Discretization of Natanzon potentials

Artur Ishkhanyan[1,2,3] **and Vladimir Krainov**[4]

[1]Institute for Physical Research, NAS of Armenia, Ashtarak 0203, Armenia
[2]Institute of Physics and Technology, National Research Tomsk Polytechnic University, Tomsk 634050, Russia
[3]Armenian State Pedagogical University, Yerevan 0010, Armenia
[4]Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Russia

We show that the Natanzon family of potentials is necessarily dropped into a restricted set of distinct potentials involving a fewer number of independent parameters if the potential term in the Schrödinger equation is proportional to an energy-independent parameter and if the potential shape is independent of both energy and that parameter. In the hypergeometric case only six such potentials exist, all five-parametric. Among these, only two (Eckart, Pöschl-Teller) are independent in the sense that each cannot be derived from the other by specifications of the involved parameters. Discussing the solvability of the Schrödinger equation in terms of the single-confluent Heun functions, we show that in this case there exist in total fifteen seven-parametric potentials, of which independent are nine. Six of the independent potentials present different generalizations of the hypergeometric or confluent hypergeometric ones, while three others do not possess hypergeometric sub-potentials. The result for the double- and bi-confluent Heun equations produces the three independent double- and five independent bi-confluent six-parametric Lamieux-Bose potentials, and the general five-parametric quartic oscillator potential for the tri-confluent Heun equation.

PACS numbers: 03.65.Ge Solutions of wave equations: bound states, 02.30.Ik Integrable systems, 02.30.Gp Special functions
Keywords: Stationary Schrödinger equation, integrable potentials, confluent Heun equations

1. Introduction

We consider the reduction of the stationary Schrödinger equation to a standard linear second-order ordinary differential equation having rational coefficients via transformation of the independent and dependent variables. If the coordinate transformation $z(x)$ is such that $dz/dx = \Pi_i (z-z_i)^{A_i}$ with constant z_i and A_i, the potentials derived in this way can be referred to as the Manning potentials who discussed the reduction of the Schrödinger equation to a rather large class of target equations by a transformation of this form involving three factors: $z'(x) = z^{A_1}(1-z)^{A_2}(z-z_3)^{A_3}$ [1]. Examples of potentials derived by the Manning scheme include the harmonic oscillator, inverse square and Coulomb potentials discussed by Schrödinger [2], Manning [1], Eckart [3], Rosen-Morse [4], Manning-Rosen [5], Hulthén [6], Woods-Saxon [7], Pöschl-Teller [8] Scarf [9], Kratzer [10], Morse [11], Morse-Stückelberg [12], Manning [13], and many others explored in the past [14-16]. Recent examples include, e.g., the reduction of the Schrödinger equation to the five Heun equations or their certain extensions [17-27].

A further specification of the potentials concerns the cases when the coordinate transformation does not depend on the energy. The potentials obeying this restriction may be referred to as the Natanzon potentials [16], who explored the general conditions for the potentials to be solvable, under this supposition, in terms of the ordinary and confluent hypergeometric functions. For the hypergeometric equation, Natanzon constructed a parametrically given potential with the coordinate transformation defined by the equation

$$z'(x) = \frac{z(1-z)}{\sqrt{r_0 + r_1 z + r_2 z^2}} \,. \tag{1}$$

The polynomial under the root can be written as r_0, $r_1(z-z_1)$ or $r_2(z-z_1)(z-z_2)$, hence, this transformation can be viewed as a Manning transformation with at most four factors. Since the potential is given in parametric form, some of its properties are not immediately seen.

In the present paper we show that if the potential in a coordinate system is proportional to a parameter which is independent of energy E and has a shape which is independent of both energy and that parameter, that is, the potential term in the Schrödinger equation is presented as $mV(x) = \mu S(x)$ with $\mu \neq \mu(E)$ and $S \neq S(\mu, E)$, then some parameters involved in a Natanzon transformation necessarily adopt discrete values from a restricted permissible set.

This specification concerns the roots of the polynomial $r(z)$, and it comes from the singularities of the target equation to which the Schrödinger equation is reduced. More specifically, the assertion is that $r_{0,1,2}$ should be such that the roots of the polynomial $r(z)$ coincide with the singularities of the target equation. In particular, in the hypergeometric case only the roots $z_{1,2} = 0,1$ are permitted. Then, there are six possibilities: $r(z) \sim 1, z, 1-z, z^2$, $z(1-z), (1-z)^2$. Because of the symmetry with respect to transposition $z \leftrightarrow 1-z$, the number of independent cases is reduced to four. Interestingly, it further turns out that there exist only two independent hypergeometric potentials which cannot be transformed into each other by specification of the remaining three involved parameters. As such independent potentials, one may choose the Eckart and Pöschl-Teller potentials derived, e.g., by $r = r_0$ and $r = r_1 z$, respectively. In the confluent hypergeometric case there are three independent potentials (Morse, harmonic oscillator, Kratzer), which are derived by choosing $r = r_0$, $r = r_1 z$ and $r = r_2 z^2$, respectively.

Mathematically, we first show that under the supposition that the parameter μ does not depend on energy and $S(x) \neq S(\mu, E)$, the parameters r_i cannot depend on μ. Note that for equations more general than the hypergeometric equations the polynomial $r(z)$ generally becomes of higher degree so that the number of involved parameters is increased (see below the examples of the single- and multiple-confluent Heun equations when the polynomials are of the fourth degree). It further follows that all the roots z_i of polynomial $r(z)$ should necessarily coincide with the (finite) singular points of the target equation to which the Schrödinger equation is reduced. Then, the coordinate transformation becomes of the form $z'(x) = z^{m_1}(z-1)^{m_2}(z-z_3)^{m_3}... = \Pi_i (z-z_i)^{m_i}$ with z_i being the singular points of the target equation. Finally, we show that the exponents m_i should be integers or half-integers.

Being the main message of the present paper, this result suggests an advanced technique for search for new potentials for which the Schrödinger equation is solvable in terms of a given special function. Applying the developed approach, we have recently introduced the inverse square root [23], the Lambert-W step [24] and singular [26] potentials which are solved in terms of the confluent hypergeometric functions, as well as the third independent Gauss hypergeometric potential [27] after those by Eckart and Pöschl-Teller.

