Tensor Representation of Color Images and Fast 2-D Quaternion Discrete Fourier Transform

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ABSTRACT
In this paper, a general, efficient, split algorithm to compute the two-dimensional quaternion discrete Fourier transform (2-D QDFT), by using the special partitioning in the frequency domain, is introduced. The partition determines an effective transformation, or color image representation in the form of 1-D quaternion signals which allow for splitting the \( N \times M \)-point 2-D QDFT into a set of 1-D QDFTs. Comparative estimates revealing the efficiency of the proposed algorithms with respect to the known ones are given. In particular, a proposed method of calculating the \( 2^r \times 2^r \)-point 2-D QDFT uses \( 18N^2 \) less multiplications than the well-known column-row method and method of calculation based on the symplectic decomposition. The proposed algorithm is simple to apply and design, which makes it very practical in color image processing in the frequency domain.

Keywords: Discrete quaternion Fourier transform, 2-D Fourier transform, 2-D discrete tensor transform, tensor representation of the image.

1. INTRODUCTION

The Fourier analysis has become one of the most frequently used tools in signal and image processing\(^1\)\(^-\)\(^9\),\(^21\)\(^-\)\(^26\). For color images the traditional approach of processing images in the frequency domain is reduced to processing each color channel separately. After introducing the quaternion Fourier transform by Ell,\(^11\) it becomes clear that the discrete quaternion Fourier transform is well-suited for color image processing applications, since it processes all three color components (R,G,B) simultaneously, it captures the inherent correlation between the components, it does not generate color artifacts or blending, and it does not need an additional color restoration process.

In this paper we describe and analyze different methods of calculation of the 2-D QDFT\(^11\)\(^-\)\(^14\) and present a new approach in processing the color images in the frequency domain. Between two different spatial and frequency spaces exists an intermediate space, the so-called frequency-and-time space, or representation of the image in the form of one-dimensional signals which are generated by a specific set of frequencies\(^19\),\(^31\)\(^-\)\(^39\). We introduce one of such representation for color images, which is called the color tensor representation when three components of the image in the RGB space are described by one dimensional signal in the quaternion algebra. The tensor representation is effective, since it allows us to process the color image by 1-D quaternion signals which can be processed separately. This representation also splits the algebraic structure of the 2-D QDFT, since each of the quaternion signals defines the 2-D QDFT in the corresponding subset of frequency-points. These subsets cover the entire Cartesian lattice of frequency-points on which the 2-D QDFT is defined. Therefore the 1-D quaternion signals are called the splitting-signals of the color image in tensor representation. The tensor transform-based algorithm of the 2-D QDFT is effective and simple, and uses less number of multiplications than the known methods. For instance, the tensor algorithm for the \( 2^r \times 2^r \)-point 2-D QDFT uses \( 18N^2 \) less multiplications than the well-known method of calculation based on the symplectic decomposition\(^16\). The proposed algorithm is simple to apply and design, which makes it very practical in color image processing in the frequency domain.
2. THE QUATERNION NUMBERS IN IMAGING

In recent years, quaternions\textsuperscript{10} have been utilized more and more in color image processing\textsuperscript{11,15,20,44}. The quaternion can be considered as a four-dimensional generation of a complex number with one real part and three imaginary parts. The imaginary dimensions are represented as \(i, j,\) and \(k\). In practice, the \(i, j,\) and \(k\) are orthogonal to each other and to the real numbers. Any quaternion may be represented in a hyper-complex form as

\[
Q = a + bi + cj + dk = a + (bi + cj + dk),
\]

where \(a, b, c,\) and \(d\) are real numbers and \(i, j,\) and \(k\) are three imaginary units with multiplication laws:

\[
i j = -ji = k, \quad jk = -kj = i, \quad ki = -ik = -j, \quad i^2 = j^2 = k^2 = ijk = -1.
\]

