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NEW CHARACTERIZATION OF Σ-INJECTIVE MODULES

K. I. BEIDAR, S. K. JAIN, AND ASHISH K. SRIVASTAVA

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Abstract. We provide a new characterization for an injective module to be Σ-injective.

1. Introduction

In his paper [4], Carl Faith introduced the concept of Σ-*injectivity* and defined an injective module M to be Σ-injective if every direct sum of copies of M is injective. It turns out that such an R-module M provides a good deal of information about the structure of a ring R. For example, R is right noetherian if and only if every injective right R-module is Σ-injective [5]. If R is an integral domain, then the injective hull $E(R_R)$ of R is Σ-injective if and only if R is a right Ore domain [4]. Goursaud-Valette showed that if a ring R admits a faithful Σ-injective module, then R is a right Goldie ring [6].

The following characterizations are well-known for an injective module to be Σ-injective.

Theorem 1 (Cailleau [3], Faith [4]). *For an injective module M_R, the following are equivalent:*

(1) *M is Σ-injective.*

(2) *M is countably Σ-injective.*

(3) *R satisfies ACC on the set of right ideals I of R that are annihilators of subsets of M.*

(4) *M is a direct sum of indecomposable Σ-injective modules.*

The purpose of this paper is to provide the following new characterization for an injective module to be Σ-injective.

Theorem 2. *Let M_R be an injective module. Then the following statements are equivalent:*

(a) *M is Σ-injective.*

(b) *Every essential extension of $M^{(\aleph_0)}$ is a direct sum of injective modules.*

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The first author is deceased.

2. Preliminaries

All rings considered in this paper have unity, and all modules are right unital. We denote by $E(M)$ the injective hull of M. We shall write $N \subseteq_e M$ whenever N is an essential submodule of M. A submodule L of M is called an essential closure of a submodule N of M if it is a maximal essential extension of N in M. A submodule K of M is called a complement if there exists a submodule U of M such that K is maximal with respect to the property that $K \cap U = 0$. Given a cardinal α and a module N, we denote by $N^{(\alpha)}$ the direct sum of α copies of the module N. A module N is said to be Σ-injective provided that $N^{(\alpha)}$ is injective for any cardinal α. We say that the Goldie dimension $G\dim_U(N)$ of N with respect to U is finite, written as $G\dim_U(N) < \infty$, if N does not contain an infinite independent family of nonzero submodules which are isomorphic to submodules of U. A module N is said to be *q.f.d.* relative to U if for any factor module $\bar{N}$ of N, $G\dim_U(\bar{N}) < \infty$. We say R is right *q.f.d.* relative to U if R_R is *q.f.d.* relative to U.

We first start with a key lemma.

Lemma 3. *Let M be an injective module and suppose that every essential extension of $M^{(\aleph_0)}$ is a direct sum of injective modules. Then*

(a) *Given a direct sum $G = \bigoplus_{i\in\mathbb{N}} M_i$, $M_i \cong M$, and nonzero injective submodules V_i of M_i, there exists an infinite subset $\mathcal{J} \subseteq \mathbb{N}$ and nonzero injective submodules $V_j^{'} \subseteq V_j$, $j \in \mathcal{J}$, such that $\bigoplus_{j\in\mathcal{J}} V_j^{'}$ is injective.*

In particular, if $\{V_i : i \in \mathbb{N}\}$ is an independent family of uniform injective submodules of M, then $\bigoplus_{j\in\mathcal{J}} V_j$ is injective for some infinite subset $\mathcal{J} \subseteq \mathbb{N}$.