Here we consider the reduction of the Schrödinger equation to the four confluent Heun equations [28,29]. The result for the single-confluent Heun equation is that there exist fifteen possible choices for the coordinate transformation each leading to a seven-parametric potential. Because of the symmetry of the confluent Heun equation with respect to the transposition $z \leftrightarrow 1-z$, however, the number of independent potentials is effectively reduced to nine. Five of these potentials present different generalizations of either ordinary or confluent hypergeometric potentials, and one of the potentials has hypergeometric sub-cases of both types. Unlike the hypergeometric case, the confluent Heun potentials in general cannot be transformed into each other by specification of the involved parameters. Among the three potentials that do not possess hypergeometric sub-cases, a potential is explicitly written through a coordinate transformation given in terms of the Lambert W-function [30], which is an implicitly elementary function also known as the product logarithm. Discussing the reducibility to the Mathieu equation, which is a particular case of the confluent Heun equation, we show that there exists only one such potential. We note that another Mathieu potential can be constructed by a transformation of non-Manning form.

Finally, discussing the multiple-confluent Heun equations, we show that the Schrödinger equation is reduced to the double-confluent Heun equation for three independent potentials, five independent potentials allow reduction to the bi-confluent Heun equation, and for the tri-confluent Heun equation there exists only one potential, the quartic oscillator. The double- and bi-confluent Heun potentials have been presented by Lamieux and Bose [15].

2. The Natanzon potentials

The one-dimensional stationary Schrödinger equation for a particle of mass m and energy E in a potential $V(x)$ is written as

$$\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}\left(E-V(x)\right)\psi=0. \tag{2}$$

Applying the transformation of the independent variable $z=z(x)$, this equation is rewritten for the argument z as

$$\psi_{zz}+\frac{\rho_z}{\rho}\psi_z+\frac{2m}{\hbar^2}\frac{E-V(z)}{\rho^2}\psi=0, \tag{3}$$

where (and hereafter) the lowercase Latin index denotes differentiation and $\rho=dz/dx$. Further transformation of the dependent variable $\psi=\varphi(z)\,u(z)$ reduces this equation to the following one for the new dependent variable:

$$u_{zz}+\left(\frac{2\varphi_z}{\varphi}+\frac{\rho_z}{\rho}\right)u_z+\left(\frac{\varphi_{zz}}{\varphi}+\frac{\rho_z}{\rho}\frac{\varphi_z}{\varphi}+\frac{2m}{\hbar^2}\frac{E-V}{\rho^2}\right)u=0. \tag{4}$$

We explore the cases when this equation becomes a target equation given as

$$u_{zz}+f(z)u_z+g(z)\,u=0. \tag{5}$$

Thus, we demand

$$2\frac{\varphi_z}{\varphi}+\frac{\rho_z}{\rho}=f(z) \tag{6}$$

and

$$\frac{\varphi_{zz}}{\varphi}+\frac{\rho_z}{\rho}\frac{\varphi_z}{\varphi}+\frac{2m}{\hbar^2}\frac{E-V(z)}{\rho^2}=g(z). \tag{7}$$

From equation (6) we have

$$\varphi(z)=\rho^{-1/2}\exp\left(\int f(z)dz/2\right). \tag{8}$$

With this, equation (7) is rewritten as

$$g-\frac{f_z}{2}-\frac{f^2}{4}=-\frac{1}{2}\left(\frac{\rho_z}{\rho}\right)_z-\frac{1}{4}\left(\frac{\rho_z}{\rho}\right)^2+\frac{2m}{\hbar^2}\frac{E-V(z)}{\rho^2}. \tag{9}$$

On the left-hand side of this equation we recognize the *invariant* of equation (5) if it is transformed to its *normal* form:

$$I(z) = g - \frac{f_z}{2} - \frac{f^2}{4}, \tag{10}$$

and the first two terms on the right-hand side are identified as

$$-\frac{1}{2}\left(\frac{\rho_z}{\rho}\right)_z - \frac{1}{4}\left(\frac{\rho_z}{\rho}\right)^2 = -\frac{\{z,x\}}{2\rho^2}, \tag{11}$$

where $\{z,x\}$ is the Schwartzian derivative.

Thus, equation (9) is rewritten as

$$z'(x)^2 I(z) + \frac{1}{2}\{z,x\} = \frac{2m}{\hbar^2}(E - V(x)) \tag{12}$$

(the prime denotes differentiation with respect to x). This is the equation considered by Bose [14] and Natanzon [16]. Sometimes it is convenient to use equation (9) which employs the *canonical* forms of equations obeyed by the standard special functions. The latter forms are written in an intuitive way that reveals the singularity structure of the equations, and it is easier to apply, without intermediate transformations, the standard mathematical knowledge. It is, however, understood that the two techniques are completely equivalent.

The Natanzon approach for constructing solvable potentials rests on the supposition that the coordinate transformation is energy-independent. In general, this is not a necessary condition; many non-Natanzon potentials solvable in terms of standard special functions are known. Still, the approach provides a major set of known exactly solvable potentials, including the classical cases. Within the supposition that $z(x)$ is E-independent, it is immediately seen that the energy and potential terms in (12) should independently match the left-hand side term involving the invariant $I(z)$. For the Gauss hypergeometric equation:

$$u_{zz} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1}\right)u_z + \frac{\alpha\beta}{z(z-1)}u = 0, \tag{13}$$

the invariant is a second-degree polynomial in z divided by $z^2(1-z)^2$. Hence, one straightforwardly arrives at the transformation (1). The Natanzon potential is then given as

$$2mV(x)/\hbar^2 = \frac{v_0 + v_1 z + v_2 z^2}{r_0 + r_1 z + r_2 z^2} - \frac{\{z,x\}}{2}. \tag{14}$$

With arbitrary $r_{0,1,2}$, $v_{0,1,2}$ and the integration constant of Eq. (1), this is a seven-parametric family of potentials. However, the classical hypergeometric potentials are achieved by such specifications of the parameters $r_{0,1,2}$ that $r(z) \sim 1$, z, $1-z$, z^2, $z(1-z)$ or $(1-z)^2$ [14].

The confluent hypergeometric potentials are derived in the same manner starting from the *scaled* Kummer confluent hypergeometric equation of the form

$$u_{zz}+\left(\frac{\gamma}{z}+\varepsilon\right)u_z+\frac{\alpha}{z^2}u=0\,. \tag{15}$$

The result just slightly differs from that for the previous case: $z'(x)=z/\sqrt{r(z)}$ and $V(x)$ is given by equation (14). Hence, this is also a seven-parametric potential, however, the classical confluent hypergeometric potentials are derived by rather severe specifications: $r(z)\sim 1$, z or z^2 [14].

Because of this specification of the parameters $r_{0,1,2}$ the potentials become five-parametric. Since three of the parameters stand for the coordinate origin, the space scale and the origin of the energy, there remain only two parameters for characterization of the potentials. In the next section we consider a situation that necessarily leads to such discretization of the Natanzon potentials.

