)^h$ such that $z(y(b)) = a$. It follows that $x(z(y(b))) = x(a) \neq 0$. Thus $xzy \neq 0$. This shows that R is a graded prime ring. Furthermore, one can choose a homogeneous element $e \in R = \mathrm{END}_D(A)$ such that $eRe \cong_{\mathrm{gr}} D$. Thus $\mathrm{End}_R(eR, eR) \cong_{\mathrm{gr}} eRe \cong_{\mathrm{gr}} D$ and the fact that R is graded prime, implies that eR is graded simple. Thus $\mathrm{soc}^{\mathrm{gr}}(R) \neq 0$.

For the converse of the theorem, since $\mathrm{soc}^{\mathrm{gr}}(R) \neq 0$, one can choose a homogeneous idempotent $e \in R$ such that eR is graded right simple. Thus, $eRe \cong_{\mathrm{gr}} \mathrm{End}_R(eR, eR)$ is a graded division ring. Since R is graded prime, $B^{\mathrm{gr}}(R) = \{0, 1\}$. Thus by the graded version of [15, Theorem 9.8], the graded ring homomorphism $R \to \mathrm{END}_{eRe}(Re)$ is an isomorphism. □

Proposition 7.10. *Let R be a Γ-graded prime, regular and right self-injective ring. Then R is graded type I if and only if R is graded isomorphic to $\mathrm{END}_D(A)$, where D is a graded division ring and A is a graded right D-module.*

Proof. Suppose $R \cong_{\mathrm{gr}} \mathrm{END}_D(A)$, where D is a graded division ring and A is a graded D-module. Then $\mathrm{soc}^{\mathrm{gr}}(R) \neq 0$ by Theorem 7.9. Thus there is a homogeneous idempotent $e \in R$ such that eR is a graded right simple R-module. Thus $\mathrm{End}_R(eR, eR) \cong eRe$ is a graded division ring. Thus e is a graded abelian idempotent. Since R is graded prime, $B^{\mathrm{gr}}(R) = \{0, 1\}$. Hence e is faithful as well, and so R is graded type I.

For the converse of the theorem, suppose R is graded type I. Thus there is a homogeneous idempotent $e \in R$ which is faithful and abelian. Thus eRe is graded prime, regular and abelian. It follows that R is a graded division ring (see the argument before Theorem 7.9) and consequently, we have that eR is graded simple. Thus $\mathrm{soc}^{\mathrm{gr}}(R) \neq 0$. Now Theorem 7.9 implies that $R \cong_{\mathrm{gr}} \mathrm{END}_D(A)$. □

We are in a position to determine the graded types of graded self-injective Leavitt path algebras.

Theorem 7.11. *Let $L_K(E)$ be a Leavitt path algebra associated to an finite graph E, where K is a field. Then the following are equivalent.*

(a) *$L_K(E)$ is graded von Neumann regular and graded self-injective;*

(b) *$L_K(E)$ is an algebra of graded type I;*

(c) *No cycle in E has an exit;*

(d) *$L_K(E)$ is a graded Σ-V ring;*

(e) *There is a graded isomorphism*

$$L_K(E) \cong_{\mathrm{gr}} \bigoplus_{v_i \in X} \mathbb{M}_{n_i}(K)((|\overline{p^{v_i}}|) \oplus \bigoplus_{w_j \in Y} \mathbb{M}_{m_j}(K[x^{t_j}, x^{-t_j}])(|\overline{q^{w_j}}|)$$

where n_i,m_j are suitable positive integers, the t_j are positive integers, X is the set of sinks in E and Y is the set of all distinct cycles (without exits) in E.

Proof. This follows from Proposition 7.10, Theorem 4.15 and [20, Theorem 6.7]. □

Remark 7.12. Similar to the theory of von Neumann regular self injective rings [15, §10], one can develop the type theory for the graded von Neumann regular graded self injective rings. In fact one can develop this theory for the rings with local units. Namely, for a Γ-graded von Neumann regular graded right self-injective ring R, we have $R \cong_{\mathrm{gr}} R_1 \times R_2 \times R_3$, where R_i are Γ-graded rings of graded type I, II and III, respectively (Definition 7.2). Furthermore, this decomposition is unique. This will be established in the upcoming paper.

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