(b) *R is right q.f.d. relative to M.*

Proof. (a) Set $E = E(G)$. Since V_i is an injective submodule of M_i, $M_i = V_i \oplus M_i^{'}$ for some submodule $M_i^{'} \subseteq M_i$. Therefore, $G = (\bigoplus_{i\in\mathbb{N}} V_i) \oplus (\bigoplus_{i\in\mathbb{N}} M_i^{'})$. Let H and $H^{'}$ be essential closures of $\bigoplus_{i\in\mathbb{N}} V_i$ and $\bigoplus_{i\in\mathbb{N}} M_i^{'}$ in E, respectively. Clearly, $E = H \oplus H^{'}$. If $\bigoplus_{i\in\mathbb{N}} V_i = H$, then there is nothing to prove.

Now consider the case when $\bigoplus_{i\in\mathbb{N}} V_i \neq H$. Pick $x \in H \setminus \bigoplus_{i\in\mathbb{N}} V_i$. Let Q be a submodule of H maximal with respect to the properties that $\bigoplus_{i\in\mathbb{N}} V_i \subseteq Q$ and $x \notin Q$. Set $P = Q \oplus H^{'}$ and note that $E/P = (H \oplus H^{'})/(Q \oplus H^{'}) \cong H/Q$ is a subdirectly irreducible module.

Now, as $G \subseteq_e E$ and $G \subseteq P \subset E$, we have $G \subseteq_e P$. Hence, by our assumption, $P = \bigoplus_{k\in\mathcal{K}} W_k$, where each W_k is a nonzero injective module. Since $P \subset_e E$ and $P \neq E$, P is not injective, and so $|\mathcal{K}| = \infty$.

We claim that for any finite subset $\mathcal{L}$ of $\mathcal{K}$ and for any positive integer n there exists $i > n$ such that $V_i \cap (\bigoplus_{k\in\mathcal{L}} W_k)$ is not essential in V_i.

Suppose the above claim is not true. Then there exists a finite subset $\mathcal{L} \subseteq \mathcal{K}$ and an integer $n \geq 1$ such that $V_i \cap (\bigoplus_{k\in\mathcal{L}} W_k) \subset_e V_i$ for all $i > n$. Let A be an essential closure of $\bigoplus_{i>n}(V_i \cap (\bigoplus_{k\in\mathcal{L}} W_k))$ in $\bigoplus_{k\in\mathcal{L}} W_k$ which is injective, and so A is also injective.

We have $\bigoplus_{i>n}(V_i \cap \bigoplus_{k\in\mathcal{L}} W_k) \subset_e A \subset \bigoplus_{k\in\mathcal{L}} W_k$. Setting $B = V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus A$, we have $V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus_{i>n} (V_i \cap \bigoplus_{k\in\mathcal{L}} W_k) \subset_e B \subset E = H \oplus H^{'}$. Now, $((\bigoplus_{i\leq n} V_i) \oplus_{i>n} (V_i \cap (\bigoplus_{k\in\mathcal{L}} W_k))) \cap H \subset_e B \cap H \subset H$, which gives $(\bigoplus_{i\leq n} V_i) \oplus_{i>n} (V_i \cap (\bigoplus_{k\in\mathcal{L}} W_k)) \subset_e B \cap H \subset H$. Since $V_i \cap (\bigoplus_{k\in\mathcal{L}} W_k) \subset_e V_i$ for all $i > n$, we have $(\bigoplus_{i\leq n} V_i) \oplus_{i>n} (V_i \cap (\bigoplus_{k\in\mathcal{L}} W_k)) \subset_e \bigoplus_{i\in\mathbb{N}} V_i \subset_e H$. Thus $B \cap H$ is an essential submodule of H. Furthermore, as $(\bigoplus_{i\leq n} V_i)\oplus_{i>n}(V_i\cap(\bigoplus_{k\in\mathcal{L}} W_k))\subset_e B$, we have $B\cap H\subset_e B$.

Since $B\cap H\subset_e B$, we have $B\cap H^{'}=0$. As $B\cap H\subset_e H$, we have $(B\cap H)\oplus H^{'}\subset_e H\oplus H^{'}=E$. Therefore, $B\oplus H^{'}\subset_e E$. But since both B and $H^{'}$ are injective, $B\oplus H^{'}$ is injective. Thus $E=B\oplus H^{'}=(V_1\oplus V_2\oplus ...\oplus V_n\oplus A)\oplus H^{'}\subseteq Q+P+H^{'}=P$, a contradiction because $P\subset E$ and $P\neq E$.