3. Discretization of Natanzon potentials

We consider the particular case when the potential term in the Schrödinger equation is proportional to a parameter which does not depend on energy and suppose that the potential *shape* is independent of both energy and that parameter. Thus, we suppose

$$2mV(x)/\hbar^2=\mu S(x) \tag{16}$$

with $\mu\neq\mu(E)$ and $S(x)\neq S(\mu,E)$. Formally, the mass m can also be considered as such a parameter, however, the limit $\mu\to 0$ that we apply in due course should then be treated with special care because of its singular nature (m acts also on the kinetic energy term).

Consider the hypergeometric case. To show the discretization, one needs to prove that the roots of the polynomial $r(z)$ coincide with the singularities of the hypergeometric equation $z=0,1$. As a first step we ask if it is possible for the potential given by equation (14) to be of the form (16) if the coordinate transformation defined by equation (1) is μ-dependent. The answer is no.

To show this, we consider the behavior of the μ-independent functions $S(x)$ and $s(x)=\ln S'(x)$. The latter function provides the following identity

$$s'(x)=\frac{S''(x)}{S'(x)}=\frac{V''(x)}{V'(x)}=\frac{V_{zz}(z)}{V_z(z)}\rho(z)+\rho_z(z)\,, \tag{17}$$

which is rather useful for what follows. Indeed, since $\rho(0)=0$ and $\rho_z(0)=\pm 1/\sqrt{r(0)}$, by taking the limit $z\to 0$, it is instantly seen that $r(0)$ is μ-independent. Similarly, $r(1)$ is also μ-independent. These two conditions already impose two restrictions on the parameters $r_{0,1,2}$

involved in the coordinate transformation. It is then well understood that the number of independent parameters involved in a Natanzon hypergeometric potential is less than seven.

Let $r_0 \neq 0$. Then, since $r(0) = r_0$, we reveal that r_0 does not depend on μ. According to equation (1), we have the following power-series expansion in the vicinity of $z = 0$:

$$\frac{dx}{dz} = \frac{1}{z'(x)} = \frac{\sqrt{r_0}}{z} + a_0 + a_1 z + a_2 z^2 + \dots . \tag{18}$$

Retaining only the first term on the right-hand side of this equation, we get that the leading asymptote is

$$z\big|_{x \to -\infty} \sim \exp\left(\frac{x - x_0}{\sqrt{r_0}}\right) = c_0 \exp\left(x / \sqrt{r_0}\right), \tag{19}$$

which allows establishing the following useful limit for any n:

$$\lim_{x \to -\infty} e^{-n x/\sqrt{r_0}} z^n(x) = c_0^n , \quad c_0 = \exp\left(-x_0 / \sqrt{r_0}\right). \tag{20}$$

Using equation (14), the functions $S(x)$ and $s(x)$ allow the expansions

$$S(x) = \frac{2m}{\mu \hbar^2}(V_0 + V_1 z + V_2 z^2 + \dots) = S_0 + S_1 z + S_2 z^2 + \dots , \tag{21}$$

$$s(x) = s_0 + s_1 z + s_2 z^2 + \dots . \tag{22}$$

Considering the partial sums $P_n(x) = \Sigma_{k=0}^{n} S_k z^k$, with $P_{-1} \equiv 0$, and taking the successive limits

$$\lim_{x \to -\infty} e^{-n x/\sqrt{r_0}} \left(S(x) - P_{n-1}(x)\right) = S_n c_0^n \tag{23}$$

for $n = 0, 1, 2, \dots$, we conclude, since $S(x)$ is μ-independent, that $S_n c_0^n$ do not depend on μ. Similarly, all $s_n c_0^n$ are μ-independent. The coefficients s_n can be expressed in terms of S_n. The first two coefficients read

$$s_0 = \frac{2 S_2 \rho(0)}{S_1} + \rho_z(0) = \frac{1}{\sqrt{r_0}}, \quad s_1 = \frac{2 S_2}{\sqrt{r_0} S_1} - \frac{2 r_0 + r_1}{r_0^{3/2}} . \tag{24}$$

These are informative equations. The first equation confirms that r_0 is μ-independent. Further, the second equation and the one for the next coefficient, produce the equations:

$$\frac{d(c_0 s_1)}{d\mu} = 0 = \frac{d}{d\mu}\left(c_0 (2 r_0 + r_1)\right), \tag{25}$$

$$\frac{d(c_0^2 s_2)}{d\mu} = 0 = \frac{d}{d\mu}\left(c_0^2 \left(A(r_{0,1}) + B(r_{0,1,2})\right)\right). \tag{26}$$

from which, together with the information that $r_0 + r_1 + r_2 = r(1)$ is μ-independent, it is readily established that $r_{0,1,2}$ and c_0 are μ-independent. Thus, if $r_0 \neq 0$, the coordinate transformation is μ-independent. The case $r_0 = 0$ is almost trivial because the μ-independence of $r(0)$ and $r(1)$ immediately shows that $r_{1,2}$ are μ-independent so that equation (17) suffices to see that c_0, and thus the coordinate transformation $z(x)$, is μ-independent. A last remark is that though we used information from the singular point $z=1$, in fact, it is sufficient to consider only the behavior in the vicinity of $z=0$. Indeed, instead of the μ-independence of $r(1)$ one can apply the equation $d(c_0^3 s_3)/d\mu = 0$.

The above derivations are exact, and the same approach is applied to any differential equation having a singularity located at a finite point of complex z-plane. The μ-independence of the coefficients of polynomial $r(z)$ as well as of the integration constant c_0 is revealed by considering the infinite set of equations the first two of which are equations (26) and (27). Hence, the μ-independence for $z(x)$ is a general result for the Natanzon potentials if the potential term is proportional to an energy-independent parameter μ, and if the potential shape is both E- and μ-independent. The result holds with the proviso that the Schrödinger equation is reduced to an equation, which has a singularity located at a finite point, and that the variation range of z includes the vicinity of this point.

Thus, the transformation $z(x)$ is μ-independent. The last step is then straightforward. Indeed, taking the limits $E \to 0$ and $\mu \to 0$ in equation (9), we have

$$\frac{1}{2}\left(\frac{\rho_z}{\rho}\right)_z + \frac{1}{4}\left(\frac{\rho_z}{\rho}\right)^2 = -\left(g - \frac{f_z}{2} - \frac{f^2}{4}\right)_{E,\mu\to 0}. \tag{27}$$

It is then immediately seen, that the function ρ_z/ρ cannot have poles other than the singularities of the invariant $I(z)$. *Consequently, the roots of the polynomial $r(z)$ should coincide with the singularities of the target equation* (5). Furthermore, comparing the E/ρ^2 term with the invariant, we conclude that *for the target equations that have rational coefficients the exponents m_i of the Manning transformation $dz/dx = \Pi_i (z - z_i)^{m_i}$ all are integers or half-integers.*

This is the central result of the present paper. We note that this observation is rather productive in applications. For instance, as we have already shown, the successive application of this theorem offers a possibility for construction of several new exactly solvable hypergeometric potentials, both confluent and ordinary (see [23,24] and [26,27]).