The number \(a\) is considered to be the real part of \(Q\) and \((bi + cj + dk)\) is the “imaginary” part of \(Q\). The quaternion conjugate and modulus of \(Q\) equal \(\bar{Q} = a - (bi + cj + dk)\) and \(|Q| = \sqrt{a^2 + b^2 + c^2 + d^2}\), respectively. The property of commutativity does not hold in quaternion algebra. When multiplying quaternion numbers, it should be noted that commutativity does not hold in quaternion algebra, i.e., \(Q_1Q_2 \neq Q_2Q_1\). The quaternion also can be represented in classic polar form as \(Q = |Q| \exp(\mu \vartheta)\), where \(\mu\) is a unit pure quaternion \(\mu = i\mu_i + j\mu_j + k\mu_k\), such that \(|\mu| = 1, \mu^2 = -1,\) and \(\vartheta\) is a real angle in the interval \([0, \pi]\). For instance, the number \(\mu = (i + j + k)/\sqrt{3}\) is the unit pure quaternion. The exponential number is defined as \(\exp(\mu \vartheta) = \cos(\vartheta) + \mu \sin(\vartheta)\).

### 2.1 Color image models: RGB color space

We consider the RGB model of colors that are used widely in color imaging.\textsuperscript{1} A quaternion number has four components, a real part and three imaginary parts, which naturally coincides with the three components, \(R(ed), G(reen),\) and \(B(lue)\) of a color pixel. Therefore, a discrete color image \(f(n, m)\) in the RGB color space can be transformed into imaginary part of quaternion numbers form by encoding the red, green, and blue components of the RGB value as a pure quaternion (with zero real part):

\[
f_{n, m} = 0 + (r_{n, m}i + g_{n, m}j + b_{n, m}k).
\]

Figure 1 shows the color map of the colors \((r, g, b)\) in the quaternion space \((1, i, j, k)\). In this additive model, colors are produced by adding components \(C = rR + gG + bB\) and reproduce a broad range of colors. Practically, the color is expressed as the triplet \((R, G, B)\), each component of which can vary from zero to a defined maximum value. The name of the model comes from the initials of the three primary colors, red, green, and blue, each of which stimulates one of the three types of the eye’s color receptors with as little stimulation as possible of the other two. The RGB model is mostly used for recording colors in digital cameras/scanners, including still image and video cameras. The RGB model is the most common way to encode color in computing, and several different binary digital representations are in use.

The advantage of using quaternion based operations to manipulate color information in an image is that we do not have to process each color channel independently, but rather, treat each color triple as a whole unit. We believe, by using quaternion operations, higher color information accuracy can be achieved because a color is treated as an entity. Another advantage of the use of quaternion-type representation is that a color image can be treated as a vector field or the hyper-complex Fourier transforms can handle color image pixels as vectors.
and thus offer scope to process color images holistically; rather than as separated luminance and chrominance, or separate color space components (example: red, green, blue).

The idea of computing the Fourier transform of a color image has only recently been realized. As the generalization of the traditional Fourier transform, the quaternion Fourier transform was first defined by Ell to process quaternion signals. Later, some practical works related to the discrete quaternion Fourier transform (DQFT) and its application in color image processing are presented in and.

3. CALCULATION OF THE 1-D QDFT

In this section, we describe briefly the algorithm for calculating the 1-D quaternion Fourier transform. Let \( f_n = (a_n, b_n, c_n, d_n) = a_n + ib_n + jc_n + kd_n \) be the quaternion signal of length \( N \). The 1-D quaternion DFT (QDFT) is defined as

\[
F_p = \sum_{n=0}^{N-1} f_n W_n^p, \quad p = 0: (N-1),
\]

where the quaternion unit number \( \mu = m_1i + m_2j + m_3k, \mu^2 = -1 \). The kernel of the transformation equals

\[
W_n = W_{N;\mu} = \exp(-\mu 2\pi/N) = \cos(2\pi/N) - \mu \sin(2\pi/N).
\]