This proves that for any finite subset $\mathcal{L}$ of $\mathcal{K}$ and for any positive integer n there exists $i>n$ such that $V_i\cap(\bigoplus_{k\in\mathcal{L}} W_k)$ is not essential in V_i.

We now proceed by induction to construct a sequence of submodules $\{W^{'}_{k_j}: j=1,2,...,n,...\}$ such that each $W^{'}_{k_j}$ is a nonzero injective submodule of W_{k_j} isomorphic to a submodule $V^{'}_{i_j}$ of V_{i_j}, where $k_1,k_2,...,k_n,...$ are distinct elements of $\mathcal{K}$ and $1\leq i_1<i_2<...<i_n<....$

Let $i_1\geq 1$ be arbitrary. Now $V_{i_1}\subset\bigoplus_{k\in\mathcal{K}} W_k$ implies that there exists a nonzero submodule $V^{'}_{i_1}$ of V_{i_1} such that $V^{'}_{i_1}$ is isomorphic to a submodule $W^{'}_{k_1}$ of W_{k_1} for some $k_1\in\mathcal{K}$. Clearly, we may choose $V^{'}_{i_1}$ to be an injective submodule of V_{i_1}.

For $n\geq 1$, assume that we have a sequence $\{W^{'}_{k_j}: j=1,2,...,n\}$ with the above stated property. By the fact proved above, there exists $i_{n+1}>i_n$ such that $X=V_{i_{n+1}}\cap(\bigoplus_{k\in\mathcal{K}_1} W_k)$ is not essential in $V_{i_{n+1}}$, where $\mathcal{K}_1=\{k_1,k_2,...,k_n\}$. Let $X^{'}$ be a complement of X in $V_{i_{n+1}}$. Then $X^{'}\neq 0$ and $X^{'}\cap(\bigoplus_{k\in\mathcal{K}_1} W_k)=X^{'}\cap X=0$. We have $X^{'}\subset V_{i_{n+1}}\subset(\bigoplus_{k\in\mathcal{K}_1} W_k)\oplus(\bigoplus_{k\in\mathcal{K}_2} W_k)$, where $\mathcal{K}_2=\mathcal{K}\backslash\mathcal{K}_1$. Let $\pi:(\bigoplus_{k\in\mathcal{K}_1} W_k)\oplus(\bigoplus_{k\in\mathcal{K}_2} W_k)\longrightarrow\bigoplus_{k\in\mathcal{K}_2} W_k$ be the projection. Then $ker(\pi|_{X'})=X^{'}\cap(\bigoplus_{k\in\mathcal{K}_1} W_k)=0$. Therefore, $X^{'}$ is isomorphic to some submodule of $\bigoplus_{k\in\mathcal{K}_2} W_k$. So, $X^{'}$ contains a nonzero submodule which is isomorphic to a submodule F of $W_{k_{n+1}}$ for some $k_{n+1}\in\mathcal{K}_2$. Denote by $W^{'}_{k_{n+1}}$ an essential closure of F in $W_{k_{n+1}}$. Since F is isomorphic to a submodule of the injective module $V_{i_{n+1}}$, we conclude that $W^{'}_{k_{n+1}}$ is isomorphic to a submodule of $V_{i_{n+1}}$ as well. Obviously the family $\{W^{'}_{k_j}: j=1,2,...,n+1\}$ satisfies the required property. This completes the induction argument.