4. Confluent Heun potentials

As a representative example, we apply the above theorem to the single-confluent Heun equation which is a second-order differential equation having regular singularities at $z = 0,1$ and an irregular singularity at $z = \infty$. In its canonical form, the equation is written as [28,29]

$$u_{zz} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \varepsilon\right)u_z + \frac{\alpha z - q}{z(z-1)}u = 0 . \tag{28}$$

Among the five Heun equations, this equation is of special interest because it directly incorporates the hypergeometric and (scaled) confluent hypergeometric equations widely applied in quantum mechanics in the past. The ordinary hypergeometric equation is achieved by putting $\varepsilon = \alpha = 0$, while the confluent one is the case if $\delta = 0$ and $q = \alpha$.

Since the finite singularities are $z = 0,1$, the Natanzon family of the confluent Heun potentials is constructed by the coordinate transformation $z(x)$ of the form

$$z'(x) = \rho = z^{m_1}(z-1)^{m_2} / \sigma \tag{29}$$

with integer or half-integer $m_{1,2}$ and arbitrary constant σ. Substituting this into equation (9) and examining the energy term, we see that $z^2(z-1)^2 / \rho^2$ is a polynomial in z of at most fourth degree. This imposes the inequalities $1 \ge m_{1,2}$, $m_1 + m_2 \ge 0$, which lead to 15 possible sets of $m_{1,2}$ shown in Fig.1 by points in the 2D space (m_1, m_2). The cases possessing ordinary hypergeometric sub-potentials are marked by squares, and those having confluent hypergeometric sub-potentials are marked by triangles. There exist two cases when the potentials have hypergeometric sub-potentials of both types. These cases are marked by rhombs. The case $m_{1,2} = 1/2$ possessing a Mathieu sub-potential is marked by a dotted circle. Since the Heun equation preserves its form when transposing $z \leftrightarrow 1-z$, the number of independent potentials is only nine. These nine cases are indicated in Fig. 1 by filled shapes.

Finally, examining the potential term in equation (9), we see that $z^2(z-1)^2 V(z) / \rho^2$ is also a polynomial in z of at most fourth degree. Thus, we get

$$V(z) = z^{2m_1-2}(z-1)^{2m_2-2}(v_0 + v_1 z + v_2 z^2 + v_3 z^3 + v_4 z^4) . \tag{30}$$

With $m_{1,2}$ from Fig. 1, this equation defines 15 seven-parametric potentials, the independent 9 of which can be written in the form presented in Table 1. Six of these potentials, those possessing hypergeometric sub-potentials, were presented by Lamieux and Bose [15] (the ten potentials listed in Table II, p. 265 of [15] are particular cases of the mentioned six, e.g., the last three rows are specifications of the potential with $m_{1,2} = (1,1)$). Some non-hypergeometric representatives of these six potentials have been discussed by many authors on several occasions, e.g., in connection with the two-centre Coulomb problem or Teukolsky equation [28,31]. It seems that the three remaining potentials have not been treated before. One of these potentials, that with $m_{1,2} = (1, -1)$, is explicitly written in terms of the Lambert function, which is an implicitly elementary function that resolves the equation $W \exp(W) = x$ [30].

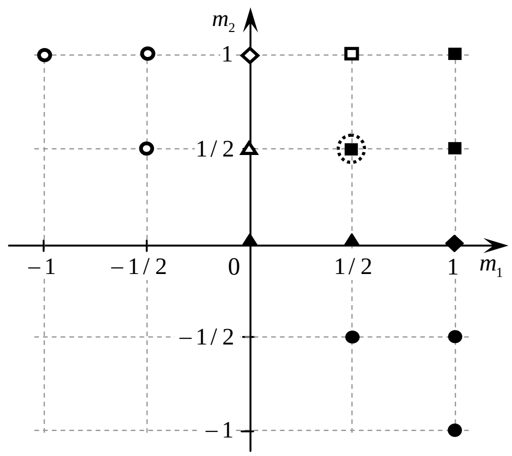

Fig. 1. Fifteen possible pairs (m_1, m_2). The cases possessing ordinary or confluent hypergeometric sub-potentials are marked by squares or triangles, respectively. The rhombs indicate the two cases that have hypergeometric sub-potentials of both types. The dotted circle marks the case possessing a Mathieu sub-potential.

The solution of the Schrödinger equation for the presented potentials is written in terms of the confluent Heun function as

$$\psi = e^{\alpha_0 z} z^{\alpha_1} (z-1)^{\alpha_2} H_C(\gamma, \delta, \varepsilon; \alpha, q; z) . \tag{31}$$

Substituting this into equations (6) and (7) and collecting the coefficients at powers of z, we get eight equations which are linear for the five parameters of the confluent Heun function and are quadratic for the three parameters $\alpha_{0,1,2}$ of the pre-factor. Resolving these equations is straightforward [32]. Examining then the final solution of the Schrödinger equation we note that the involved confluent Heun functions in general are not reduced to simpler special functions, however, they allow several quasi-exactly solvable reductions for specific choices of the parameters. Such solutions are derived by termination of the corresponding power-series expansions.

Thus, for any row of Table 1, we have a potential initially given parametrically as a pair of functions $x(z), V(z)$. The transformation $x(z)$ in general can be written in terms of the incomplete Beta-functions. However, since the parameters $m_{1,2}$ are integers or half-integers, the Beta-functions are always reduced to elementary functions. In the six cases possessing hypergeometric sub-potentials the inverse transformation $z = z(x)$ is written in terms of elementary functions presented in the third column of Table 1.

