Mean convergence of Lagrange interpolation for Freud’s weights with application to product integration rules

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Abstract: The connection between convergence of product integration rules and mean convergence of Lagrange interpolation in $L_p$ ($1 < p < \infty$) has been thoroughly analysed by Sloan and Smith [37]. Motivated by this connection, we investigate mean convergence of Lagrange interpolation at the zeros of orthogonal polynomials associated with Freud weights on $\mathbb{R}$. Our results apply to the weights $\exp(-x^m/2)$, $m = 2, 4, 6 \ldots$, and for the Hermite weight ($m = 2$) extend results of Nevai [28] and Bonan [2] in at least one direction. The results are sharp in $L_p$, $1 < p \leq 2$. As a consequence, we can improve results of Smith, Sloan and Opie [38] on convergence of product integration rules based on the zeros of the orthogonal polynomials associated with the Hermite weight. In the process, we prove a new Markov-Stieltjes inequality for Gauss quadrature sums, and solve a problem of Nevai on how to estimate certain quadrature sums.

Keywords: Quadrature estimates, Lagrange interpolation, mean convergence, singular integrands, product integration rules.

1. Introduction

In [28], Nevai proved:

**Theorem 1.** Let $W^2$ be the Hermite weight, so that

$$W(x) = \exp(-x^2/2), \quad x \in \mathbb{R}. \quad (1.1)$$

Let $f$ be continuous in $\mathbb{R}$ and assume

$$|f(x)|W(x) = o(|x|^{-1}), \quad |x| \to \infty. \quad (1.2)$$

Let $L_n(f, x)$ denote the Lagrange interpolation polynomial of degree at most $n - 1$ to $f$ at the zeros of the orthogonal polynomial of degree $n$ associated with $W^2$. Then, for $1 < p < \infty$,

$$\lim_{n \to \infty} \|f(x) - L_n(f, x)\|_{L_p(\mathbb{R})} = 0. \quad (1.3)$$

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Subsequently, Bonan [2] improved and extended Nevai’s results, and considered the more general weight $|x|^\lambda \exp(-x^2/2)$, $\lambda > -1$. We quote his result for $\lambda = 0$.

**Theorem 2.** Let $W$ be given by (1.1). Let $f$ be continuous in $\mathbb{R}$ and assume

$$|f(x)|W(x) = o(|x|^{-2}), \quad |x| \to \infty,$$  

where

$$\delta \left\{ \begin{array}{ll}
< 1 - 1/p, & 0 < p \leq 2, \\
\geq 1 - (3p - 1)/2, & p > 2.
\end{array} \right. \quad (1.4)$$

Let $L_n(f, x)$ be as above and $0 < p < \infty$. Then,

$$\lim_{n \to \infty} \|f(x) - L_n(f, x)\|_{L^p(\mathbb{R})}^p = 0,$$  

where

$$0 < p \leq 2, \quad p > 2, \quad |x| \to \infty.$$  

Neither Nevai nor Bonan attended to the question of whether the growth conditions (1.2) and (1.4) are best possible, a question which had been raised by Nevai in [26, p. 190]. Further, Askey [1, p. 75] questioned in a related context whether continuity could be replaced by Riemann integrability, as in the classical Erdős–Turan theorem. Here we shall show that (1.3) persists if $f$ is Riemann integrable in each bounded interval and (1.2) is replaced by:

If $1 < p \leq 2$, for some $\epsilon > 0$,

$$|f(x)|W(x) = O(|x|^{-1/p}(\log|x|)^{-1/p} \cdots (\log \cdots \log|x|)^{-1/p-\epsilon}),$$  

as $|x| \to \infty$, while if $p > 2$,

$$|f(x)|W(x) = o(|x|^{-1+1/p})$$  

as $|x| \to \infty.$

Note that (1.7) is ‘best possible’: If we let $\epsilon = 0$, we can choose $f$ satisfying (1.7) with $\epsilon = 0$, but such that $fW \not\in L^p(\mathbb{R})$, so that even the norms in (1.3) are infinite. For $p = 2$, (1.7) essentially appears in Lubinsky and Sidi [19], where the Erdős–Turan theorem was extended to include functions with finitely many square integrable singularities. For $p > 2$, it seems likely that $-1 + 1/p$ in (1.8) can be replaced by $\frac{1}{2}$.

We shall also show that $f$ may be allowed to have finitely many singularities of suitably restricted growth, provided one replaces $L_n(f, x)$ in (1.3) by $L_n^*(f, x)$ which interpolates to $f$ except at the closest interpolation point to each singularity of $f$. At such interpolation points, $L_n^*(f, x)$ is chosen to interpolate to $0$. This notion of ‘avoiding the singularity’ was first used by Rabinowitz [35] in a related context, namely in convergence of Gauss quadrature.

The above results for the Hermite weight are special cases of Theorems 3 and 4, which deal with Freud weights $W(x) = \exp(-Q(x))$ whose orthonormal polynomials satisfy certain bounds. The most typical Freud weights are $W_\lambda(x) = \exp(-|x|^\lambda/2)$, $\lambda > 1$, and for $\lambda$ a positive even integer, Bonan [3,4] and Nevai [30] have obtained suitable bounds on the associated orthonormal polynomials. Magnus [20,21] proved the related Freud’s Conjecture for $\lambda$ a positive even integer. For a lengthy, but entertaining, survey of Freud weights, the reader may refer to Nevai [32]. Shorter surveys, with a different perspective, appear in Levin and Lubinsky [11], Lubinsky [15], Mhaskar [23], Nevai [31] and Nevai and Totik [33,34].

To state our results, we need some notation. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of $n$ and $x$, and of all polynomials $P$ of degree at most $n$ or a constant.
times \( n \). The same symbol does not necessarily denote the same constant from line to line. Further, we use \( o, O \) and \( \sim \) as in Nevai [27]. Thus, for example, \( f(x) \sim g(x) \) if there exist \( C_1 \) and \( C_2 \) such that \( C_1 \leq f(x)/g(x) \leq C_2 \) for all \( x \) considered.

**Definition 1.** We say \( W \) is a Freud weight of finite order, and write \( W \in \mathcal{F}^* \) if \( W(x) = \exp(-Q(x)), x \in \mathbb{R} \), where \( Q \) is even and continuous in \( \mathbb{R} \), and \( Q' \) and \( Q'' \) are continuous for large positive \( x \), while

\[
Q'(x) > 0, \quad x \in [C_1, \infty)
\]

and

\[
0 \leq xQ''(x)/Q'(x) \leq C_2, \quad x \in [C_1, \infty),
\]

and either

(i) \( Q''(x) \) is positive and nondecreasing in \([C_1, \infty)\), or

(ii) there exists \( \alpha \in (1, 2) \) such that

\[
\lim_{x \to \infty} xQ'(x)/Q(x) = \alpha.
\]

Note that if \( \lambda > 1 \), \( W_\lambda(x) = \exp(-|x|^\lambda/2) \in \mathcal{F}^* \). The class \( \mathcal{F}^* \) has been studied in [17] and is a subclass of the class \( \mathcal{F} \) considered in [16]. Freud weights of various types have been intensively studied in [7–17, 22–25, 31–34]. Associated with \( Q \) are the numbers \( q_n \), extensively used by Freud, and defined for \( n \) large enough to be the positive root of the equation

\[
q_n Q'(q_n) = \alpha.
\]

For small \( n \), we set \( q_n = 1 \). Throughout, \( p_j(x) = p_j(W^2; x) \) \( j = 0, 1, 2, \ldots \), denote the orthonormal polynomials associated with \( W^2 \), so that

\[
\int_{-\infty}^{\infty} p_m(W^2, x)p_n(W^2, x)W^2(x) \, dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}
\]

**Definition 2.** Let \( W \in \mathcal{F}^* \). We write \( W \in \mathcal{F}^*B \) and say \( W \) is a bounded Freud weight of finite order, if there exists \( D > 0 \) independent of \( n \) and \( x \) such that

\[
\{p_n(W^2; x)W(x)\}^2 \leq C_1/q_n, \quad |x| \leq Dq_n.
\]

For \( W_m(x) = \exp(-x^m/2) \), \( m \) a positive even integer, (1.15) was proved first by Bonan [3] and subsequently in a sharper form by Nevai [30]. As discussed in Bonan, Lubinsky and Nevai [5], one can show that (1.15) is valid if \( W(x) = \exp(-Q(x)) \), where \( Q \) is any even polynomial of positive even degree with positive leading coefficient.