Now set $\mathcal{K}^{'}=\{k_1,k_2,...,k_n,...\}$. Choose disjoint subsets $\mathcal{K}^{'}_1$ and $\mathcal{K}^{'}_2$ of $\mathcal{K}$ such that $\mathcal{K}=\mathcal{K}^{'}_1\cup\mathcal{K}^{'}_2$ and $\mathcal{K}^{'}\cap\mathcal{K}^{'}_1=\{k_1,k_3,...,k_{2n+1},...\}$. Clearly, $\mathcal{K}^{'}\cap\mathcal{K}^{'}_2=\{k_2,k_4,...,k_{2n},...\}$.

Now we claim that either $\bigoplus_{k\in\mathcal{K}'_1} W_k$ is injective or $\bigoplus_{k\in\mathcal{K}'_2} W_k$ is injective.

Set $V=\bigoplus_{k\in\mathcal{K}'_1} W_k$ and $W=\bigoplus_{k\in\mathcal{K}'_2} W_k$. We have $P=V\oplus W$. Let $\widehat{V}$ and $\widehat{W}$ be essential closures of V and W, respectively, in E. Clearly, $E=\widehat{V}\oplus\widehat{W}$. Therefore, $E/P=(\widehat{V}\oplus\widehat{W})/(V\oplus W)\cong(\widehat{V}/V)\times(\widehat{W}/W)$. Since E/P is shown to be subdirectly irreducible in the beginning of the proof, we have either $V=\widehat{V}$ or $W=\widehat{W}$. This proves our claim.

Thus, we may assume, without loss of generality, that the module $\bigoplus_{k\in\mathcal{K}'_1} W_k$ is injective. Since $\bigoplus_{n=0}^{\infty} W^{'}_{k_{2n+1}}$ is a direct summand of $\bigoplus_{k\in\mathcal{K}'_1} W_k$, we get that $\bigoplus_{n=0}^{\infty} W^{'}_{k_{2n+1}}$ is injective. Recalling that $\bigoplus_{n=0}^{\infty} V^{'}_{i_{2n+1}}\cong\bigoplus_{n=0}^{\infty} W^{'}_{k_{2n+1}}$, we conclude that $\bigoplus_{n=0}^{\infty} V^{'}_{i_{2n+1}}$ is an injective module. This completes the proof.

(b) Assume to the contrary that R is not right $q.f.d.$ relative to M. Then there exists a cyclic right R-module C with an infinite independent family $\{C_i: i\in\mathbb{N}\}$ of nonzero submodules of C such that each C_i is isomorphic to a submodule B_i of M. Set D_i equal to an essential closure of B_i in M. Then $\{D_i : i \in \mathbb{N}\}$ is a family of injective submodules of M. Therefore by (a), there exists an infinite subset $\mathcal{J} \subseteq \mathbb{N}$ and nonzero injective submodules $D_j^{'} \subseteq D_j$, $j \in \mathcal{J}$, such that $\bigoplus_{j\in\mathcal{J}} D_j^{'}$ is injective. Set $B_j^{'} = B_j \cap D_j^{'}$, $j \in \mathcal{J}$ and note that $B_j^{'} \neq 0$. Let $C_j^{'}$ be the inverse image of $B_j^{'}$ under the isomorphism $C_j \longrightarrow B_j$ stated above. This induces a canonical isomorphism between $\bigoplus_{j\in\mathcal{J}} C_j^{'}$ and $\bigoplus_{j\in\mathcal{J}} B_j^{'}$, say θ. Let σ be the inclusion map $\bigoplus_{j\in\mathcal{J}} B_j^{'} \longrightarrow \bigoplus_{j\in\mathcal{J}} D_j^{'}$. Then, since $\bigoplus_{j\in\mathcal{J}} D_j^{'}$ is injective, the map $f = \sigma\theta : \bigoplus_{j\in\mathcal{J}} C_j^{'} \longrightarrow \bigoplus_{j\in\mathcal{J}} D_j^{'}$ can be extended to a homomorphism $\widehat{f} : C \longrightarrow \bigoplus_{j\in\mathcal{J}} D_j^{'}$. Because C is cyclic, there exists a finite subset $\mathcal{K} \subseteq \mathcal{J}$ such that $\widehat{f}(C) \subseteq \bigoplus_{k\in\mathcal{K}} D_k^{'}$. Now, $\widehat{f}(C_j^{'}) = f(C_j^{'}) = \sigma\theta(C_j^{'}) = \sigma(B_j^{'}) = B_j^{'}$. But $\widehat{f}(C_j^{'}) \subseteq \widehat{f}(C) \cap D_j^{'} = 0$ for all $j \notin \mathcal{K}$, a contradiction.