m_1, m_2	Potential $V(z)$	Coordinate transformation	Hypergeometric sub-potential
0, 0	$V_0 + \frac{V_1}{z} + \frac{V_2}{z^2} + \frac{V_3}{z-1} + \frac{V_4}{(z-1)^2}$	$z = \frac{x - x_0}{\sigma}$	$_1F_1$ [1,10]
$1/2, 1/2$	$V_0 + V_1 z + V_2 z^2 + \frac{V_3}{z} + \frac{V_4}{z-1}$	$z = \cosh^2\left(\frac{x-x_0}{2\sigma}\right)$	$_2F_1$ [8] Mathieu
$1/2, 0$	$V_0 + V_1 z + \frac{V_2}{z} + \frac{V_3}{z-1} + \frac{V_4}{(z-1)^2}$	$z = \frac{(x-x_0)^2}{4\sigma^2}$	$_1F_1$ [1]
$1/2, -1/2$	$V_0 + \frac{V_1}{z} + \frac{V_2}{z-1} + \frac{V_3}{(z-1)^2} + \frac{V_4}{(z-1)^3}$	$\frac{x-x_0}{\sigma} = \sqrt{z(z-1)} - \sinh^{-1}\left(\sqrt{z-1}\right)$	-----
1, 1	$V_0 + V_1 z + V_2 z^2 + V_3 z^3 + V_4 z^4$	$z = \frac{1}{e^{(x-x_0)/\sigma} + 1}$	$_2F_1$ [3]
$1, 1/2$	$V_0 + V_1 z + V_2 z^2 + V_3 z^3 + \frac{V_4}{z-1}$	$z = \sec^2\left(\frac{x-x_0}{2\sigma}\right)$	$_2F_1$ [8]
1, 0	$V_0 + V_1 z + V_2 z^2 + \frac{V_3}{z-1} + \frac{V_4}{(z-1)^2}$	$z = e^{\frac{x-x_0}{\sigma}}$	$_1F_1$ [11] $_2F_1$ [3]
$1, -1/2$	$V_0 + V_1 z + \frac{V_2}{z-1} + \frac{V_3}{(z-1)^2} + \frac{V_4}{(z-1)^3}$	$\frac{x-x_0}{\sigma} = 2\sqrt{z-1} - 2\tan^{-1}\left(\sqrt{z-1}\right)$	-----
$1, -1$	$V_0 + \frac{V_1}{z-1} + \frac{V_2}{(z-1)^2} + \frac{V_3}{(z-1)^3} + \frac{V_4}{(z-1)^4}$	$z = -W\left(-e^{-(x-x_0)/\sigma}\right)$	Lambert W

Table 1. Nine independent seven-parametric confluent Heun potentials.

Among the cases having hypergeometric sub-potentials specific is the case with $m_{1,2} = (1,0)$ for which $z = \exp((x - x_0)/\sigma)$ and the confluent Heun potential presents the sum of the Morse confluent hypergeometric and the Eckart ordinary hypergeometric potentials:

$$V = V_0 + V_1 e^{\frac{x-x_0}{\sigma}} + V_2 e^{2\frac{x-x_0}{\sigma}} + \frac{V_3}{e^{\frac{x-x_0}{\sigma}} - 1} + \frac{V_4}{(e^{\frac{x-x_0}{\sigma}} - 1)^2}. \tag{32}$$

Among the three cases of the lower right quadrant in Fig.1 that do not possess hypergeometric sub-potentials, we note the potential with $m_{1,2} = (1,-1)$:

$$V(z)=\sum_{n=0}^{4}\frac{V_n}{(z-1)^n}, \tag{33}$$

for which the transformation $z(x)$ is written in terms of an *implicitly elementary* function, the Lambert W-function [30]:

$$z=-W\left(-e^{-(x-x_0)/\sigma}\right). \tag{34}$$

For a positive σ and $x_0=-\sigma$ this is a potential defined on the positive half-axis $x>0$. The potential has a singularity at the origin and tends to $V_\infty=\Sigma_{n=0}^{n=4}(-1)^n V_n$ at infinity. In the vicinity of the origin applies the expansion

$$V\big|_{x\to 0}=\frac{d_{-4}}{x^2}+\frac{d_{-3}}{x^{3/2}}+\frac{d_{-2}}{x}+\frac{d_{-1}}{x^{1/2}}+d_0+\ldots \tag{35}$$

Because equation (33) involves five independent parameters, the first five parameters in this expansion may adopt arbitrary values. The behavior of the potential at infinity is

$$V\big|_{x\to +\infty}=V_\infty-A\,e^{-(x+\sigma)/\sigma}, \tag{36}$$

where $A=V_1-2V_2+3V_3-4V_4$.

It is worth mentioning that there exists a four-parametric sub-potential of the general Lambert-W potential (33), presenting an asymmetric step-barrier, for which the Schrödinger equation is exactly solvable using combinations of the confluent hypergeometric functions [24]. Another example of an exactly solvable sub-case of potential (33), the singular Lambert-W potential, is presented in [26]. Furthermore, there exist several five-parametric sub-potentials that are conditionally exactly solvable in terms of the confluent hypergeometric functions. The term "conditionally" implies that for these potentials the problem is solvable under some restrictions imposed on the parameters $V_{0,1,2,3,4}$ that define the potential (33). Three examples of such potentials are those presented in [24], [33] and [34]. A final remark is that the solutions for these potentials are derived by means of termination of the advanced expansions of the involved confluent Heun functions in terms of the confluent or generalized hypergeometric functions [35-38].

5. Mathieu potentials

The next after the hypergeometric functions in the hierarchy of the special functions generated by second-order linear differential equations are the Mathieu functions. Actually, the theory of these functions is a part of the theory of the Heun functions [28]. Indeed, in its *algebraic* form the Mathieu equation is a particular case of the confluent Heun equation achieved by the specification $\gamma = \delta = 1/2$ and $\varepsilon = 0$ [28,29]. It is then readily checked that these specifications for the above presented confluent Heun potentials are satisfied only if $m_{1,2} = (1/2, 1/2)$ and $V_{2,3,4} = 0$. Thus, there exists only one Mathieu potential which in current notations is written as $V = V_0 + V_1 \cosh^2\left((x - x_0)/2\sigma\right)$ (Table 1). This case is marked in Fig. 1 by a dotted circle. We note that for the imaginary $\sigma = i\sigma_0$ the hyperbolic cosine is changed to the trigonometric one so that this is the potential that is immediately derived from the *standard* form of the Mathieu equation which is written as $u_{zz} + \left(a - 2q\cos(2z)\right)u = 0$.

