Note on "α-admissible mappings and related fixed point theorems"

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Abstract
In this paper gave a theorem that theorems (4,6,8)of [1], obtain from it.

Keywords: fixed point, invariant set , 1-1-subclass, 1–upclass , α-F-admissible

1-Introduction and Preliminaries

2- Main result

Definition1: we say that $h: R^2 \times [0,\infty) \to R$ is a function of 1-1-subclass if is continuous and

$$x \geq 1, y \geq 1 \Rightarrow h(1,1,z) \leq h(x,y,z)$$

Example:

(1) $h(x,y,z) = (z+l)^{xy} \quad l > 1$
(2) $h(x,y,z) = (xy+l)^z \quad l > 1$
(3) $h(x,y,z) = xyz$
(4) $h(x,y,z) = \left(\frac{x+y}{2}\right)z$

Definition 2: we say that $f: [0,1] \times R^+ \to R$ is a 1–upclass with h a 1-1subclass if is a continuous function such that

$$0 \leq x \leq 1 \Rightarrow f(x,y) \leq f(1,y)$$

$$h(1,1,z_1) \leq f(x,z_2) \Rightarrow z_1 \leq xz_2$$

Examples:

(1) $h(x,y,z) = (z+l)^{xy} \quad l > 1, f(x,y) = xy + l$
(2) $h(x,y,z) = (xy+l)^z \quad l > 1, f(x,y) = (1+l)^{xy}$,
(3) $h(x,y,z) = xyz, f(x,y) = xy$
(4) $h(x,y,z) = \left(\frac{x+y}{2}\right)z \ , f(x,y) = xy$

Definition3:let $T: X \to X$,The subset F is *invariant* under the T if $Tx \in F$ for every $x \in F$.

Definition4: Let $T: X \to X$ and $\alpha: F \times F \to R^+$. We say that T is an α-F-admissible mapping if $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1, x,y \in F$.

Note: if in Definition4 let $F = X$ We say that T is an α- admissible mapping ,see([2],[1])

Theorem5: *Let (X,d) be a complete metric space and $T: X \to X$ be an α-F-admissible mapping*.and *invariant* under the T *.Assume that there exists a function $\beta: [0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$* and h a 1-1-subclass, f a 1-upclass of h and ψ is altering distance function, F closed subset of X, such that

$$h(\alpha(x,Tx), \alpha(y,Ty), \psi d(Tx,Ty)) \leq f(\beta(d(x,y)), \psi d(x,y)) \quad (2.1)$$

for all $x,y \in F$ .Suppose that either

(a) *T is continuous, or*
(b) *if $\{x_n\}$ is a sequence in F such that $x_n \to x$, $\alpha(x_n,x_{n+1}) \geq 1$ for all n, then* $\alpha(x,Tx) \geq 1$
If there exists $x_0 \in F$ such that $\alpha(x_0,Tx_0) \geq 1$, then T has a fixed point.

Proof: Let $x_0 \in F$ *such that* $\alpha(x_0, Tx_0) \geq 1$, we construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$, $n = 1, 2, \dots$,we have $\alpha(x_n, x_{n+1}) = \alpha(x_n, Tx_n) \geq 1$,and $\{x_n\} \subset F$.Substituting $x = x_{n-1}$ and $y = x_n$ in (2.1) , we obtain

$$h\left(1,1,\psi\left(d(x_n, x_{n+1})\right)\right) \leq h\left(\alpha(x_{n-1}, x_n), \alpha(x_n, x_{n+1}), \psi\left(d(x_n, x_{n+1})\right)\right) \leq$$
$$\leq f(\beta\left(d(x_n, x_{n-1})\right), \psi\left(d(x_n, x_{n-1})\right))$$

And this equal with

$$\frac{\psi\left(d(x_n, x_{n+1})\right)}{\psi\left(d(x_n, x_{n-1})\right)} \leq \beta\left(d(x_n, x_{n-1})\right) \leq 1 \qquad (2.2)$$

$\psi\left(d(x_n, x_{n+1})\right) \leq \psi\left(d(x_{n-1}, x_n)\right)$

ψ is altering distance function therefore

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \qquad (2.3)$$

for every $n \in N$, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing so that for the nonnegative decreasing sequence $\{d(x_n, x_{n+1})\}$, there exists some $r \geq 0$,such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \qquad (2.4)$$

On letting $n\to\infty$ in (2.2) implies,$\lim_{n\to\infty} \beta\left(d(x_n, x_{n-1})\right) = 1$,and this

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \qquad (2.5)$$

now prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exists $\delta > 0$ for which we can find subsequences $\{x_{n_k}\}$and $\{x_{m_k}\}$of $\{x_n\}$with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \delta \qquad (2.6)$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (2.6).

Then

$$d(x_{n_k-1}, x_{m_k}) < \delta \qquad (2.7)$$

Then we have

$$o < \delta \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) < \delta + d(x_{n_k}, x_{n_k-1}) \quad (2.8)$$

Letting $k\to\infty$and using (2.6),

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \delta \qquad (2.9)$$

Again,

$d\left(x_{n_k}, x_{m_k}\right) \leq d\left(x_{n_k}, x_{n_k-1}\right) + d\left(x_{n_k-1}, x_{m_k-1}\right) + d\left(x_{m_k-1}, x_{m_k}\right)$,

$d\left(x_{n_k-1}, x_{m_k-1}\right) \leq d\left(x_{n_k}, x_{n_k-1}\right) + d\left(x_{n_k}, x_{m_k}\right) + d\left(x_{m_k-1}, x_{m_k}\right)$.