Therefore, R is right $q.f.d.$ relative to M. □

3. Proof of Theorem 2

Proof. (b) $\Longrightarrow$ (a). Suppose that $M^{(\lambda)}$ is not injective for some infinite cardinal λ. Set $E = E(M^{(\lambda)})$, pick $x \in E\backslash M^{(\lambda)}$ and let $L = xR$. By Lemma 3 (b), R is right $q.f.d.$ relative to M. From this it follows that every nonzero cyclic and hence every nonzero submodule of M contains a uniform submodule. Now, consider the set $\mathcal{S}$ of independent families $(M_k)_{k\in\mathcal{K}}$ of uniform injective modules $0 \neq M_k \subseteq M$. Suppose $\mathcal{S}$ is partially ordered by $(M_k)_{k\in\mathcal{K}} \leq (N_l)_{l\in\mathcal{L}}$ if and only if $\mathcal{K} \subseteq \mathcal{L}$ and $M_k = N_k$ for $k \in \mathcal{K}$. By Zorn's lemma we get a maximal independent family $(M_i)_{i\in\mathcal{I}}$ of uniform injective submodules. Clearly $\bigoplus_{i\in\mathcal{I}} M_i \subseteq_e M$, because otherwise we will get a contradiction to the maximality of this independent family of submodules. This yields that we have an independent family $\{W_i : i \in \mathcal{I}\}$ of uniform injective submodules of $M^{(\lambda)}$ such that each W_i is isomorphic to a submodule of M and $\bigoplus_{i\in\mathcal{I}} W_i \subseteq_e M^{(\lambda)}$.

Now we proceed to show that there is a sequence of pairwise distinct elements $i_1, i_2, ...$ in $\mathcal{I}$ and an independent family of direct summands $V_1, V_2, ...$ of E such that $V_j \cong W_{i_j}$ with $V_j \oplus (\bigoplus_{i\in\mathcal{I}_j} W_i) = \bigoplus_{i\in\mathcal{I}_{j-1}} W_i$, $E = E_j \oplus (\bigoplus_{k=1}^{j} V_k)$ and $\pi_{j-1}(L) \cap V_j \neq 0$ for all $j \in \mathbb{N}$, where $\mathcal{I}_0 = \mathcal{I}$, $\mathcal{I}_j = \mathcal{I}_{j-1}\backslash\{i_j\}$ for $i_j \in \mathcal{I}$, $E_0 = E$, E_j is an essential closure of $\bigoplus_{i\in\mathcal{I}_j} W_i$ in E_{j-1}, $\pi_0 = id_E$, and π_j is the projection of E onto E_j along $V_1 \oplus ... \oplus V_j$.