The direct calculations show, that the multiplication of quaternion numbers \( f_n = a_n + ib_n + jc_n + kd_n \) and \( \exp(-\mu \phi) \) can be written as

\[
f_n \exp(-\mu \phi) = a_n \cos(\phi) + (b_n m_1 + c_n m_2 + d_n m_3) \sin(\phi) \\
+ i \left[ b_n \cos(\phi) - (a_n m_1 - d_n m_2 + c_n m_3) \sin(\phi) \right] \\
+ j \left[ c_n \cos(\phi) - (d_n m_1 + a_n m_2 - b_n m_3) \sin(\phi) \right] \\
+ k \left[ d_n \cos(\phi) - (-c_n m_1 + b_n m_2 + a_n m_3) \sin(\phi) \right].
\]

(1)

If we denote the \( N \)-point 1-D DFTs of the parts \( a_n, b_n, c_n, \) and \( d_n \) of the quaternion signal \( f_n \) by \( A_p, B_p, C_p, \) and \( D_p \), respectively, we can calculate of the 1-D QDFT as

\[
F_p = Q_1(p) + iQ_i(p) + jQ_j(p) + kQ_k(p), \quad p = 0: (N - 1).
\]

with components calculated in the following way:

\[
Q_1(p) = \text{real}(A_p) + m_1 \text{imag}(B_p) + m_2 \text{imag}(C_p) + m_3 \text{imag}(D_p), \\
Q_i(p) = -m_1 \text{imag}(A_p) + \text{real}(B_p) - m_3 \text{imag}(C_p) + m_2 \text{imag}(D_p), \\
Q_j(p) = -m_2 \text{imag}(A_p) + m_3 \text{imag}(B_p) + \text{real}(C_p) - m_1 \text{imag}(D_p), \\
Q_k(p) = -m_3 \text{imag}(A_p) - m_2 \text{imag}(B_p) + m_1 \text{imag}(C_p) + \text{real}(D_p).
\]

(2)

If the real part is zero, \( a_n = 0 \), and \( f_n = 0 + ib_n + jc_n + kd_n \), the number of operations of multiplication and addition can be estimated as \( m_{QFT}(N) = 3m_{FT}(N) + 9N \) and \( a_{QFT}(N) = 3a_{FT}(N) + 9N \), respectively. Here, \( m_{FT} \) and \( a_{FT} \) are the number of operations of multiplication and addition in the \( N \)-point DFT of the real signal. Thus, per one sample, the 1-D QDFT of this signal uses nine more operations multiplication and addition, than the 1-D DFTs of three signals \( b_n, c_n, \) and \( d_n \), when calculates separately. As an example, Figure 2 shows separately the three components of the “color” signal \( f_n = 0 + ib_n + jc_n + kd_n \). The components of the imaginary parts \( b_n, c_n, \) and \( d_n \) are shown in parts a, b, and c, respectively. Each signal is of length \( N = 512 \).

The 1-D Fourier transforms of the signals parts \( b_n, c_n, \) and \( d_n \) are shown in parts a, b, and c in Figure 3, respectively. Thus, the vector signal \( \{(b_n, c_n, d_n); n = 0 : 511\} \) is represented by three spectral characteristics in the frequency domain. The same signal in the quaternion algebra, \( f_n = 0 + ib_n + jc_n + kd_n \) is represented by four spectral characteristics in the frequency domain in the cost of \( 9N \) operations of multiplication and addition. Figure 4 shows four components of the \( N \)-point 1-D QDFT in parts a, b, c, and d. These four characteristics are real, and the three characteristics in 3 are complex.
As follows from (2), in the general case of the quaternion signal \( f_n \), the number of operations of multiplication and addition can be estimated as

\[
4 \times \mu_{FT}(N) = 4 \times [2^{r-1}(r - 3) + 2] \quad \text{and} \quad 2 \times \alpha_{FT}(N) = 2 \times [(2^r6 - r^2 - 3r - 6) + \mu_{FT}(N)].
\] (3)