Throughout, we denote the zeros of \( p_n(W^2; x) \) by

\[-\infty > x_{1n} > x_{2n} > \cdots > x_{nn} > -\infty.\]

The Lagrange interpolation polynomial to \( f \) at the zeros of \( p_n \) will be denoted by

\[
L_n(f, x) = \sum_{j=1}^{n} f(x_{jn})l_{jn}(x),
\]

(1.16)
where $l_j$ is the $j$th fundamental polynomial, satisfying

$$l_j(x_k) = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

Suppose now that $f$ has finitely many singularities, say

$$-\infty < y_1 < y_2 < \cdots < y_l < \infty,$$

in the sense that $f$ becomes unbounded at these points, but is bounded in each compact subinterval of $(y_i, y_{i+1})$, $i = 0, 1, \ldots, l$, where $y_0 = -\infty$ and $y_{l+1} = \infty$. Motivated by Rabinowitz’ idea of avoiding the singularity [35], we modify the definition of $L_n(f, x)$ as in Lubinsky and Sidi [19]: For each positive integer $n$, we let

$$x_{c(1, n)}$, $x_{c(2, n)}$, \ldots, $x_{c(l, n)}$$

denote the closest elements from \{x_{1n}, x_{2n}, \ldots, x_{ln}\} to $y_1$, $y_2$, \ldots, $y_l$ respectively. If $x_{c(j, n)}$ is not uniquely defined, that is $y_j$ lies midway between two abscissas, we let $x_{c(j, n)}$ denote the closest abscissa on the left. Finally, let

$$L_n(f, x) = \sum_{j \notin \mathcal{Y}(n)} f(x_{c(n)}) l_j(x).$$

Thus $L_n^*(f, x)$ interpolates to $f$ at the zeros of $p_n$, except at the closest abscissa to a singularity of $f$, where it interpolates to 0. Our first result deals with $L_n$, $1 < p \leq 2$, and is sharp in the sense that we cannot let $\epsilon = 0$ in the growth conditions (1.19) or (1.20).

**Theorem 3.** Let $W \in \mathcal{F}^*$. Let $f: \mathbb{R} \to \mathbb{R}$ and let there exist

$$-\infty = y_0 < y_1 < y_2 < \cdots < y_l < y_{l+1} = \infty$$

such that $f$ is bounded and Riemann integrable in each compact subinterval of $(y_i, y_{i+1})$, $i = 0, 1, 2, \ldots, l$. Let $1 < p \leq 2$. Assume that for some $\epsilon > 0,$

$$|f(x)| = O(|x|^{-1/p}|\log|x - y_i||^{-1/p} \cdots |\log|x - y_l||^{-1/p - \epsilon})$$

as $x \to y_i$, $i = 1, 2, \ldots, l$, so that in particular $f$ has $L_p$ integrable singularities at $y_i$, $i = 1, 2, \ldots, l$. Assume, further, that for some $\epsilon > 0,$

$$|f(x)|W(x) = O(|x|^{-1/p}|\log|x||^{-1/p} \cdots |\log|x||^{-1/p - \epsilon}),$$

as $|x| \to \infty$, so that in particular $fW \in L_p(\mathbb{R})$. Then

$$\lim_{n \to \infty} \|f(x) - L_n^*(f, x)\|_{L_p(\mathbb{R})} = 0.$$

In particular, if $l = 0$, so that $f$ is bounded and Riemann integrable in each compact interval,

$$\lim_{n \to \infty} \|f(x) - L_n(f, x)\|_{L_p(\mathbb{R})} = 0.$$

Using ideas from Bonan’s thesis [2], one can extend the above result to $0 < p \leq 1$, but in a
weaker form. Since it is not of much interest for product integration rules, we shall omit the extension. The following is our result for \( p > 2 \).

**Theorem 4.** Let \( W \in \mathcal{F}^* B \). Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) and let there exist

\[-\infty = y_0 < y_1 < y_2 < \cdots < y_i < y_{i+1} = \infty\]

such that \( f \) is bounded and Riemann integrable in each compact subinterval of \((y_i, y_{i+1})\), \( i = 0, 1, 2, \ldots , l \). Let \( p > 2 \). Assume that for some \( \epsilon > 0 \), (1.19) holds as \( x \rightarrow y_i, i = 1, 2, \ldots , l \). Assume further that for some \( \beta > 0 \),

\[ |f(x)|W(x) = O(|x|^{-\beta}), \quad |x| \rightarrow \infty \]  

(1.23)

where

\[ \limsup_{n \rightarrow \infty} \left( \frac{n}{q_n} \right)^{1/2} \leq \frac{1}{q_n} \frac{1-p}{1-\beta} \leq \infty . \]  

(1.24)

(It is always possible to find \( \beta \) satisfying (1.24)). Further, assume that \( \gamma \geq 0 \) satisfies

\[ \lim_{n \rightarrow \infty} \left\{ \int_1^{q_n} u^\beta \, du \right\} = \frac{1}{q_n} \frac{1-p}{1-\beta} \leq \infty . \]  

(1.25)

(It is always possible to find \( \gamma \) satisfying (1.25)). Then

\[ \lim_{n \rightarrow \infty} \| (f(x) - L_n(f, x))W(x)(1 + |x|)^{-\gamma} \|_{L_p(\mathbb{R})} = 0 . \]  

(1.26)

In particular, if \( l = 0 \), so that \( f \) is bounded and Riemann integrable in each compact interval

\[ \lim_{n \rightarrow \infty} \| (f(x) - L_n(f, x))W(x)(1 + |x|)^{-\gamma} \|_{L_p(\mathbb{R})} = 0 . \]  

(1.27)

Let us consider the rather technical conditions (1.24) and (1.25) for the weight \( W(x) = W_\lambda(x) = \exp(-|x|^\lambda \lambda /2) \), \( \lambda > 1 \). Of course we at present only know \( W_\lambda \in \mathcal{F}^* B \) if \( \lambda \) is a positive even integer, but it seems certain that \( W_\lambda \in \mathcal{F}^* B \) for arbitrary \( \lambda > 1 \). From (1.13), we see that

\[ q_n = (2n/\lambda)^{1/\lambda}, \quad n = 1, 2, 3, \ldots . \]

Consequently (1.24) becomes

\[ \beta \geq \frac{1}{2} + \left( \frac{1}{\lambda} - 1/p \right)(\lambda - 1) . \]  

(1.28)

In particular, for the Hermite weight (\( \lambda = 2 \)), (1.24) is satisfied if \( \beta = 1 - 1/p \), so that (1.23) becomes (1.8). Further for the Hermite weight, it is known (Szegö [40]) that

\[ \| p_n W \|_{L_\alpha(\mathbb{R})} = O(1), \quad n \rightarrow \infty \]  

(1.29)

and hence that for \( p > 2 \),

\[ \| p_n W \|_{L_p(\mathbb{R})} = O(1), \quad n \rightarrow \infty . \]  

(1.30)

It is then easy to check that (1.25) is valid with \( \beta = 1 - 1/p \) and \( \gamma = 0 \). More generally, Bonan and Clark [4] showed that if \( W(x) = W_\lambda(x) \) and \( \lambda \) is a positive even integer, then

\[ \| p_n W \|_{L_\alpha(\mathbb{R})} = O(n^{1/6-1/(2\lambda)}) , \quad n \rightarrow \infty . \]  

(1.31)

One can use this to show that for \( \lambda \leq 6 \), both (1.24) and (1.25) are valid with \( \gamma = 0 \) and \( \beta \).
satisfying (1.28). An alternative approach, which is more generally applicable, but yields weaker results, is to use weighted Nikolskii inequalities (Mhaskar [22], Nevai and Totik [34]) which show that

$$\|p_n W\|_{L_p(R)} < C(n/q_n)^{1/2-1/p}, \quad n = 1, 2, 3, \ldots, p \geq 2.$$ 

Next, we turn to convergence of product integration rules. For rules based on zeros of orthogonal polynomials associated with weights on $[-1, 1]$, a thorough analysis was performed by Sloan and Smith [37]. See also the references in [37], Lubinsky and Sidi [19] and Smith, Sloan and Opie [38]. The latter two papers contain (among other things) results for product integration rules associated with the Hermite weight. Rabinowitz and Sloan [36] dealt with convergence of product integration rules based on spline quadratures.