Since $\bigoplus_{i\in\mathcal{I}} W_i \subseteq_e M^{(\lambda)} \subset_e E$ and L is a nonzero submodule of E, we have $L \cap (\bigoplus_{i\in\mathcal{I}} W_i) \neq 0$. So $L \cap (\bigoplus_{i\in\mathcal{I}} W_i)$ contains a nonzero cyclic uniform submodule, say, C_1. This implies that there exists a finite subset $\mathcal{K}_1 \subset \mathcal{I}$ such that $C_1 \subseteq \bigoplus_{i\in\mathcal{K}_1} W_i$. Let V_1 be an essential closure of C_1 in $\bigoplus_{i\in\mathcal{K}_1} W_i$. Since $\bigoplus_{i\in\mathcal{K}_1} W_i$ is injective, V_1 is injective. So, $\bigoplus_{i\in\mathcal{K}_1} W_i = V_1 \oplus D_1$ for some submodule D_1 of $\bigoplus_{i\in\mathcal{K}_1} W_i$. Since V_1 is injective, it has the exchange property. Therefore, $\bigoplus_{i\in\mathcal{K}_1} W_i = V_1 \oplus (\bigoplus_{i\in\mathcal{K}_1} W_i^{'})$ for some submodules $W_i^{'}$ of W_i. Since $W_i^{'}$ are injective and each W_i is indecomposable, either $W_i^{'} = 0$ or $W_i^{'} = W_i$. We recall that V_1 is uniform because it is the closure of the uniform module C_1. Comparing the Goldie dimension on each side of $\bigoplus_{i\in\mathcal{K}_1} W_i = V_1 \oplus (\bigoplus_{i\in\mathcal{K}_1} W_i^{'})$, we get that there exists exactly one index, say $i_1 \in \mathcal{K}_1$, such that $W_{i_1}^{'} = 0$, and for all $i(\neq i_1) \in \mathcal{K}_1$, $W_i^{'} = W_i$. So, $\bigoplus_{i\in\mathcal{K}_1} W_i = V_1 \oplus (\bigoplus_{i\in\mathcal{K}_1\backslash\{i_1\}} W_i)$. This yields

$V_1 \cong (\bigoplus_{i\in\mathcal{K}_1} W_i)/(\bigoplus_{i\in\mathcal{K}_1\setminus\{i_1\}} W_i) \cong W_{i_1}$. Also, we have $V_1 \oplus (\bigoplus_{i\in\mathcal{K}_1\setminus\{i_1\}} W_i) \oplus (\bigoplus_{i\in\mathcal{I}\setminus\mathcal{K}_1} W_i) = (\bigoplus_{i\in\mathcal{K}_1} W_i) \oplus (\bigoplus_{i\in\mathcal{I}\setminus\mathcal{K}_1} W_i)$. This yields $V_1 \oplus (\bigoplus_{i\in\mathcal{I}_1} W_i) = \bigoplus_{i\in\mathcal{I}} W_i$. Taking injective hulls of both sides, we get $E_1 \oplus V_1 = E$. Clearly, $L \cap V_1 \neq 0$ as it contains C_1.