Therefore

$$d\left(x_{n_k}, x_{m_k}\right) \leq d\left(x_{n_k}, x_{n_k-1}\right) + d\left(x_{n_k-1}, x_{m_k-1}\right) + d\left(x_{m_k-1}, x_{m_k}\right) \leq$$
$$\leq 2d\left(x_{n_k}, x_{n_k-1}\right) + d\left(x_{n_k}, x_{m_k}\right) + 2d\left(x_{m_k-1}, x_{m_k}\right) \quad (2.10)$$

Letting $k\to\infty$in (2.10) and using (2.5), (2.9), we get

$$\lim_{k\to\infty} d\left(x_{n_k-1}, x_{m_k-1}\right) = \delta. \qquad (2.11)$$

Setting $x = x_{m_k-1}$ and $y = x_{n_k-1}$ in (2.1), we obtain

$$h\left(1,1,\psi\left(d\left(x_{m_k}, x_{n_k}\right)\right)\right) \leq h\left(\alpha\left(x_{m_k-1}, x_{m_k}\right), \alpha\left(x_{n_k-1}, x_{n_k}\right), \psi\left(d\left(x_{m_k}, x_{n_k}\right)\right)\right) \leq$$

$$\leq f(\beta\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right), \psi\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right)$$

Therefore

$$\frac{\psi\left(d\left(x_{m_k}, x_{n_k}\right)\right)}{\psi\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right)} \leq \beta\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right) \leq 1 \qquad (2.12)$$

Letting k→∞, utilising (2.11) and (2.9), we obtain

$$\lim_{k\to\infty} d\left(x_{m_k-1}, x_{n_k-1}\right) = 0 \qquad (2.13)$$

which is a contradiction if $\delta > 0$

This shows that $\{x_n\}$ is a Cauchy sequence and hence is convergent in the complete set F. Let

$$x_n \rightarrow z \in F \text{ as } n \rightarrow \infty. \qquad (2.14)$$

First, we suppose that T is continuous. then

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = T z.$$

Next, we suppose that (b) holds. Then, $\alpha(z, Tz) \geq 1$. Now, by (2.1) we have

$$h\left(1,1,\psi\left(d(Tz, x_{n+1})\right)\right) \leq h\left(\alpha(z, x_n), \alpha(x_n, T x_n), \psi\left(d(Tz, x_{n+1})\right)\right) \leq$$
$$\leq f(\beta\left(d(Z, x_n)\right), \psi\left(d(Z, x_n)\right))$$

thus

$$\psi\left(d(Tz, x_{n+1})\right) \leq \beta\left(d(Z, x_n)\right)\psi\left(d(Z, x_n)\right)$$

And this equal with

$$d(T z, x_{n+1}) \leq d(z, x_n).$$

and so we get

$$d(Tz, z) \leq d(Tz, x_{n+1}) + d(z, x_{n+1}) \leq d(z, x_n) + d(z, x_{n+1}).$$

By taking the limit as $n\to\infty$, we get $d(Tz, z) = 0$, *i.e.*, $z = Tz$. ■

CONSEQUENCE

In theorem (5) let $h(x, y, z) = (z + l)^{xy}$ $l > 1$, $f(x, y) = xy + l$, $\psi(t) = t$, $F = X$, this trun to theorem (4) of [1].

Corollary6: *Let* (X, d) *be a complete metric space and* $T: X \to X$ *be an* α*-admissible mapping .Assume that there exists a function* $\beta: [0, \infty) \to [0, 1]$ *such that, for any bounded sequence* $\{t_n\}$ *of positive real ,* $\beta(t_n) \to 1$ *implies* $t_n \to 0$, such that

$$(d(Tx, Ty) + l)^{\alpha(x,Tx)\alpha(y,Ty)} \leq \beta\left(d(x, y)\right)d(x, y) + l$$

for all $x, y \in X$ *.Suppose that either*

(a) *T is continuous, or*

(b) *if* $\{x_n\}$ *is a sequence in X such that* $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ *for all n, then* $\alpha(x, Tx) \geq 1$

If there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$, *then T has a fixed point..*

In theorem (5) let $h(x, y, z) = (xy + l)^{z}$ $l > 1$, $f(x, y) = (1 + l)^{xy}$, $l = 1$, $\psi(t) = t$, $F = X$, this trun to theorem (6) of [1].

Corollary7: *Let* (X, d) *be a complete metric space and* $T: X \to X$ *be an* α*-admissible mapping.Assume that there exists a function* $\beta: [0, \infty) \to [0, 1]$ *such that, for any bounded sequence* $\{t_n\}$ *of positive reals,* $\beta(t_n) \to 1$ *implies* $t_n \to 0$, such that

$$(\alpha(x, Tx)\alpha(y, Ty) + 1)^{d(Tx,Ty)} \leq 2^{\beta(d(x,y))d(x,y)}$$

for all $x, y \in X$ *.Suppose that either*

(a) *T is continuous, or*

(b) *if* $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n, *then* $\alpha(x, Tx) \geq 1$
If there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$, *then* T *has a fixed point..*
In theorem (5) let $h(x, y, z) = xyz$, $f(x, y) = xy$, $\psi(t) = t$, $F = X$, this trun to theorem (8) of [1].

Corollary8: *Let* (X, d) *be a complete metric space and* $T: X \to X$ *be an* α*-admissible mapping.Assume that there exists a function* $\beta: [0, \infty) \to [0, 1]$ *such that, for any bounded sequence* $\{t_n\}$ *of positive reals,* $\beta(t_n) \to 1$ *implies* $t_n \to 0$, such that

$$\alpha(x, Tx)\alpha(y, Ty)d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$$

for all $x, y \in X$ *.Suppose that either*
(a) *T is continuous, or*
(b) *if* $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n, *then* $\alpha(x, Tx) \geq 1$
If there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$, *then* T *has a fixed point..*

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