Let $k(x)$ be a measurable function for which all the integrals $\int_{-\infty}^\infty |k(x)| |x|^j \, dx$, $j = 0, 1, 2, \ldots$, are finite. The product integration rule $I_n[k, f]$ based on the zeros of $p_n(W^2; x)$ is

$$I_n[k, f] = \sum_{j=1}^n w_{jn} f(x_{jn}), \quad (1.32)$$

where the weights $w_{jn}$ are chosen so that

$$I_n[k, P] = \int_{-\infty}^\infty P(x)k(x) \, dx, \quad (1.33)$$

for all polynomials $P$ of degree at most $n - 1$. It can easily be shown [19,37] that

$$I_n[k, f] = \int_{-\infty}^\infty L_n(f, x)k(x) \, dx. \quad (1.34)$$

When $f$ has singularities $y_1, y_2, \ldots, y_l$, we define modified integration rules $I_n^*[k, f]$ by

$$I_n^*[k, f] = \sum_{j \notin \mathcal{S}(n)}^n w_{jn} f(x_{jn}), \quad (1.35)$$

where $\mathcal{S}(n)$ is as in (1.17). It is easy to see that

$$I_n^*[k, f] = \int_{-\infty}^\infty L_n^*(f, x)k(x) \, dx. \quad (1.36)$$

As previously noted, $I_n^*$ is an extension of Rabinowitz' idea [35] of dropping the closest abscissa to a singularity. We remark that the modified rules $I_n^*$ used in Lubinsky and Sidi [19] may omit one more abscissa for each singularity than the rules used here.

**Theorem 5.** Let $W \in \mathcal{F}^* B$. Let $f: \mathbb{R} \to \mathbb{R}$ and let there exist

$$-\infty = y_0 < y_1 < y_2 < \cdots < y_{l} < y_{l+1} = \infty,$$

such that $f$ is bounded and Riemann integrable in each compact subinterval of $(y_i, y_{i+1})$, $i = 0, 1, 2, \ldots, l$. Let $1 < p < \infty$ and let $q > 1$ satisfy $1/p + 1/q = 1$. Assume that for some $\varepsilon > 0$, (1.19) is valid as $x \to y_i$, $i = 1, 2, \ldots, l$. Further, if $1 < p \leq 2$, assume that (1.20) is valid as $|x| \to \infty$ for some $\varepsilon > 0$, while if $p > 2$, we assume that (1.23), (1.24) and (1.25) are satisfied for some $\beta \geq 0$ and $\gamma \geq 0$. 


Let \( k(x) \) be measurable in \( \mathbb{R} \), and assume that
\[
\| k(x) W^{-1}(x) \|_{L_q(\mathbb{R})} < \infty \quad \text{if } 1 < p \leq 2 ,
\]
while
\[
\| k(x) W^{-1}(x)(1 + |x|)^{\gamma} \|_{L_q(\mathbb{R})} < \infty \quad \text{if } p > 2 .
\]

Then
\[
\lim_{n \to \infty} I_n^a[k, f] = \int_{-\infty}^{\infty} f(x)k(x) \, dx .
\]

In particular, if \( l = 0 \), so that \( f \) is bounded and Riemann integrable in each compact interval,
\[
\lim_{n \to \infty} I_n[k, f] = \int_{-\infty}^{\infty} f(x)k(x) \, dx .
\]

We remark that for the Hermite weight, Theorem 5 improves a result of Smith, Sloan and Opie [38] in the sense that their growth condition on \( f \), namely (1.2), is replaced by (1.7) if \( p \leq 2 \) and by (1.8) if \( p > 2 \). Further, the above result allows \( f \) to have singularities. For \( p = 2 \), essentially the above result appears in Lubinsky and Sidi [19]. Exactly as in Smith, Sloan and Opie [38], one may introduce and prove convergence of the ‘companion rules’, modified as in [19] to deal with singularities. Such results indicate ‘asymptotic stability’ of the integration rules.

One of our auxiliary results is a new Markov–Stieltjes inequality for Gauss quadrature sums involving even weights and integrands—Lemma 3.2 below. Using it and results from [14], one can resolve a problem posed by Nevai in 1976 [26, p. 170] on how to estimate sums such as
\[
\sum_{k=1}^{n} \lambda_k W^{-2}(x_k) .
\]

**Theorem 6.** Let \( W \in \mathcal{F}^* \). Let \( \log^+ r = \max\{1, \log |r|\}, r \in \mathbb{R} \), and
\[
\phi(r) = (1 + |r|)^a(\log^+ |r|)^b(\log^+ \log^+ |r|)^c \ldots , \quad r \in \mathbb{R} ,
\]
where \( a, b, c, \ldots \) are arbitrary real numbers of which at most finitely many are non-zero. Then, as \( n \to \infty \),
\[
\sum_{k=1}^{n} \lambda_k W^{-2}(x_k) \phi(x_k) \sim \int_{\lambda_n}^{\lambda_n} \phi(u) \, du .
\]

The paper is organised as follows: In Section 2, we introduce more notation. In Section 3, we state some properties of weights \( W \in \mathcal{F}^* \) and prove some estimates and convergence results for Gauss quadrature sums, including Theorem 6. In Section 4, we turn to the proof of Theorems 3, 4 and 5.

The proofs use results and ideas of Askey [1], Bonan [3,4], Freud [7–10], Levin and Lubinsky [11,12], Lubinsky and Sidi [19], Lubinsky, Mate and Nevai [16,17], Mhaskar [22] and Nevai [27–30].
2. Notation

For the reader's convenience, we shall not only introduce new notation, but also recap on the notation from Section 1. Let $W \in F^*$. Throughout, $p_n(x) = p_n(W^2; x)$, $n = 0, 1, 2, \ldots$, denote the orthonormal polynomials associated with $W^2$, so that (1.14) holds. Throughout the zeros of $p_n$ are denoted by

$$\infty > x_{1n} > x_{2n} > \cdots > x_{nn} > -\infty$$

and $\gamma_n > 0$ denotes the leading coefficient of $p_n$. Further, the $n$th kernel function is

$$K_n(x, t) = K_n(W^2, x, t) = \sum_{j=0}^{n-1} p_j(W^2; x)p_j(W^2; t)$$

and the $n$th Christoffel function is

$$\lambda_n(x) = \lambda_n(W^2; x) = 1/K_n(x, x),$$

while

$$\lambda_{jn} = \lambda_n(x_{jn}), \quad j = 1, 2, \ldots, n.$$

The Gauss quadrature formula is

$$I_n[f] = \sum_{j=1}^{n} \lambda_{jn} f(x_{jn}).$$

Whenever $f$ has singularities $-\infty < y_1 < y_2 < \cdots < y_j < \infty$, in the sense that $f$ becomes unbounded at these points, we let $x_{c(1, n)n}, x_{c(2, n)n}, \ldots, x_{c(l, n)n}$ denote the closest abscissas from \{\{x_1, x_2, \ldots, x_n\}\} to $y_1, y_2, \ldots, y_j$ respectively. As at (1.17), we let

$$\mathcal{F}(n) = \{c(1, n), c(2, n), \ldots, c(l, n)\}, \quad n = 1, 2, \ldots,$$

and let

$$I_n^*[f] = \sum_{j=1}^{n} \lambda_{jn} f(x_{jn}), \quad n = 1, 2, \ldots.$$

Of course, if $f$ is bounded in each finite interval, $I_n^*[f] = I_n[f]$. Given a function $k(x)$ that is measurable and real valued and such that

$$\int_{-\infty}^{\infty} |k(x)||x|^j \, dx < \infty, \quad j = 0, 1, 2, \ldots,$$

the product integration rule $I_n[k, \cdot]$ is given by

$$I_n[k, f] = \sum_{j=1}^{n} w_{jn} f(x_{jn}),$$

where the weights $w_{jn}$ are chosen so that (1.33) is valid for all polynomials $P$ of degree at most $n - 1$. Given a function $f$ with singularities $y_1, y_2, \ldots, y_j$, the modified rule $I_n^*[f, \cdot]$ may be defined analogously to $I_n^*[f]$, as at (1.35) to (1.36).

The fundamental polynomials of Lagrange interpolation may be given by any of the
formulae (Freud, [6, p. 23], Nevai [27, p. 6])

\[ l_{jn}(x) = \lambda_{jn} K_n(x, x_{jn}) = \lambda_{jn} (\gamma_{n-1}/\gamma_n) p_{n-1}(x_{jn}) p_n(x)/(x - x_{jn}), \quad j = 1, 2, \ldots, n, \]  
(2.1)

while the Lagrange interpolation polynomial to \( f \) at the zeros of \( p_n(W^2; x) \) is

\[ L_n(f, x) = \sum_{j=1}^{n} f(x_{jn}) l_{jn}(x). \]

If \( f \) has singularities \( y_1, y_2, \ldots, y_l \), the (modified) Lagrange interpolation polynomial is

\[ L_n^*(f, x) = \sum_{j=1}^{n} f(x_{jn}) l_{jn}(x) \]

which interpolates to \( f \) at the zeros of \( p_n \), except at the closest abscissa to a singularity, where it interpolates to zero.