For $n \geq 1$, assume that we have a sequence $\{V_j\}$, $1 \leq j \leq n$, of submodules of E with the above stated properties. Since $x \notin M^{(\lambda)}$, $L = xR \nsubseteq \bigoplus_{i=1}^n V_i = \ker(\pi_n)$, if $x \in \bigoplus_{i=1}^n V_i$, then $V_1 \oplus ... \oplus V_n \oplus (\bigoplus_{i\in\mathcal{I}_n} W_i) = \bigoplus_{i\in\mathcal{I}_0} W_i$ implies that x belongs to $\bigoplus_{i\in\mathcal{I}_0} W_i$ and hence to $M^{(\lambda)}$, a contradiction. So $\pi_n(L) \neq 0$. Now $\bigoplus_{i\in\mathcal{I}_n} W_i \subset_e E_n$ and because $\pi_n : E \longrightarrow E_n$, we have $\pi_n(L) \cap (\bigoplus_{i\in\mathcal{I}_n} W_i) \neq 0$. So $\pi_n(L) \cap (\bigoplus_{i\in\mathcal{I}_n} W_i)$ contains a nonzero cyclic uniform submodule, say, C_{n+1}. This implies, there exists a finite subset $\mathcal{K}_{n+1} \subseteq \mathcal{I}_n$ such that $C_{n+1} \subseteq \bigoplus_{i\in\mathcal{K}_{n+1}} W_i$. Let V_{n+1} be an essential closure of C_{n+1} in $\bigoplus_{i\in\mathcal{K}_{n+1}} W_i$. Since $\bigoplus_{i\in\mathcal{K}_{n+1}} W_i$ is injective, V_{n+1} is injective. So, $\bigoplus_{i\in\mathcal{K}_{n+1}} W_i = V_{n+1} \oplus D_{n+1}$ for some submodule D_{n+1} of $\bigoplus_{i\in\mathcal{K}_{n+1}} W_i$. Since V_{n+1} is injective, it has the exchange property. Therefore, $\bigoplus_{i\in\mathcal{K}_{n+1}} W_i = V_{n+1} \oplus (\bigoplus_{i\in\mathcal{K}_{n+1}} W_i^{'})$ for some submodules $W_i^{'}$ of W_i. Since $W_i^{'}$ are injective and each W_i is indecomposable, either $W_i^{'} = 0$ or $W_i^{'} = W_i$. Again note that V_{n+1} is uniform because it is the closure of the uniform module C_{n+1}. Comparing the Goldie dimension on each side of $\bigoplus_{i\in\mathcal{K}_{n+1}} W_i = V_{n+1} \oplus (\bigoplus_{i\in\mathcal{K}_{n+1}} W_i^{'})$, we get that there exists exactly one index, say $i_{n+1} \in \mathcal{K}_{n+1}$, such that $W_{i_{n+1}}^{'} = 0$, and for all $i(\neq i_{n+1}) \in \mathcal{K}_{n+1}$, $W_i^{'} = W_i$. So, $\bigoplus_{i\in\mathcal{K}_{n+1}} W_i = V_{n+1} \oplus (\bigoplus_{i\in\mathcal{K}_{n+1}\setminus\{i_{n+1}\}} W_i)$. This yields $V_{n+1} \cong (\bigoplus_{i\in\mathcal{K}_{n+1}} W_i)/(\bigoplus_{i\in\mathcal{K}_{n+1}\setminus\{i_{n+1}\}} W_i) \cong W_{i_{n+1}}$. Also, we get $V_{n+1} \oplus (\bigoplus_{i\in\mathcal{K}_{n+1}\setminus\{i_{n+1}\}} W_i) \oplus (\bigoplus_{i\in\mathcal{I}_n\setminus\mathcal{K}_{n+1}} W_i) = (\bigoplus_{i\in\mathcal{K}_{n+1}} W_i) \oplus (\bigoplus_{i\in\mathcal{I}_n\setminus\mathcal{K}_{n+1}} W_i)$. This yields $V_{n+1} \oplus (\bigoplus_{i\in\mathcal{I}_{n+1}} W_i) = \bigoplus_{i\in\mathcal{I}_n} W_i$. Taking injective hulls of both sides, we get $E_{n+1} \oplus V_{n+1} = E_n$. Thus, we have $E = E_{n+1} \oplus (\bigoplus_{k=1}^{n+1} V_k)$. Note that $\pi_n(L) \cap V_{n+1} \neq 0$ as it contains C_{n+1}. Thus, we have obtained a sequence of submodules $\{V_j\}$, $j = 1, 2, ...$, with the required properties. This completes the induction argument.

Now we claim that there exists a properly ascending chain $N_0 \subset N_1 \subset ... \subset N_j \subset ...$ of submodules of L such that $N_0 = 0$ and $E(N_j/N_{j-1}) \cong V_j$ for all $j \geq 1$.