For fast \( N \)-point discrete paired FFT transform, the estimation for multiplications is \( \mu_{FT}(N) = 2^{r-1}(r - 3) + 2 \) and \( \alpha_{FT}(N) = (2^r6 - r^2 - 3r - 6) \) is the number of additions. Here, we assume that the complex multiplication is performed with two additions and four multiplications. The calculation of two \( N \)-point DFTs over the real signals can be reduced to one \( N \)-point DFT of a complex signal. Therefore, we consider that

\[
2m_{FT}(N) = 4 \times \mu_{FT}(N) \quad \text{and} \quad 2a_{FT}(N) = 2 \times \alpha_{FT}(N).
\]

The number of operations for calculating the \( N \)-point DFT of the real signal is twice less than the number of operations for the transform of complex signal. Two 1-D DFTs with real inputs can be calculated by one DFT with complex input. The number of operations for the 1-D QDFT can be estimated as

\[
\begin{align*}
m_{QFT}(N) &= 8\mu_{FT}(N) + 12N = 4Nr + 16, \\
a_{QFT}(N) &= 4\alpha_{FT}(N) + 12N = 2N(r + 15) - 4(r^2 + 3r + 4).
\end{align*}
\] (4)

4. TENSOR REPRESENTATION OF GRAY-SCALE IMAGE

In this section, we consider the tensor representation of the discrete image \( \{f_{n,m}\} \) of size \( N \times N \), for the case when \( N = 2^r \), and \( r > 1 \) is an integer. In tensor representation, which is the 2-D frequency and 1-D time
representation of the 2-D image, the image is represented and processed by one-dimensional signal of length \( N \) each. The 2-D discrete Fourier transform (DFT) of the image is defined as

\[
F_{p,s} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f_{n,m} W^{np+ms}, \quad p, s = 0 : (N - 1),
\]

with the exponential kernel \( W = W_N = \exp(-2\pi j/N) \). The signals which describe the image, or the set of 1-D splitting-signals is defined as

\[
\chi : \{f_{n,m}\} \rightarrow \{f_{T_{p,s}} = \{f_{p,s,t}; t = 0 : (N - 1)\}\}_{(p,s) \in J_{N,N}}.
\]

The components \( f_{p,s,t} \) are periodic by \( t \), i.e., \( f_{p,s,t+N} = f_{p,s,t} \), for \( t = 0 : (N - 1) \).

For a given frequency-point \((p, s)\), the components of the splitting-signal \( f_{T_{p,s}} \) are calculated by the sums of the image \( f_{n,m} \) along the parallel lines on the Cartesian lattice \( X_{N,N} = \{(n, m); n, m = 0 : (N - 1)\} \),

\[
f_{p,s,t} = \sum_{(n,m) \in X} \{f_{n,m}; np + ms = t \text{ mod } N\}.
\]

The following holds for the splitting-signal:

\[
F_{kp \mod N, ks \mod N} = \sum_{t=0}^{N-1} f_{p,s,t} W^{kt}, \quad k = 0 : (N - 1).
\]

The signal carries the information about the 2-D DFT at \( N \) frequency-points of the set

\[
T_{p,s} = \{(kp \mod N, ks \mod N); k = 0 : (N - 1)\}.
\]

Therefore, we denote the splitting-signal by this set. The set \( J_{N,N} \) of frequency-points \((p, s)\), or generators, of the splitting-signals is selected in a way that covers the Cartesian lattice \( X_{N,N} \) with a minimum number of subsets \( T_{p,s} \). In other words, the sets cover the lattice with minimum number of intersections. This set \( J_{N,N} \) contains \( 3N/2 \) generators and can be defined as

\[
J_{N,N} = \{(1, s); s = 0 : (N - 1)\} \cup \{(2p, 1); p = 0 : (N/2 - 1)\}.
\]

As an example, Figure 5 shows the gray-scale image of size \( 512 \times 512 \) in part a, and the splitting-signal \( \{f_{4,1,t}; t = 0 : 511\} \) in b. This signal is generated by the frequency-point \((p, s) = (4, 1)\). The 512-point 1-D DFT of this signal in absolute scale is shown in c. The 1-D DFT is equals to the part of the 2-D DFT of the image in the frequency-points of the set \( T_{4,1} \). The location of 512 frequency-points of this set is shown in d.