It is known that there exist several other potentials for which the solution of the Schrödinger equation is written in terms of the Mathieu functions (see, e.g., [39-41]). However, these potentials are out of the class we discuss here. For instance, the solution for the inverse fourth-power potential $V = V_4 / x^4$ is achieved by the coordinate transformation $z = \ln(x/\sigma)$ [39] for which $\rho(z) = 1/x = \exp(-z)/\sigma$ so that it is not of the Manning form that we consider. Some of the potentials do not belong to the Natanzon class since they apply energy-dependent coordinate transformations. Finally, there are conditionally exactly solvable potentials involving a fixed parameter. Among these, we mention the following four-parametric potential well/barrier:

$$V(x) = V_0 + \frac{V_2}{z} + \frac{3\hbar^2/(32m\sigma^2)}{z^2}, \tag{37}$$

$$z = 1 + \frac{(x - x_0)^2}{4\sigma^2}, \tag{38}$$

for which the general solution of the Schrödinger equation is explicitly written as

$$\psi = z^{1/4}\left(c_1 S\left(4q - 2\alpha, \alpha, \arccos(\sqrt{z})\right) + c_2 C\left(4q - 2\alpha, \alpha, \arccos(\sqrt{z})\right)\right), \tag{39}$$

$$\alpha = \frac{2m\sigma^2(E - V_0)}{\hbar^2}, \quad q = \frac{1}{16} + \frac{2m\sigma^2 V_2}{\hbar^2}, \tag{40}$$

where S and C are the Mathieu functions. This is a sub-case of the confluent Heun potential with $m_{1,2} = (0, 1/2)$ (see Table 1). It is straightforwardly derived if the involved parameters are allowed to depend on each other or to be fixed. A five-parametric generalization of this potential achieved by a transformation of non-Manning form is presented in [41].

6. Double-, bi- and tri-confluent Heun potentials

To treat the cases of the double- confluent (DHE), bi-confluent (BHE) and tri-confluent (THE) Heun equations, it is convenient to adopt the following canonical forms for these equations which differ from those applied in the standard references [28,29] in that they provide unified *five-parametric* representation for all four confluent Heun equations:

DHE: $$u_{zz}+\left(\frac{\gamma}{z^2}+\frac{\delta}{z}+\varepsilon\right)u_z+\frac{\alpha z-q}{z^2}u=0 \quad (41)$$

BHE: $$u_{zz}+\left(\frac{\gamma}{z}+\delta+\varepsilon z\right)u_z+\frac{\alpha z-q}{z}u=0 \quad (42)$$

THE: $$u_{zz}+\left(\gamma+\delta z+\varepsilon z^2\right)u_z+(\alpha z-q)u=0 \quad (43)$$

(compare with equation (28)). Though the number of irreducible parameters in these three equations is less than five (four for DHE and BHE and three for THE), however, these forms provide a convenient unified description of results which allows one to easily follow the links between different equations.

The reduction of the Schrödinger equation to equations (41)-(43) is straightforward. Indeed, since the double- and bi-confluent equations possess only one finite singularity, conventionally, located at $z=0$, the permissible coordinate transformation for these equations is given through the equation

$$z'(x)=\rho=z^{m_1}/\sigma \quad (44)$$

with integer or half-integer m_1. Hence,

$$z\sim x^{1/(1-m_1)} \text{ if } m_1\neq 1 \quad \text{and} \quad z\sim e^{x/\sigma} \text{ if } m_1=1. \quad (45)$$

The invariants of the double- confluent and bi-confluent equations are fourth-degree polynomials divided by z^4 and z^2, respectively. Accordingly, for the exponent m_1 we have

$$0\le 4-2m_1\le 4 \text{ and } 0\le 2-2m_1\le 4 \quad (46)$$

for DHE and BCE, respectively. Hence, $m_1=0,1/2,1,3/2,2$ for DHE, and for BCE we have $m_1=-1,-1/2,0,1/2,1$.

The corresponding potentials are constructed by applying the equation

$$V=z^{2m_1-d}(v_0+v_1z+v_2z^2+v_3z^3+v_4z^4), \quad (47).$$

where $d=4$ for the double-confluent case and $d=2$ for the bi-confluent equation. Using equations (45), the potentials are written as explicit functions of x. It turns out that for the double-confluent case independent are only the first three potentials with $m_1=0,1/2,1$.

m_1	Double-confluent Heun potentials		m_1	Bi-confluent Heun potentials
			-1	$V(x)=\frac{V_0}{x^2}+\frac{V_1}{x^{3/2}}+\frac{V_2}{x}+\frac{V_3}{x^{1/2}}+V_4$
			$-1/2$	$V(x)=\frac{V_0}{x^2}+\frac{V_1}{x^{4/3}}+\frac{V_2}{x^{2/3}}+V_3+V_4x^{2/3}$
0	$V_0+\frac{V_1}{x}+\frac{V_2}{x^2}+\frac{V_3}{x^3}+\frac{V_4}{x^4}$		0	$V(x)=\frac{V_0}{x^2}+\frac{V_1}{x}+V_2+V_3x+V_4x^2$
$1/2$	$V_0x^2+V_1+\frac{V_2}{x^2}+\frac{V_3}{x^4}+\frac{V_4}{x^6}$		$1/2$	$V(x)=\frac{V_0}{x^2}+V_1+V_2x^2+V_3x^4+V_4x^6$
1	$V_0e^{-2x}+V_1e^{-x}+V_2+V_3e^{x}+V_4e^{2x}$		1	$V(x)=V_0+V_1e^{x}+V_2e^{2x}+V_3e^{3x}+V_4e^{4x}$

Table 2. Three double-confluent and five bi-confluent potentials.

In general, all the double- and bi-confluent Heun potentials involve six independent parameters. All these potentials have been previously presented by Lamieux and Bose [15]. For the convenience of the reader, we reproduce the potentials in Table 2 where we have omitted the parameters x_0,σ (everywhere one should replace $x\to(x-x_0)/\sigma$).

We note that the above-mentioned inverse fourth-power Mathieu potential [39] is a member of both the first double-confluent potential with $m_1=0$ and the second double-confluent potential with $m_1=1/2$. Another relevant observation is that the bi-confluent potential $m_1=-1$ simulates the first five terms of the expansion (35) of the Lambert-W confluent Heun potential (33), (34), for which the first five expansion coefficients $d_{0,-1,-2,-3,-4}$, as it has been already mentioned above, can be chosen arbitrary. One more point worth mentioning here is that the recently reported inverse square root potential [23] that is exactly solved through a two-term ansatz presenting an irreducible linear combination of two confluent hypergeometric functions is a member of this first bi-confluent Heun family. Finally, we note that the centrifugal barrier potential $1/x^2$ is common for all double- and bi-confluent Heun potentials except the ones derived by an exponential coordinate transformation.

The case of the tri-confluent Heun equation is different because this equation does not possess a finite singularity. It is then understood that in this case the polynomial $r(z)$ should not have roots at all. Then, the only possibility is $r(z)=r_0=\text{const}$ so that the only possible coordinate transformation is $z=(x-x_0)/\sigma$. The corresponding tri-confluent Heun potential is the general quartic oscillator:

$$V(x)=V_0+V_1x+V_2x^2+V_3x^3+V_4x^4, \tag{43}$$

where one may replace $x \to (x - x_0)/\sigma$. However, it is seen that this replacement does not change the general form of equation (43), hence, this is a five-parametric potential.