Let \( f(x) \) be a function that is real valued and measurable and such that

\[ \int_{-\infty}^{\infty} |f(x)||x|^j \, dx < \infty, \quad j = 0, 1, 2, \ldots. \]

The \( n \)th partial sum of the orthonormal expansion of \( f \) in \( p_0, p_1, p_2, \ldots \) admits the representation

\[ S_n(f, x) = \int_{-\infty}^{\infty} K_n(x, t) f(t) W^2(t) \, dt, \quad n = 1, 2, 3, \ldots. \]

Throughout, \( v(x) \) is the Chebyshev weight

\[ v(x) = \begin{cases} (1 - x^2)^{-1/2}, & x \in (-1, 1), \\ 0, & \text{otherwise} \end{cases} \]

and \( p_j(v; x), j = 0, 1, 2, \ldots \), are the orthonormal polynomials associated with \( v \), while the \( n \)th kernel for the Chebyshev weight is

\[ K_n(v, x, t) = \sum_{j=0}^{n-1} p_j(v; x) p_j(v; t), \quad n = 1, 2, \ldots. \]

Finally, if \( \mathcal{A} \) is a measurable subset of \( \mathbb{R} \), and \( g \) is real valued and measurable

\[ \|g\|_{L_q(\mathcal{A})} = \begin{cases} \left\{ \int_{\mathcal{A}} |g(u)|^q \, du \right\}^{1/q}, & 1 < q < \infty, \\ \text{ess sup}\{ |g(u)| : u \in \mathcal{A} \}, & q = \infty. \end{cases} \]

For brevity, we also use \( \|g\|_q = \|g\|_{L_q(\mathbb{R})} \).

3. Quadrature estimates and convergence of quadratures

Following are some properties of weights \( W \in \mathcal{F}^* \):

Lemma 3.1. Let \( W \in \mathcal{F}^* \). There exist \( C_1, C_2, \ldots, C_{12} \) with the following properties:
(i) Let \( r \geq 0 \). Then for all polynomials \( P \) of degree at most \( n \),
\[
\| x^r P(x)W(x) \|_p \leq C_1 q_n^{r} \| P(x)W(x) \|_{L_p((-1)^2n,11q_2n)}.
\] (3.1)

(ii) \( \gamma_{n-1}/\gamma_n \leq C_2 q_n \).

(iii) \( \lambda_n(W^2; x) = C_3(q_n/n)W^2(x), \quad x \in \mathbb{R} \).

(iv) \( \lambda_n(W^2; x) \sim (q_n/n)W^2(x), \quad |x| \leq C_4 q_n \).

(v) \( x_{kn} - x_{k+1,n} \sim q_n/n \)

uniformly for \( k \) and \( n \) such that \( |x_{kn}| \leq C_5 q_n \).

(vi) \( x_{1n} \sim q_n \sim x_{2n} \).

(vii) There exists \( C_6 > 1 \) such that
\[
|x|^C \leq Q(x) \leq |x|^C', \quad |x| > C_8.
\] (3.7)

(viii) There exists \( C_0 < 1 \) such that
\[
n^{C_{10}} \leq q_n \leq n^{C_0}, \quad n \text{ large enough.}
\] (3.8)

(ix) There exists \( C_{11} > 1 \) such that
\[
C_{11} \leq q_{2n}/q_n \leq 2, \quad n \text{ large enough.}
\] (3.9)

(x) Let \( w > 1 \). Then uniformly for \( 1 \leq u \leq w \),
\[
Q'(ux) \sim Q'(x), \quad x \geq C_{12}.
\] (3.10)

**Proof.** (i) Since \( W \) is even and nonincreasing for large \( |x| \), this follows from Theorem A in Lubinsky [13, p. 264]. For sharp \( L_\infty \) results, consult Mhaskar and Saff [24,25].

(ii) This follows easily from (i) with \( W \) replaced by \( W^2 \) (note that \( W \in \mathcal{F}^* \) implies \( W' \in \mathcal{F}^* \) for all \( s > 0 \)) and from the identity
\[
\gamma_{n-1}/\gamma_n = \int_{-\infty}^{\infty} x p_{n-1}(W^2; x)p_n(W^2; x)W^2(x) \, dx.
\]

Compare [7].

(iii) This follows from (3.5) in Lemma 3.3. in Lubinsky, Maté and Nevai [16], since \( \mathcal{F}^* \) is contained in the class \( \mathcal{F} \) considered in [16] and since (see (1.27) in [16])
\[
\lambda_n(W^2; x) = (\lambda_{n,2}(W, 0, x))^2.
\]

(iv) This follows from (3.4) in Lemma 3.3 in [16].

(v) This is Lemma 3.5 in [16].

(vi) The fact that \( x_{1n} \geq x_{2n} \geq C_{13} q_n \), \( n \text{ large enough.} \), follows easily from (v). The corresponding upper bound follows easily from the infinite-finite range inequality (i) and the formula
\[
x_{1n} = \sup \int_{-\infty}^{\infty} xP^2(x)W^2(x) \, dx \left/ \int_{-\infty}^{\infty} P^2(x)W^2(x) \, dx \right.,
\]
where the sup is taken over all polynomials $P$ of degree at most $\nu - 1$. See [7,8].

(vii) The upper bound follows from Lemma 7(v) in [14], since $\mathcal{F}$ is contained in the class of Freud weights considered in [14]. To prove the lower bound, let us suppose first that $Q$ satisfies (1.11). Then it is easy to see that $Q(x) \geq C_1|x|^2$ for large enough $|x|$. On the other hand, if $Q$ satisfies (1.12), it is easily seen that for each $\epsilon > 0$, $Q(x) \geq C_1|x|^{n-\epsilon}$ for large enough $|x|$. Hence the lower bound in (3.7) holds with $C_6 > 1$.

(viii) By Lemma 7(vi) in [14], $Q(x) \sim xQ'(x)$, $x \geq C_6$. Hence from (1.13), $Q(q_n) \sim n, n$ large enough. By (3.7), for some $C_6 > 1$, $C_6 q_n^{C_6} \leq n \leq C_1 q_n^{C_7}$. Then (3.8) follows if $1 > C_9 > 1/C_6$.

(ix) This follows from Lemma 3.1 (i) and (ii) in [16].

(x) This is Lemma 7(ix) in [14]. □

We next prove a new Posse-Markov-Stieltjes inequality for even weights and integrands.

**Lemma 3.2.** Let $d\alpha(x) = W^2(x) \, dx$, where $W$ is an even non-negative function on $\mathbb{R}$ such that all moments of $d\alpha$ are finite. Let $G(x)$ be entire, with

$$G(x) = \sum_{j=0}^{\infty} g_j x^{2j}, \quad x \in \mathbb{R}, \quad (3.11)$$

where

$$g_j \geq 0, \quad j = 0, 1, 2, \ldots \quad (3.12)$$

Let $0 < X < Y$ be two consecutive positive zeros of $p_n(W^2; x)$ or let $X = x_{1n}$ and $Y = \infty$. Then

$$\int_{-X}^{X} G(x) W^2(x) \, dx \leq \sum_{|x_j| < Y} \lambda_j G(x_j) \leq \int_{-Y}^{Y} G(x) W^2(x) \, dx. \quad (3.13)$$

**Proof.** We first prove (3.13) for odd positive integers $n$, say $n = 2k + 1$. Define another weight $d\alpha^*(u) = (W^*(u))^2 \, du$ on $[0, \infty)$ by

$$(W^*(u))^2 = \begin{cases} u^{1/2} (W(u^{1/2}))^2, & u \in [0, \infty), \\ 0, & \text{otherwise}. \end{cases} \quad (3.14)$$