5. THE 2-D DQFTS

The quaternion multiplication is not commutative and the definition of the 2-D DQFT is not unique,\(^{12–14}\) Different DQFT can be used in image processing, including the two-side and left-side DQFTs. These two transforms are described similarly. Therefore, we consider the right-side 2-D DQFT. The color image \( f_{n,m} \) is considered to be of size \( N \times M \times 3 \), and in the quaternion space this image is associated with a multi-dimension image \( N \times M \times 4 \) each component of which is a pure quaternion.

5.1 The right-side 2-D QDFTs

The definitions of the 2-D QDFTs have been originated from the traditional 2-D DFT over the real image \( f_{n,m} \), which is defined as

\[
F_{p,s} = \sum_{n=0}^{N-1} \left( \sum_{m=0}^{M-1} f_{n,m} W_M^{ms} \right) W_N^{np}, \quad p = 0 : (N - 1), \quad s = 0 : (M - 1),
\]
where $W_N = \exp(-2\pi j/N)$ and $W_M = \exp(-2\pi j/M)$. We define $N_0$ the g.c.d.$(N,M)$ and $N = N_0 N_1, M = N_0 M_1$, and $K = M_1 N = N_1 M$, where integers $N_1, M_1 \geq 1$. The 2-D DFT can be written in different forms because the exponential kernel $W_M^{n}W_N^{m} = W_K^{M_1np+N_1ms}$ and the complex algebra is commutative,

$$F_{p,s} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{n,m}W_K^{M_1np+N_1ms} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} W_K^{M_1np+N_1ms}f_{n,m} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} W_N^{np}f_{n,m}W_M^{ms}.$$ 

With the first form of the 2-D DFT, the following generalized quaternion Fourier transform is defined\textsuperscript{14}:

$$F_{p,s} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{n,m}W_{\mu}^{M_1np+N_1ms}, \quad p = 0 : (N - 1), \quad s = 0 : (M - 1).$$

Here, for a given quaternion unit number $\mu$, the basis exponential function is $W_{\mu} = W_{\mu;K} = \exp(-2\pi \mu/K) = \cos(2\pi/K) - \mu \sin(2\pi/K)$. This is the right-side 2-D QDFT. The inverse right-side 2-D QDFT is calculated by

$$f_{n,m} = \frac{1}{NM} \sum_{p=0}^{N-1} \sum_{s=0}^{M-1} F_{p,s}W_{\mu}^{-(M_1np+N_1ms)}, \quad n = 0 : (N - 1), \quad m = 0 : (M - 1).$$

For the simplicity of calculation, we consider the $N = M$ case, i.e., when $K = N$ and the 2-D $N \times N$-point QDFT of the color-in-quaternion image $f_{n,m}$ is

$$F_{p,s} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{n,m}W_{\mu}^{np+ms}, \quad p, s = 0 : (N - 1).$$
5.1.1 Column-row wise calculation of the 2-D QDFT

The calculation of the separable 2-D QDFT by the formula

\[ F_{p,s} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f_{n,m} W_n^m W_m^s, \quad p, s = 0 : (N - 1), \]

requires \(2N\) \(N\)-point 1-D QDFTs. Each of the 1-D QDFT requires two \(N\)-point 1-D DFTs. Therefore, the column-row method uses \(4N\) \(N\)-point 1-D DFTs and the multiplications in number

\[ m_{QFT}(N, N) = 8N^2 \log_2 N + 32N. \]