7. Discussion

The solution of the Schrödinger equation in terms of special mathematical functions has a long history, e.g., [1-34,39-42]. The general algebraic form of the independent variable transformation was discussed by Manning [1] who, however, did not consider the details due to the pre-factor in the dependent variable transformation.

Discussing the reducibility to the ordinary and confluent hypergeometric equations, Natanzon presented a general analysis, with the pre-factor, for the case when the coordinate transformation is energy-independent [16]. His result for each hypergeometric case is a potential involving seven parameters, all supposed to be continuous.

The main result of the present paper is that if the potential is proportional to a parameter μ, which does not depend on the energy, and if the potential shape is independent of both energy and this parameter, the Natanzon family of potentials is dropped into a finite set of separate potentials involving fewer continuous parameters. We first show that the coordinate transformation $z(x)$ applied to construct the Natanzon potentials via reduction of the Schrödinger equation to an equation, which has a finite singularity, should be μ-independent. Further, we show that then the coordinate transformation for the target equations having rational coefficients should necessarily be of the Manning form with factors involving only the singularities of the target equation. We note that our result is general and applies to any equation possessing a singularity located at a finite point of complex z-plane. The result implies that the variation range of z includes the vicinity of the finite singular point of the considered equation.

The discretization of the Natanzon potentials has been noticed on several occasions. For example, the authors of [42], have shown that the discretization is necessarily the case if the potential term is proportional to an energy-independent parameter μ; however, their discussion rests on the *presupposition* that the independent variable transformation is μ-independent. The complementary observation we report here is that the μ-independence of the coordinate transformation is necessarily the case for any target equation possessing a finite singular point if the variable z is allowed to vary in a region that includes the vicinity of this singularity.

As a representative example, we have discussed the single-confluent Heun equation which presents a natural generalization of the two hypergeometric equations. Like the quantum two-state problem [43-45], there exist fifteen seven-parametric potentials for which the Schrödinger equation can be reduced to the confluent Heun equation. However, only nine potentials are independent. Six of these independent cases suggest different generalizations of the hypergeometric sub-potentials, while the other three do not allow hypergeometric reductions. Unlike the case of the ordinary hypergeometric sub-potentials, no confluent Heun potential can in general be transformed into another one by means of specification of the involved parameters. Among the potentials that do not posses hypergeometric sub-cases, a potential has an explicit representation through a coordinate transformation written in terms of the Lambert W-function, which is an implicitly elementary function known also as the product logarithm. We note that the recently introduced two Lambert-W potentials, the step-barrier [24] and the singular [26] potentials, which are exactly solvable via a two-term ansatz presenting an irreducible linear combination of two confluent hypergeometric functions, are members of this family of the single-confluent Heun potentials.

Further, we have shown that the μ-independent Natanzon-type potentials for which the Schrödinger equation is reduced to the double-confluent, bi-confluent and tri-confluent Heun equations are the three six-parametric double- and five six-parametric bi-confluent Lamieux-Bose potentials [15] and one five-parametric tri-confluent Heun potential (the quartic oscillator). It should be noted that in a recent paper Batic and co-workers [25] presented a general discussion of the Heun potentials for all five Heun equations (for a detailed discussion for particularly the tri-confluent case see [46]); however, they did not discuss the discretiztion of the potentials.

We would like to conclude by noting that both the Manning form of the coordinate transformation and the Natanzon energy-independent specification for this transformation are not the only options for construction of potentials solvable in terms of known special functions. The inverse fourth-power Mathieu potential [39] and the two conditionally exactly solvable potentials by Stillinger [47] are direct examples supporting this observation. Furthermore, the widely applied in the past one-term ansatz $\psi = \varphi(z)\,u(z)$ involving a single special function $u(z)$ is also not necessary. There are many potentials that allow solution as a linear combination with constant or non-constant coefficients of two or more hypergeometric functions, for instance, the inverse square root potential [23], the Lambert-W step-potential [24], as well as the singular Lambert-W potential [26] and the third independent ordinary hypergeometric potential [27]. Thus, there is a need to systematically study the alternative approaches. A step in this direction is presented in our recent work [48].

Acknowledgments

This research has been conducted within the scope of the International Associated Laboratory IRMAS (CNRS-France & SCS-Armenia). The work has been supported by the Armenian State Committee of Science (SCS Grants No. 13RB-052 and No. 15T-1C323), the Ministry of Education and Science of the Russian Federation (state assignment No. 3.679.2014/K) and the project “Leading Russian Research Universities” (Grant No. FTI_24_2016 of the Tomsk Polytechnic University).

References

1. M.F. Manning, "Exact solutions of the Schrödinger equation", Phys. Rev. **48**, 161-164 (1935).
2. E. Schrödinger, "Quantisierung als Eigenwertproblem (Erste Mitteilung)", Annalen der Physik **76**, 361-376 (1926).
3. C. Eckart, "The penetration of a potential barrier by electrons", Phys. Rev. **35**, 1303-1309 (1930).
4. N. Rosen and P.M. Morse, "On the vibrations of polyatomic molecules", Phys. Rev. **42**, 210-217 (1932).
5. M.F. Manning and N. Rosen, "A potential function for the vibrations of diatomic molecules", Phys. Rev. **44**, 953-953 (1933).
6. L. Hulthén, Ark. Mat. Astron. Fys. **28A**, 5 (1942); L. Hulthén, "Über die Eigenlösungen der Schrödinger-Gleichung der Deuterons", Ark. Mat. Astron. Fys. **29B**, 1-12 (1942).
7. R.D. Woods and D.S. Saxon, "Diffuse surface optical model for nucleon-nuclei scattering", Phys. Rev. **95**, 577-578 (1954).
8. G. Pöschl, E. Teller, "Bemerkungen zur Quantenmechanik des anharmonischen Oszillators", Z. Phys. **83**, 143-151 (1933).
9. F. Scarf, "New soluble energy band problem", Phys. Rev. **112**, 1137-1140 (1958).
10. A. Kratzer, "Die ultraroten Rotationsspektren der Halogenwasserstoffe", Z. Phys. **3**, 289-307 (1920).
11. P.M. Morse, "Diatomic molecules according to the wave mechanics. II. Vibrational levels", Phys. Rev. **34**, 57-64 (1929).
12. P.M. Morse and E.C.G. Stückelberg, "Lösung des Eigenwertproblems eines Potentialfeldes mit zwei Minima", Helv. Phys. Acta **4**, 337-354 (1931).
13. M.F. Manning, "Energy levels of a symmetrical double minima problem with applications to the NH_3 and ND_3 molecules", J. Chem. Phys. **3**, 136-138 (1935).
14. A.K. Bose, "Solvable potentials", Phys. Lett. **7**, 245-246 (1963).
15. A. Lamieux and A.K. Bose, "Construction de potentiels pour lesquels l’équation de Schrödinger est soluble", Ann. Inst. Henri Poincaré A **10**, 259-270 (1969).
16. G.A. Natanzon, "General properties of potentials for which the Schrödinger equation can be solved by means of hypergeometric functions", Theor. Math. Phys. **38**, 146-153 (1979).
17. R.L. Hall, N. Saad, and K.D. Sen, "Soft-core Coulomb potentials and Heun’s differential equation", J. Math. Phys. **51**, 022107 (2010).
18. C.A. Downing, "On a solution of the Schrödinger equation with a hyperbolic double-well potential", J. Math. Phys. **54**, 072101 (2013).