It is known (see [6, p. 50, Problem 14]) that the orthonormal polynomials $p_k^*(u)$ for $d\alpha^*$, their zeros $x_j^*$ and Christoffel numbers $\lambda_j^*$ may be expressed in terms of the corresponding quantities for $d\alpha$ by

$$p_k^*(u) = u^{-1/2} p_{2k+1}(W^2; u^{1/2}), \quad (3.15)$$

$$x_j^* = x_{j, 2k+1}, \quad j = 1, 2, \ldots, k, \quad (3.16)$$

$$\lambda_j^* = 2 \lambda_{j, 2k+1} x_{j, 2k+1}^2, \quad j = 1, 2, \ldots, k. \quad (3.17)$$

Both (3.15) and (3.16) follow by the substitution $x = u^{1/2}$ in

$$2 \int_0^w p_{2k+1}(W^2; x)p_{2m+1}(W^2; x)W^2(x) \, dx = \delta_{mk}, \quad m, k = 0, 1, 2, \ldots \; \text{To prove (3.17), let } P \text{ be a polynomial of degree at most } 2k - 1. \text{ By the}$$
Gauss quadrature formulae of order $k$ for $d\alpha^*$, and of order $2k+1$ for $d\alpha$, and by (3.16),

$$
\sum_{j=1}^{k} \lambda_j^{*} P(x_{j,2k+1}) = \int_{0}^{\infty} P(u) \, d\alpha^*(u)
$$

$$
= \int_{-\infty}^{\infty} P(x^2) x^2 W^2(x) \, dx \quad \text{(by (3.14))}
$$

$$
= \sum_{j=1}^{2k+1} \lambda_j^{*} P(x_{j,2k+1}) x_{j,2k+1}^2 .
$$

(3.18)

Using symmetry of the zeros of $p_{2k+1}(W^2; x)$ about 0, and applying (3.18) to suitable polynomials, we obtain (3.17).

Next, let us suppose $g_0 = 0$ and define

$$
H(u) = G(u^{1/2})/u , \quad u \in [0, \infty) ,
$$

so that by (3.11) and (3.12)

$$
H^{(j)}(u) \geq 0 , \quad u \in [0, \infty) , \quad j = 0, 1, 2, \ldots .
$$

(3.20)

We can then apply the usual Posse–Markov–Stieltjes inequality for the weight $d\alpha^*$ and the function $H(u)$ [6, p. 33, ineq. (5.10); p. 92, Lemma 1.5] to deduce that

$$
\int_{0}^{U} H(u) \, d\alpha^*(u) \leq \sum_{x_{ik} < V} \lambda_{ik}^{*} H(x_{ik}^{*}) \leq \int_{0}^{V} H(u) \, d\alpha^*(u)
$$

(3.21)

where $U < V$ are two consecutive zeros of $p_k^{*}$ related to the zeros $X, Y$ of $p_k(W^2; x)$ by (3.16), or $U = x_{ik}^{*}$ and $V = \infty$. Although Freud stated the inequality under the assumption that the derivatives in (3.20) are positive, he remarks [6, p. 50, Problem 16] that it suffices to have the derivatives non-negative. Alternatively, the reader may refer to [18, Lemma 3.2] for a full proof. Finally, if we make the substitution $u = x^2$ in (3.21) and use (3.14), (3.16), (3.17), (3.19) and $n = 2k + 1$, we obtain

$$
\int_{-X}^{X} G(x) W^2(x) \, dx \leq \sum_{|x_{jn}| < Y} \lambda_{jn} G(x_{jn}) \leq \int_{-Y}^{Y} G(x) W^2(x) \, dx ,
$$

which is (3.13). To complete the proof of (3.13) for $n$ odd, if suffices to prove (3.13) with $G$ replaced by 1 and then to add the inequalities for $G(u) - g_0$ and for $g_0 \cdot 1$.

To this end, we use the usual Markov–Stieltjes inequality [6, p. 33, inequality (5.10)]:

$$
\sum_{|x_{jn}| < Y} \lambda_{jn} = \sum_{x_{jn} < Y} \lambda_{jn} - \sum_{x_{jn} < -Y} \lambda_{jn}
$$

$$
\leq \int_{-\infty}^{Y} d\alpha(u) - \int_{-Y}^{-\infty} d\alpha(u) = \int_{-Y}^{Y} W^2(u) \, du .
$$

Similarly

$$
\sum_{|x_{jn}| < Y} \lambda_{jn} \geq \int_{-X}^{X} W^2(u) \, du .
$$
Thus (3.13) is valid for \( G = 1 \) also.

When \( n \) is even, say \( n = 2k \), the proof is similar but easier, as \( G(0) \) is irrelevant. One uses the weight \( d\hat{\alpha}(u) = (\hat{W}(u))^2 \, du \), where

\[
(\hat{W}(u))^2 = \begin{cases} 
  u^{-1/2}(W(u^{1/2}))^2, & u \in [0, \infty), \\
  0, & \text{otherwise}.
\end{cases}
\]

Further the associated zeros \( \hat{\nu}_{jk} \) and Christoffel numbers \( \hat{\lambda}_{jk} \) satisfy

\[
\hat{\nu}_{jk} = x_{j,2k}^2 \quad \text{and} \quad \hat{\lambda}_{jk} = 2\lambda_{j,2k}, \quad j = 1, 2, \ldots, k.
\]

One then applies the usual Posse–Markov–Stieltjes inequality to

\[
H(u) = G(u^{1/2}) \quad \text{and} \quad d\hat{\alpha}.
\]

We turn to the

**Proof of Theorem 6.** As remarked previously, \( \mathcal{F}^* \) is contained in the class of Freud weights considered in [14]. By Theorem 1 in [14], there exists an even entire function \( G(x) \) with all even order derivatives non-negative and such that for \(|x|\) large enough,

\[
G(x) \sim W^{-2}(x)\phi(|x|). \tag{3.22}
\]

Then \( G \) has a representation in the form (3.11) and (3.12) and we may clearly assume \( g_0 > 0 \). Since both sides of (3.22) are positive and continuous in \( \mathbb{R} \), we see that (3.22) holds for all \( x \in \mathbb{R} \). Then by Lemma 3.2, with \( X = x_{1n} \) and \( Y = \infty \),

\[
\sum_{k=1}^{n} \lambda_{kn} W^{-2}(x_{kn})\phi(x_{kn}) \geq C_1 \sum_{k=1}^{n} \lambda_{kn} G(x_{kn}) \geq C_1 \int_{-X}^{X} G(u)W^2(U) \, du
\]

\[
\geq C_2 \int_{-X}^{X} \phi(|u|) \, du \quad \text{(by (3.22))}
\]

\[
\sim \int_{0}^{q_n} \phi(u) \, du, \tag{3.23}
\]

since \( X = x_{1n} \sim q_n \) (by Lemma 3.1(vi)) and since \( \phi(u/C) \sim \phi(u) \) in \( \mathbb{R} \), for any \( C > 0 \). In a similar manner, Lemma 3.2 yields

\[
\sum_{k=2}^{n} \lambda_{kn} W^{-2}(x_{kn})\phi(x_{kn}) \leq C_3 \int_{0}^{q_n} \phi(u) \, du. \tag{3.24}
\]

To deal with the residual term

\[
\tau_n = \lambda_{1n} W^{-2}(x_{1n})\phi(x_{1n}) + \lambda_{nn} W^{-2}(x_{nn})\phi(x_{nn}),
\]

we choose a \( G_1(x) \) which is even, entire, satisfies (3.11) and (3.12), and

\[
G_1(x) \sim W^{-2}(x)(1 + x^2)^{-1}, \quad x \in \mathbb{R}. \tag{3.25}
\]
This is possible by Theorem 1 in [14]. Then as $x_{1n} = -x_{nn} \sim q_n$,

$$
\tau_n \leq C_4 \phi(q_n) q_n^2 (\lambda_{1n} G_1(x_{1n}) + \lambda_{nn} G_1(x_{nn}))
= C_4 \phi(q_n) q_n^2 \left( \sum_{k=1}^{n} \lambda_{kn} G_1(x_{kn}) - \sum_{k=2}^{n-1} \lambda_{kn} G_1(x_{kn}) \right)
\leq C_4 \phi(q_n) q_n^2 \left( \int_{-\infty}^{x_{2n}} G_1(u) W^2(u) \, du - \int_{-x_{2n}}^{x_{2n}} G_1(u) W^2(u) \, du \right)
\leq C_5 \phi(q_n) q_n^2 \int_{x_{2n}}^\infty u^{-2} \, du \leq C_6 \phi(q_n) q_n ,
$$

by (3.25) and Lemma 3.1(vi). As $\phi(q_n) \sim \phi(u)$, $u \in [q_n/2, q_n]$, we obtain

$$
\tau_n \leq C_7 \int_{q_n/2}^{q_n} \phi(u) \, du .
$$

Finally, (3.23), (3.24) and (3.26) yield the result. \qed

Next, we prove a ‘local’ quadrature sum estimate using the results and methods of Lubinsky, Maté and Nevai [16] and Lubinsky and Nevai [17].