It is not difficult to show, that the 2-D DFT of the complex image can be calculated by using the number of operations of multiplication in number

\[ m_{DFT}(N, N) = 4(2Nm_{DFT}(N)) = 8Nm_{DFT}(N) = 8N(N/2(\log_2 N - 3) + 2) = 4N^2(\log_2 N - 3) + 16N \]

when using the traditional column-row method with fast paired transform-based 1-D DFT. Thus, the number of multiplications in the \(N \times N\)-point 2-D QDFT is almost two times larger than in the \(N \times N\)-point 2-D DFT. For comparison, we consider the known method of calculation the 2-D QDFT by using the symplectic decomposition of the color image\cite{16,45}. We denote by \(m_{DFT}(N, N)\) the number of real multiplications in the complex \(N \times N\)-point 2-D DFT. If estimate \(m_{DFT}(N, N)\) by the column-row wise method and 1-D paired transform, i.e., when

\[ m_{DFT}(N, N) = 4N^2(\log_2 N - 3) + 16N, \]

we obtain the following estimation:

\[ m_{QFT}(N, N) = 18N^2 + 2[4N^2(\log_2 N - 3) + 16N] = 8N^2 \log_2 N - 6N^2 + 32N. \]  \hspace{1cm} (6)

The symplectic decomposition saves \(6N^2\) real multiplications in the right-side 2-D QDFT when comparing with the simple column-row method.

5.1.2 Tensor Representation of the ride-side 2-D QDFT

The kernel of the transform is a periodic function,

\[ W_\mu^{t+N} = \cos(2\pi(t + N)/N) - \mu \sin(2\pi(t + N)/N) = \cos(2\pi t/N) - \mu \sin(2\pi t/N) = W_\mu^t, \]

when \(t = 0 : (N - 1)\). Therefore, for calculating the right-side 2-D QDFT, we can apply the concept of the tensor representation\cite{3,17,18,33,34,38} and reduce the calculation and processing of this transform to processing of 1-D signals, which we call the color splitting-signals. To apply this concept to the quaternion image, we first consider the gray-scale image \(f_{n,m}\). The tensor representation of the discrete image is the 2-D frequency and 1-D time representation, when the image is described by a set of 1-D splitting-signals each of length \(N\),

\[ \chi: \{f_{n,m}\} \rightarrow \{f_{T_{p,s}} = \{f_{p,s,t}; t = 0 : (N - 1)\}\}_{(p,s) \in J_{N,N}}. \]

The components \(f_{p,s,t}\) are periodic by \(t\), i.e., \(f_{p,s,t+N} = f_{p,s,t}\). The set \(J_{N,N}\) of frequency-points \((p,s)\), or generators, of the splitting-signals is selected in a way that covers the Cartesian lattice \(X_{N,N}\) with a minimum number of subsets \(T_{p,s}\). The tensor representation is unique, and the image can be defined through the 2-D DFT calculated by (5), or directly from the tensor transform, as shown in\cite{2,17,28}. Since \(3N/2\) splitting-signals compose the tensor representation, the calculation of the 2-D DFT is reduced to \(3N/2\) one-dimensional \(N\)-point DFT, instead of \(2N\) such transformation in the traditional row-column method. It should be noted, that the total number of components of \(3N/2\) splitting-signals equals \(N^2 + N^2/2\), which exceeds the number \(N^2\) of points in the image. On the other side, many sets \(T_{p,s}\), with generators \((p,s) \in J_{N,N}\) have intersections at frequency-points. The tensor transform is therefore redundant. This redundancy can be removed and the 2-D DFT can be calculated in a more effective way when using the modification of the tensor representation, which is called
the paired transform$^{3,18}$. The tensor representation is very effective in another case of most interest when $N$ is a prime, since the number of required sets $T_{p,s}$ is $N + 1$ and the sets intersect only at the point $(0,0)$. Now, we implement the concept of the tensor representation for the quaternion images.