Set $N_j = L \cap (V_1 \oplus ... \oplus V_j)$. Clearly, $N_0 \subseteq N_1 \subseteq ... \subseteq N_j \subseteq ....$ Since $N_j \cap \ker(\pi_{j-1}) = N_{j-1}$, we have $N_j/N_{j-1} \cong \pi_{j-1}(N_j)$. If $l \in N_j$, then $l = v_1 + ... + v_j$ with $v_i \in V_i$, so $\pi_{j-1}(l) = v_j$ and $v_j \in \pi_{j-1}(L) \cap V_j$. This shows that $\pi_{j-1}(N_j) \subseteq \pi_{j-1}(L) \cap V_j$. Conversely, if $v_j \in \pi_{j-1}(L) \cap V_j$, then $v_j = \pi_{j-1}(l)$ with $l \in L \cap (V_1 \oplus ... \oplus V_j) = N_j$, so $v_j \in \pi_{j-1}(N_j)$. Therefore $\pi_{j-1}(N_j) = \pi_{j-1}(L) \cap V_j \neq 0$. Because $\pi_{j-1}(N_{j-1}) = 0$ and $\pi_{j-1}(N_j) \neq 0$, it follows that $N_{j-1} \subsetneq N_j$. Since $N_j/N_{j-1} \cong \pi_{j-1}(N_j) = \pi_{j-1}(L) \cap V_j$, we have $E(N_j/N_{j-1}) \cong V_j$.

Since $\{V_j : j \in \mathbb{N}\}$ is an independent family of uniform injective modules isomorphic to submodules of M, by the above lemma there exists an infinite subset $\mathcal{J} \subseteq \mathbb{N}$ such that $\bigoplus_{j\in\mathcal{J}} V_j$, and hence $\bigoplus_{j\in\mathcal{J}} E(N_j/N_{j-1})$ is injective. Set $N = \bigcup_{j\in\mathcal{J}} N_j$. Given $j \in \mathcal{J}$, the canonical map $N_j \longrightarrow N_j/N_{j-1} \subset E(N_j/N_{j-1})$ induces a map $\alpha_j : N \longrightarrow E(N_j/N_{j-1})$. Let $\alpha : N \longrightarrow \bigoplus_{j\in\mathcal{J}} E(N_j/N_{j-1})$ be defined by $\alpha(x) = \{\alpha_j(x)\}_{j\in\mathcal{J}}$ for all $x \in N$. Since $\bigoplus_{j\in\mathcal{J}} E(N_j/N_{j-1})$ is injective, we may extend α to $\alpha^* : L \longrightarrow \bigoplus_{j\in\mathcal{J}} E(N_j/N_{j-1})$. As L is finitely generated, there exists a finite subset $\mathcal{K} \subseteq \mathcal{J}$ such that $\alpha^*(L) \subseteq \bigoplus_{k\in\mathcal{K}} E(N_k/N_{k-1})$. For $j \in \mathcal{J}\setminus\mathcal{K}$ and $x \in N_j$ we have $0 = \alpha_j(x) = x + N_{j-1}$, showing that $N_{j-1} = N_j$, a contradiction.

Therefore, $M^{(\lambda)}$ is injective for any cardinal λ, and hence M is Σ-injective.

(a) $\Longrightarrow$ (b) is obvious.

This completes the proof of Theorem 2. $\square$

As a consequence of Theorem 2, we have the following characterization for a right noetherian ring.

Theorem 4. *Let R be a ring. Then the following are equivalent:*

(i) *R is right noetherian.*

(ii) *For each injective module M_R, every essential extension of $M^{(\aleph_0)}$ is a direct sum of injective modules.*

Proof. (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (i) follows from Theorem 2 and by Faith-Walker [5] that a ring R is right noetherian if and only if every injective right R-module is Σ-injective. $\square$

Remark 5. The above result generalizes a result of Beidar-Ke [2] which states that a ring R is right noetherian if and only if every essential extension of a direct sum of injective right R-modules is again a direct sum of injective right R-modules. Note that [2] indeed generalizes a result of Bass [1] that a ring is right noetherian if and only if every direct sum of injective modules is injective.

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Department of Mathematics, Ohio University, Athens, Ohio 45701

Department of Mathematics, Ohio University, Athens, Ohio 45701
E-mail address: jain@math.ohiou.edu

Department of Mathematics and Computer Science, St. Louis University, St. Louis, Missouri 63103
E-mail address: asrivas3@slu.edu