**Lemma 3.3.** Let $W \in \mathcal{F}^*$. Let $q > 0$. Let $-\infty < r < 2$. There exists $E > 0$ such that for $0 < \delta < \epsilon \leq E$, for each positive integer $l$ and for all polynomials $P$ of degree at most $ln$, $W^{*}(u) = W^{(2-r)/q}(u) = \exp(-Q^*(u))$, where $Q^*(u) = ((2 - r)/q)Q(u)$. It is easy to see that $W^* \in \mathcal{F}^*$. Further if $q_n^*$ is the root of the equation

$$
q_n^* Q^*(q_n^*) = n , \quad n \text{ large enough},
$$

then it follows from Lemma 3.1(ix) that $q_n^* \sim q_n$ for $n$ large enough. Theorem 3.2 in [17] with $\psi(t) = t$ shows that

$$
|PW^*(x)|^q \leq C_2 (n^3 q_n)^{-1} \int_{-\xi q_n}^{\xi q_n} |PW^*(u)|^q \{K_n^*(x, u)\}^4 \, du ,
$$

for $|x| \leq \chi q_n$, $0 < \chi < \xi$, and all polynomials $P$ of degree at most an arbitrary constant times $n$. Here $C_2$ is independent of $n, P$ and $x$, while

$$
K_n^*(x, u) = K_n(v, x/(\xi q_n), u/(\xi q_n)) , \quad x, u \in \mathbb{R} ,
$$

(3.29)
is the (scaled) $n$th kernel for the Chebyshev weight. In view of our definition of $W^*$, we may rewrite (3.28) as
\[
|P(x)|^q W^{-r}(x) \leq C_2(n^3 q_n)^{-1} \int_{-\xi_n}^{\xi_n} |P(u)|^q W^{-r}(u) \left( K_n^*(x, u) \right)^q du,
\]
(3.30)

Next, by Lemma 3.1(iv), there exists $E > 0$ such that
\[
\lambda_{kn} = \lambda_n(W^2; x_{kn}) \leq C_3(q_n/n) W^2(x_{kn}), \quad |x_{kn}| \leq Eq_n.
\]
Then if $0 < \delta < \epsilon < E$, (3.30) shows that if $P$ has degree at most $ln$
\[
\sum_{|x_{kn}| \leq \delta q_n} \lambda_{kn} |P(x_{kn})|^q W^{-r}(x_{kn}) \leq C_4 \int_{-\xi_n}^{\xi_n} |P(u)|^q W^{-r}(u) n^{-q} \left( \sum_{|x_{kn}| \leq \delta q_n} (K_n^*(x_{kn}, u))^q \right) du,
\]
where $K_n^*$ is given by (3.29) with $\xi = \epsilon$. To estimate the sum in the right member of (3.31), we use Lemma 2.3 in [16]. Firstly, if $E$ is small enough, Lemma 3.1(v) shows that for $E \leq \epsilon$,
\[
(x_{kn}/(\epsilon q_n)) - (x_{k+1,n}/(\epsilon q_n)) \sim n^{-1}, \quad |x_{kn}| \leq \epsilon q_n.
\]
Hence, if
\[
\theta_{kn} = \arccos(x_{kn}/(\epsilon q_n)), \quad k = 1, 2, \ldots, n,
\]
it is easily seen that for $\delta < \epsilon$,
\[
\Delta_n = \min \{\theta_{k+1,n} - \theta_{k,n} : |x_{kn}| \leq \delta q_n\} \sim n^{-1}.
\]
Then by Lemma 2.3 in [16], and by (3.29),
\[
\sum_{|x_{kn}| \leq \delta q_n} (K_n^*(x_{kn}, u))^2 \leq C_5 n^2, \quad |u| \leq Eq_n.
\]
(3.32)

Since [27, p. 108]
\[
|K_n(u, x, t)| \leq C_6 n, \quad |x|, |t| \leq 1,
\]
(3.31) and (3.32) yield the result. \(\square\)

The final result in this section, on convergence of Gauss quadrature, is essentially drawn from Lubinsky and Sidi [19, Theorem 3.5(a)].

**Lemma 3.4.** Let $W \in \mathcal{F}^*$. Let $0 < p < \infty$. Let $f: \mathbb{R} \to \mathbb{R}$ and assume there exist
\[
-\infty = y_0 < y_1 < y_2 < \cdots < y_l < y_{l+1} = \infty
\]
such that $f$ is bounded and Riemann integrable in each compact subinterval of $(y_i, y_{i+1})$, $i = 0, 1, 2, \ldots, l$. Assume that for some $\epsilon > 0$,
\[
|f(x)| = O(|x - y_i|^{-1/p} |\log |x - y_i||^{-1/p} \cdots |\log |x - y_i| ||^{-1/p - \epsilon}),
\]
(3.33)
as \( x \to y_i, \ i = 1, 2, \ldots, l \). Assume further that for some \( \epsilon > 0 \),

\[
|f(x)|W(x) = O(|x|^{-1/p} \log|x|^{-1/p} \cdots \log \cdots \log|x|^{-1/p-\epsilon})
\]  

(3.34)

as \( |x| \to \infty \). Then for any polynomial \( P \),

\[
\lim_{n \to \infty} I^*_n([f - P]^p W^{p-2}) = \|f - P\|^p_p .
\]  

(3.35)

**Proof.** We first show that \( |f - P|^p W^{p-2} \) is ‘monotone integrable’ in the sense of Definition 3.3 in Lubinsky and Sidi [19]. Let

\[
\phi(x) = |x|^{-1} \log|x|^{-1} \cdots \log \cdots \log|x|^{-1-\epsilon p},
\]

for small enough \( |x| \) and large enough \( |x| \). By Corollary 2 in [14], there exists an even nonnegative entire function \( G_0(x) \), with all even order derivatives nonnegative in \( \mathbb{R} \), satisfying

\[
\int_{-\infty}^{\infty} G_0(x)W^2(x) \, dx < \infty
\]

and

\[
G_0(x) \sim W^{-2}(x)\phi(|x|), \quad |x| \to \infty.
\]

By (3.34), we see that

\[
\lim_{n \to \infty} \sup \frac{|f - P|^p(x)W^{p-2}(x)/G_0(x)} < \infty.
\]

Next, by Corollary 4 in [14], there exist functions \( G_i(x) \) absolutely monotone in \( (-\infty, y_i) \) and completely monotone in \( (y_i, \infty) \) satisfying

\[
\int_{-\infty}^{\infty} G_i(x)W^2(x) \, dx < \infty
\]

and

\[
G_i(x) \sim \phi(|x - y_i|), \quad x \to y_i,
\]

\( i = 1, 2, \ldots, l \). Then by (3.33), we have for \( i = 1, 2, \ldots, l \),

\[
\lim_{x \to y_i} \frac{|f - P|^p(x)W^{p-2}(x)/G_i(x)} < \infty.
\]

Thus \( |f - P|^p W^{p-2} \) is ‘monotone integrable’ in the sense of [19], and Theorem 3.5(a) in [19] shows that

\[
\lim_{n \to \infty} K^*_n([f - P]^p W^{p-2}) = \int_{-\infty}^{\infty} |f - P|^p(x)W^p(x) \, dx .
\]  

(3.36)

Here \( K^*_n(\cdot) \) is the Gauss quadrature rule associated with \( W^2 \), modified as in (2.4A,B) and (2.5) in [19]. \( K^*_n(\cdot) \) is similar to the \( I^*_n[\cdot] \) of this paper but differs in the following respect: Whereas \( I^*_n[\cdot] \) used in this paper omits only the closest abscissa to each singularity \( y_i \), \( K^*_n(\cdot) \) may omit both the closest abscissas on the left and right of each singularity \( y_j \). Then (3.36) will imply (3.35) provided we can show the term included in \( I^*_n \), but possibly omitted from \( K^*_n \), converges to 0 as \( n \to \infty \). Thus if \( 1 \leq i \leq l \) and \( x_d(i, n) \) is not the closest abscissa to \( y_i \) but is
either the closest abscissa on the left or right, we must show
\[ \lim_{n \to \infty} \lambda_{d(i,n)} |f - P|^p W^{p-2}(x_{d(i,n)n}) = 0. \]  
(3.37)
Firstly, it follows from Lemma 3.1(iv) that \( \lambda_{d(i,n)n} = O(q_n/n) \) while the spacing property (3.5) ensures that \( |x_{d(i,n)n} - y_i| \sim q_n/n \). Then (3.33) yields (3.37). \( \square \)

4. Proof of Theorems 3, 4 and 5

It will be convenient to define some characteristic functions. Throughout, we let \( E \) denote the constant of Lemma 3.3 and we assume \( E \leq D \), where \( D \) is as in (1.15). The characteristic function of \( (-Eq_n/4, Eq_n/4) \) will be denoted by \( \chi_n(x) \), and the characteristic function of \( (-Eq_n/2, Eq_n/2) \) will be denoted by \( \chi^*_n(x) \). Finally,
\[ \chi^*_n(x) = 1 - \chi_n(x) \]  
(4.1)
denotes the characteristic function of \( \mathbb{R} \setminus (-Eq_n/4, Eq_n/4) \). Our strategy in proving Theorem 3 and 4 will be to write, for some polynomial \( P \),
\[ f - P = (f - P)\chi_n + (f - P)\chi^*_n. \]  
(4.2)
The two terms on the right hand side will be dealt with separately, in a manner similar to that in Nevai [28].