We consider the color image
\[ f_{n,m} = (r_{n,m}i + g_{n,m}j + b_{n,m}k) \]
with zero real part in the quaternion algebra. For a given generator $(p,s)$ from the set $J_{N,N}$, we define the following $N$ subsets in the Cartesian lattice:
\[ V_{p,s,t} = \{(n,m) \in X; np + ms = t \mod N\}, \quad t = 0 : (N - 1). \]

The quaternion splitting-signal $f_{p,s,t} = \{f_{p,s,t}; t = 0 : (N - 1)\}$ is calculated as
\[ f_{p,s,t} = (r_{p,s,t})i + (g_{p,s,t})j + (b_{p,s,t})k = \sum_{(n,m) \in V_{p,s,t}} f_{n,m} = \left( \sum_{(n,m) \in V_{p,s,t}} r_{n,m} \right)i + \left( \sum_{(n,m) \in V_{p,s,t}} g_{n,m} \right)j + \left( \sum_{(n,m) \in V_{p,s,t}} b_{n,m} \right)k. \quad (7) \]

The imaginary components of the splitting-signals are the splitting-signals of the red, green, and blue channels of the color image. The right-side 2-D QDFT at $N$ frequency-points of the cyclic group $T_{p,s}$ is defined by the 1-D QDFT of the splitting-signal,
\[ F_{kp \mod N, ks \mod N} = \sum_{t=0}^{N-1} f_{p,s,t} W_{k}^{nt}, \quad k = 0 : (N - 1). \quad (8) \]

The union of the family of disjoint subsets $V_{p,s,t}$, $t = 0 : (N - 1)$, is the Cartesian grid and, therefore, the following calculations hold:
\[ \sum_{t=0}^{N-1} f_{p,s,t} W_{k}^{nt} = \sum_{t=0}^{N-1} \sum_{np+ms=t+1 \mod N} (r_{n,m})i + (g_{n,m})j + (b_{n,m})k W_{k}^{nt} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f_{n,m} W_{k}^{n(kp+ms)} = F_{kp \mod N, ks \mod N}. \]

In the $N = 2^r$ case when $r > 1$, the color image is described by $3N/2$ quaternion splitting-signals and 2-D QDFT of the image is split by the $3N/2$ 1-D QDFT of these signals. In the case when $N$ is prime, the number of such signals and 1-D QDFT equals $(N + 1)$, because the set of generators can be taken as
\[ J_{N,N} = \{(1,s); s = 0 : (N - 1)\} \cup \{(0,1)\}. \]

As an example, Figure 6 shows the color image of size $512 \times 512$ in part a, and splitting-signals of the red, green, and blue channels, which are calculated for the frequency-point $(p,s) = (3,1)$. The set of 768 = $(512+256)$ such triplet splitting-signals of length 512 each describe the image as well as its 2-D QDFT. The splitting-signal $f_{3,1,t} = (r_{3,1,t})i + (g_{3,1,t})j + (b_{3,1,t})k$ is referred to as the color splitting-signal in the quaternion space. The 512-point 1-D QDFT of this splitting-signal is shown in Figure 7 in part a in absolute scale (and with the normalized factor of $1/N^2$). The amplitude spectra of the 2-D QDFT of the color image is shown in part b, and in c, this spectrum is shifted to the center. The 1-D QDFT of the color splitting-signal coincides with the 2-D QDFT of the image at frequency-points of the set $T_{3,1}$ which are shown in part b. All quaternion DFTs were calculated for the unit $\mu = (i + j + k)/\sqrt{3}$. When $N$ is prime, the number of $N$-point 1-D QDFTs required to calculate the $N \times N$-point 2-D QDFT equals $N + 1$. For instance, if we consider the above Mini-Anoush-Mini image inside the matrix $521 \times 521$, the number of splitting signal in tensor representation for the image will have only 522 splitting-signals of length 521 each. For comparison, the original image of size $512 \times 512$ has 768 splitting-signals.
The representation of the color image by the set of such direction images or components removes redundancy of the tensor representation by introducing the new set of color direction images as
denoted by

\[ d_{n,m} = d_{n,m;p,s} = \frac{1}{N} f_{p,s,(np+ms)} \mod N \quad n, m = 0 : (N - 1). \]  