19. B.-H. Chen, Y. Wu and Q.-T. Xie, "Heun functions and quasi-exactly solvable double-well potentials", J. Phys. A **46**, 035301 (2013).
20. R.R. Hartmann, "Bound states in a hyperbolic asymmetric double-well", J. Math. Phys. **55**, 012105 (2014).
21. J. Karwowski and H.A. Witek, "Biconfluent Heun equation in quantum chemistry: Harmonium and related systems", Theor. Chem. Acc. **133**, 1494 (2014).
22. D. Batic, D. Mills-Howell and M. Nowakowski, "Potentials of the Heun class: the triconfluent case", J. Math. Phys. **56**, 052106 (2015).
23. A.M. Ishkhanyan, "Exact solution of the Schrödinger equation for the inverse square root potential $V_0/\sqrt{x}$ ", EPL **112**, 10006 (2015).
24. A.M. Ishkhanyan, "The Lambert-W step-potential - an exactly solvable confluent hypergeometric potential", Phys. Lett. A **380**, 640-644 (2016).
25. D. Batic, R. Williams, M. Nowakowski, "Potentials of the Heun class", J. Phys. A **46**, 245204 (2013).
26. A.M. Ishkhanyan, "A singular Lambert-W Schrödinger potential exactly solvable in terms of the confluent hypergeometric functions", Mod. Phys. Lett. A **31**, 1650177 (2016).
27. A.M. Ishkhanyan, "The third exactly solvable hypergeometric quantum-mechanical potential", EPL **115**, 20002 (2016).
28. A. Ronveaux (ed.), *Heun's Differential Equations* (Oxford University Press, London, 1995).
29. S.Yu. Slavyanov and W. Lay, *Special functions* (Oxford University Press, Oxford, 2000).
30. J.H. Lambert, "Observationes variae in mathesin puram", Acta Helvitica **3**, 128-168 (1758).
31. E.W. Leaver, "Solutions to a generalized spheroidal wave equation: Teukolsky equations in general relativity, and the two-center problem in molecular quantum mechanics", J. Math. Phys. **27**, 1238 (1986).
32. A.M. Ishkhanyan, "Schrödinger potentials solvable in terms of the confluent Heun functions", Theor. Math. Phys. **188**, 980-993 (2016).
33. B.W. Williams, "Exact solutions of a Schrödinger equation based on the Lambert function", Phys. Lett. A **334**, 117 (2005).
34. A.M. Ishkhanyan, "A conditionally exactly solvable generalization of the inverse square root potential", Phys. Lett. A, in press arXiv:1511.03565
35. Th. Kurth and D. Schmidt, "On the global representation of the solutions of second-order linear differential equations having an irregular singularity of rank one in ∞ by series in terms of confluent hypergeometric functions", SIAM J. Math. Anal. **17**, 1086 (1986).
36. L.J. El-Jaick and B.D.B. Figueiredo, "Solutions for confluent and double-confluent Heun equations", J. Math. Phys. **49**, 083508 (2008).
37. T.A. Ishkhanyan and A.M. Ishkhanyan, "Expansions of the solutions to the confluent Heun equation in terms of the Kummer confluent hypergeometric functions", AIP Advances **4**, 087132 (2014).
38. C. Leroy and A.M. Ishkhanyan, "Expansions of the solutions of the confluent Heun equation in terms of the incomplete Beta and the Appell generalized hypergeometric functions", Integral Transforms and Special Functions **26**, 451-459 (2015).
39. R. Spector, "Exact solution of the Schrödinger equation for inverse fourth-power potential", J. Math. Phys. **5**, 1185 (1964).
40. L.K. Sharma, "Potentials from modified Mathieu equation", Proc. Indian Nat. Science Academy **36A**, 230-238 (1970).
41. B.F. Samsonov and I.N. Ovcharov, "Potentials reducing the one-dimensional Schrödinger equation to the Mathieu equation", Russian Phys. J. **37**, 574-577 (1994).

42. A.M. Goncharenko, V.A. Karpenko, V.N. Mogilevich, "Potentials that admit solution of the one-dimensional time-independent Schrödinger equation in terms of known functions", Theor. Math. Phys. **88**, 715-720 (1991).
43. A.M. Ishkhanyan and A.E. Grigoryan, "Fifteen classes of solutions of the quantum two-state problem in terms of the confluent Heun function", J. Phys. A **47**, 465205 (2014).
44. T.A. Shahverdyan, T.A. Ishkhanyan, A.E. Grigoryan, A.M. Ishkhanyan, "Analytic solutions of the quantum two-state problem in terms of the double, bi- and triconfluent Heun functions", J. Contemp. Physics (Armenian Ac. Sci.) **50**, 211-226 (2015).
45. A.M. Ishkhanyan, T.A. Shahverdyan, T.A. Ishkhanyan, "Thirty five classes of solutions of the quantum time-dependent two-state problem in terms of the general Heun functions", Eur. Phys. J. D **69**, 10 (2015).
46. D. Batic, D. Mills-Howell and M. Nowakowski, "Potentials of the Heun class: the triconfluent case", J. Math. Phys. **56**, 052106 (2015).
47. F.H. Stillinger, "Solution of a quantum mechanical eigenvalue problem with long range potentials", J. Math. Phys. **20**, 1891-1895 (1979).
48. T.A. Ishkhanyan and A.M. Ishkhanyan, "Solutions of the bi-confluent Heun equation in terms of the Hermite functions", arXiv:1608.02245 [quant-ph] (2016).