Lemma 4.1. Let \( W \in \mathcal{F}^*B \). Let \( 1 < q < \infty \). Let \( h : \mathbb{R} \to \mathbb{R} \) be real valued and measurable and let \( hW \in L_q(\mathbb{R}) \). Then there exists \( C \) independent of \( n \) and \( h \) such that
\[ \|S_n(h\chi^*_n, x)W\chi^*_n(x)\|_q \leq C\|hW\|_q. \]  
(4.3)

Proof. Let \( g \in L_q(\mathbb{R}) \). We use \( \sim \) to denote the Hilbert transform of \( g \), that is, for almost all \( x \in \mathbb{R} \),
\[ g^\sim(x) = \lim_{\delta \to 0^+} \int_{|t-x|<\delta} g(t)/(t-x) \, dt. \]
See [39]. It is a classical result of Riesz [39, p. 188] that \( \sim \) is bounded as an operator from \( L_q(\mathbb{R}) \) to \( L_q(\mathbb{R}) \). Now by the Christoffel–Darboux formula for \( K_n(x, t) \) [6], one may write
\[ S_n(h\chi^*_n, x) \]
\[ = (\gamma_{n-1}/\gamma_n) \int_{-\infty}^{\infty} (h\chi^*_n(t)) \{ p_n(x)p_{n-1}(t) - p_n(t)p_{n-1}(x) \} (x-t)^{-1}W^2(t) \, dt \]
\[ = (\gamma_{n-1}/\gamma_n) \{ p_n(x)(h\chi^*_nW^2)^\sim(x) - p_{n-1}(x)(h\chi^*_nW^2)^\sim(x) \}. \]
Using (1.15) and Lemma 3.1(ii), we see that for \( |x| \leq Dq_n \),
\[ |S_n(h\chi^*_n, x)W(x)| \leq C_1q_n^{1/2} \sum_{j=n-1}^n |(h\chi^*_np_jW^2)^\sim(x)|. \]
Using boundedness of \( \sim \) from \( L_q(\mathbb{R}) \) to \( L_q(\mathbb{R}) \) and the fact that \( E/2 \leq D \), we obtain
\[
\| S_n(h\chi_n^*, x)W\chi_n^*(x) \|_q \leq C_2 q^{1/2} \sum_{j=n-1}^n \| h\chi_n^*p \|_2 W^2 \|_q \\
\leq C_3 \| hW \|_q,
\]
by (1.15) and the definition of \( \chi_n^* \). □

The following lemma uses ideas due to Marcinkiewicz, as modified by Askey [1] and Nevai [28].

**Lemma 4.2.** Let \( W \in \mathcal{F}^*B \). Let \( 1 < p < \infty \). Let \( f \) be as in Lemma 3.4, let \( P \) be a polynomial, and let
\[
u_n = (f - P)\chi_n, \quad n = 1, 2, 3, \ldots.
\]
Then for some \( C_1 \) independent of \( n \) and \( P \),
\[
\limsup_{n \to \infty} \| L_n^*(u_n, x)W\chi_n^*(x) \|_p \leq C_1 \|(f - P)W\|_p.
\]

**Proof.** Let
\[
h_n(x) = \chi_n^*(x) \text{ sign}(L_n^*(u_n, x)) |L_n^*(u_n, x)|^{p-1} W^{p-2}(x) / \| L_n^*(u_n, x)W\chi_n^*(x) \|_p^{p-1},
\]
n = 1, 2, 3, \ldots. Then if \( q \) satisfies \( 1/p + 1/q = 1 \), we see that \( \| h_n W \|_q = 1 \). Further,
\[
\| L_n^*(u_n, x)W\chi_n^*(x) \|_p = \int_{-\infty}^\infty L_n^*(u_n, x)h_n(x)W^2(x) \, dx \\
= \int_{-\infty}^\infty L_n^*(u_n, x)S_n(h_n, x)W^2(x) \, dx \\
\text{(by orthogonality of } h_n - S_n(h_n, \cdot) \text{ to polynomials of degree } < n) \\
= \sum_{|xjn| \leq E^{q/p}} \lambda_j \| f - P \|_p(xjn)S_n(h_n, xjn),
\]
by the Gauss–Jacobi quadrature formula, the definition (4.4) of \( u_n \), and the interpolatory properties of \( L_n^* \). Since \((p - 2)/p + (q - 2)/q = 0\), we have
\[
W^{(p-2)/p}(xjn)W^{(q-2)/q}(xjn) = 1
\]
and so we may apply Hölder’s inequality to the right hand side of (4.6) to deduce that
\[
\| L_n^*(u_n, x)W\chi_n^*(x) \|_p \leq \{ T_1(n) \}^{1/p} \{ T_2(n) \}^{1/q},
\]
where
\[
T_1(n) = \sum_{|xjn| \leq E^{q/p}} \lambda_j \| f - P \|_p(xjn)W^{p-2}(xjn)
\]
and
\[
T_2(n) = \sum_{|xjn| > E^{q/p}} \lambda_j \| f - P \|_p(xjn)W^{q-2}(xjn).
\]
and
\[ T_2(n) = \sum_{|x_{jn}| \leq Eq_n^{1/4}} \lambda_{jn} |S_n(h_n, x_{jn})|^q W^q - 2(x_{jn}). \]  
(4.9)

Here by Lemma 3.4, as \( n \to \infty \),
\[ T_1(n)^{1/p} \leq \left\{ I_n^p \left[ |f - P|^{pW^p - 2} \right] \right\}^{1/p} \to \| (f - P)W \|_p . \]  
(4.10)

Further, Lemma 3.3 with \( r = 2 - q \), \( \delta = E/4 \) and \( \epsilon = E/2 \), shows that
\[ T_2(n)^{1/q} \leq C_2 \| S_n(h_n, x)W \|_q \leq C_3 \| h_n W \|_q = C_3 \]  
(4.11)

by Lemma 4.1 and as \( h_n = h_n \chi_n^* \). Then the result follows from (4.7) to (4.11).

We can now prove the following lemma.

**Lemma 4.3.** Let \( W \in \mathcal{F}^* B \). Let \( 1 < p < \infty \). Let \( f \) be as in Lemma 3.4. Assume further that \( \psi \) is a positive function in \( \mathbb{R} \) such that
\[ |f(x)|W(x) - O(\psi(x)) , \quad |x| \to \infty , \]  
(4.12)

where for large enough \( |x| \),
\[ \psi(x) = (1 + |x|)^a (\log |x|)^b (\log \log |x|)^c \cdots (\log \cdots \log |x|)^d , \]
and where \( a, b, c, \ldots \) are real numbers of which at most finitely many are nonzero. Finally, assume that \( \gamma \geq 0 \) is such that
\[ \lim_{n \to \infty} \left\{ \int_0^{q_n} \psi(u) \, du \right\} q_n^{-1/2 - \gamma} \| p_n W \|_p = 0 . \]  
(4.13)

Then, if \( P \) is any polynomial and \( \{ u_n \} \) is given by (4.4),
\[ \limsup_{n \to \infty} \| L_n^\ast(u_n, x)W(x)(1 + |x|)^{-\gamma} \|_p \leq C_1 \| (f - P)W \|_p , \]  
(4.14)

where \( C_1 \) is independent of \( n \) and \( P \).