As an example, Figure 8 show the color image in part a, and the direction images generated by seven different frequency-points \((p, s)\) in parts b to h. Both direction images were amplified by the factor of 2. By using all such direction images, we can calculate the original color image. Indeed, as shown in\(^{18,27,30}\), we can modify and remove redundancy of the tensor representation by introducing the new set of color direction images as

\[ d'_{n,m,p,s} = f'_{p,s,(np+ms)} \mod N = f_{p,s,(np+ms)} \mod N - f_{p,s,(np+ms+N/2)} \mod N. \]

The representation of the color image by the set of such direction images or components \(f'_{p,s,(np+ms)} \mod N\) can be referred to as the paired representation, similar to the paired representation for gray-scale images (see for more details\(^{3,4,17}\)). It is interesting to note, that the sum of the new direction images equals the image \(f_{n,m}\).

**Statement 1:** The discrete image of size \(N \times N\), where \(N = 2^r\), \(r > 1\), can be composed from its \((3N - 2)\) color direction images as

\[ f_{n,m} = \sum_{(p,s) \in J_{N,N}} d'_{n,m;p,s} = \frac{1}{2N} \sum_{k=0}^{r-1} \frac{1}{2^k} \sum_{(p,s) \in 2^k J_{N/2^k,N/2^k}} f'_{p,s,(np+ms)} \mod N + \frac{1}{N^2} f_{0,0,0}. \]

The last addendum in the formula represents the mean value of the image \(E[f]\). The set of generators of these direction images is calculated as

\[ J'_{N,N} = J_{N,N} \cup 2J_{N/2,N/2} \cup 4J_{N/4,N/4} \cup \cdots \cup N/2J_{1/2,1/2} \cup \{(0,0)\}. \]
Figure 8. (a) The color image and direction images calculated from the splitting-signals generated by the frequencies (b) (1, 1), (c) (1, 3), (d) (1, 4), (e) (3, 1), (f) (0, 1), (g) (1, 0), and (h) (7, 1).

The tensor representation of the image is very effective for the case when $N$ is prime, because all $(N + 1)$ required groups $T_{p,s}$ intersect only at the point $(0, 0)$. For instance, $N = 521$ is prime and the sum of 522 direction images $\{d_{n,m,p,s}; (p, s) \in J_{521,521}\}$ similar to images in Figure 8 for the Mike-Anoush-Mini image of size $521 \times 521$ determines the original image. Therefore, as it directly follows from (8), the following statement holds.

Statement 2: The image $f_{n,m}$ of size $N \times N$, where $N > 2$ is a prime, can be reconstructed from $(N + 1)$ directional images as follows:

$$f_{n,m} = \sum_{(p,s) \in J_{N,N}} d_{n,m,p,s} = \frac{1}{N} \left[ \sum_{s=0}^{N-1} f_{1,s,(np+m) \mod N} + f_{0,1,n} \right], \quad n, m = 0 : (N - 1).$$  \hspace{1cm} (11)

The number of real multiplications for calculation of the 2-D QDFT by $(N + 1)$ quaternion $N$-point DFT is estimated by

$$m_{QFT}(N, N) = (N + 1)m_{QFT}(N) = 2(N + 1)m_{DFT}(N) \approx 2m_{DFT}(N, N).$$  \hspace{1cm} (12)

The 2-D QDFT with symplectic decomposition\cite{16,45} requires $18N^2 + 2m_{DFT}(N, N)$ multiplications, i.e., $18N^2$ more multiplications, than the method of tensor representation.

6. CONCLUSIONS

We presented a new concept of the tensor representation for color images in the quaternion algebra. By means of this representation, the processing of the color image in the frequency domain is reduced to calculation of the 1-D signals and the 2-D QDFT is calculated through the 1-D QDFTs. It was shown that the proposed algorithm leads to a more efficient performance of the 2-D quaternion DFT than the existent fast algorithms. The discrete color image and its quaternion Fourier transform was considered on the Cartesian lattice, and the presented concept of tensor representation can also be used for other lattices, such as hexagonal lattices, as in the case of gray-scale images.\cite{2,3,7,46}
REFERENCES