**Proof.** Since \( (1 + |x|)^{-\gamma} \leq 1 \), Lemma 4.2 shows that it suffices to prove
\[ \lim_{n \to \infty} \| L_n^\ast(u_n, x)W(x)(1 + |x|)^{-\gamma} \|_{L_p(|x| \geq Eq_n^{1/2})} = 0 . \]  
(4.15)

Now, by (2.1),
\[ L_n^\ast(u_n, x) = \sum_{|x_{jn}| \leq Eq_n^{1/4}} (f - P)(x_{jn}) \lambda_{jn} (\gamma_{n-1}/\gamma_n) p_{n-1}(x_{jn}) p_n(x)/(x - x_{jn}) . \]

By Lemma 3.1(u), (1.15) and as \( E \geq D \), we obtain for \( |x| \geq Eq_n^{1/2} \),
\[ |L_n^\ast(u_n, x)W(x)(1 + |x|)^{-\gamma}| \leq C_2 q_n^{-1/2 - \gamma} \| p_n W \|_p \sum_{j \notin \mathcal{Y}(n)} \lambda_{jn} |f - P|(x_{jn})W^{-1}(x_{jn}) . \]  
(4.16)

Let \( A > 0 \) be so large that \( |y_j| \leq A, j = 1, 2, \ldots , l \). We see from (4.12) that there exists \( C_3 \)
depending on \( f, P \) and \( A \) such that
\[
|f - P|(x) \leq C_3 W^{-1}(x) \psi(x), \quad |x| \geq A.
\]
Then
\[
\sum_{|x_j| \geq A} \lambda_{jn} |f - P|(x_j) W^{-1}(x_j) \leq C_5 \sum_{j=1}^n \lambda_{jn} W^{-2}(x_j) \psi(x_j) \leq C_4 \int_0^q \psi(u) \, du,
\]
by Theorem 6. Next, it follows easily from Lemma 3.4 that
\[
\lim_{n \to \infty} \sum_{|x_j| \leq A \setminus \mathcal{G}(n)} \lambda_{jn} |f - P|(x_j) W^{-1}(x_j) = \int_{-A}^A |f - P|(x) W(x) \, dx.
\]
Since \( \psi \) is a positive function, we deduce that for some \( C_5 \) depending on \( f, P \) and \( A \), the sum in the right hand side of (4.16) is bounded above by
\[
C_5 \int_0^q \psi(u) \, du.
\]
Then (4.15) follows easily from (4.13) and (4.16).

Next, we need some weighted Nikolskii inequalities (Mhaskar [22], Mhaskar and Saff [24], Nevai and Totik [34], Levin and Lubinsky [11, Theorem 7.6]).

Lemma 4.4. Let \( W \in \mathcal{F}^* \). Let \( 1 \leq p \leq r \leq \infty \). Then for all polynomials \( P \) of degree at most \( n \),
\[
\|P W\|_p \leq C_1 q_n^{1/p - 1/r} \|P W\|_r \tag{4.17}
\]
and
\[
\|P W\|_r \leq C_2 (n/q_n)^{1/p - 1/r} \|P W\|_p \tag{4.18}
\]
Proof. If \( W \in \mathcal{F}^* \) satisfies (1.11), then \( W \) is essentially a superregular weight in the sense of Mhaskar [22]. Then Corollary 2 and Theorem 3 in [22] yield the result. If \( W \in \mathcal{F}^* \) satisfies (1.12), then Theorem 1 in [22] and Lemma 3.1 (ii) and (iii) yield the result.

We can now prove the following lemma.

Lemma 4.5. Let \( W \in \mathcal{F}^* B \). Let \( A > 0 \) and \( 1 < p < \infty \). Then
\[
\lim_{n \to \infty} (\max (\|l_{jn} W\|_p : |x_j| \leq A)) = 0. \tag{4.19}
\]
Proof. If \( p \geq 2 \) or \( q_n^2 = O(n) \), this follows easily from the weighted Nikolskii inequalities and Lemma 3.1(iv). However if \( W \) satisfies (1.12), \( n = o(q_n^2) \), so we provide a proof of (4.19) which works in general.

Much as in the proof of Lemma 4.2 up to (4.6), and as \( l_{jn} = L_n(l_{jn}) \), we see
\[
\|l_{jn}(x) W_{X_n}^*(x)\|_p = \|L_n(l_{jn}, x) W_{X_n}^*(x)\|_p = \lambda_{jn} S_n(H_n, x_{jn}) \tag{4.20}
\]
where \( H_n \) is a measurable function with \( \|H_n W\|_q = 1 \), and where \( 1/p + 1/q = 1 \). As at (4.7) to
(4.9), the right hand side of (4.20) may be bounded above by $T_1(n)^{1/p}T_2(n)^{1/q}$, where

$$T_1(n) = \lambda_n W_n^{-2}(x_{jn}) = o(1) , \quad \text{uniformly for } |x_{jn}| \leq A ,$$

by Lemma 3.1(iv). Further $T_1(n)$ may be given by (4.9), with $h_n$ replaced by $H_n$. As in the proof of Lemma 4.2, we see that (4.11) holds with some $C_3$ independent of $|x_{jn}| \leq A$ and $n$. Thus

$$\lim_{n \to \infty} \|l_m(x)W_n(x)\|_p = 0 \quad \text{uniformly for } |x_{jn}| \leq A .$$

Hence (4.19) follows if we can show that uniformly for $|x_{jn}| \leq A$,

$$\lim_{n \to \infty} \|l_m(x)W_n(x)\|_{L_p(|x| \leq Eq_n/2)} = 0 .$$

The proof of this is similar to that of Lemma 4.3. Indeed, as at (4.16), we see that for $|x| \geq Eq_n/2$, and $|x_{jn}| \leq A$,

$$|l_m(x)W_n(x)| \leq C_1 q_n^{-1/2} |p_n W|(x) \lambda_n W^{-1}(x_{jn}) \leq C_2 q_n^{-1/2} n^{-1} |p_n W|(x) ,$$

by Lemma 3.1(iv). Next, the weighted Nikolskii inequalities of Lemma 4.4 and the orthonormality of $p_n$ show that

$$\|p_n W\|_p \leq \begin{cases} C_1 q_n^{1/p - 1/2} , & p \leq 2 , \\ C_1 (n/q_n)^{1/2 - 1/p} , & p > 2 . \end{cases}$$

Together with (4.22) this easily yields (4.21), as $q_n = o(n)$.

Following is our general theorem.

**Theorem 7.** Let $W \in \mathcal{F}^* B$. Let $1 < p < \infty$. Let $f$ be as in Lemma 3.4. Assume also that $\psi$ is a positive function in $\mathbb{R}$ such that

$$|f(x)|W(x) = o(\psi(x)) , \quad |x| \to \infty ,$$

where for large enough $|x|$,

$$\psi(x) = (1 + |x|)^a(\log |x|)^b(\log \log |x|)^c \cdots (\log \cdots \log |x|)^d ,$$

and $a, b, c, \ldots$ are real numbers of which at most finitely many are non-zero. Further, assume that $\gamma \geq 0$ satisfies

$$\lim_{n \to \infty} \left\{ \int_0^{q_n} \psi(u) \, du \right\} q_n^{-1/2 - \gamma} \|p_n W\|_p = 0 ,$$

and if $p > 2$, assume also

$$\limsup_{n \to \infty} \psi(q_n) q_n^{1/2} (n/q_n)^{1/2 - 1/p} < \infty .$$

Then

$$\lim_{n \to \infty} \|\{f(x) - L_n^*(f, x)\} W(x)(1 + |x|)^{-\gamma}\|_p = 0 .$$

**Proof.** Let $P$ be a polynomial. For $n = 1, 2, 3, \ldots$, let

$$u_n = (f - P) \chi_n \quad \text{and} \quad w_n = (f - P) \chi_n^* .$$
with the notation of (4.1). If \( n \) exceeds the degree of \( P \),

\[
P(x) = L_n(P, x) = \sum_{j \in \mathcal{F}(n)} P(x_j) I_{jn}(x) + \sum_{j \in \mathcal{F}(n)} P(x_j) I_{jn}(x).
\]

Then if \( n \) is large enough, we see from (4.1), (4.29) and the definition of \( L_n^*(f, x) \) that

\[
f(x) - L_n^*(f, x) = (f - P)(x) - L_n^*(f - P, x) + \sum_{j \in \mathcal{F}(n)} P(x_j) I_{jn}(x)
\]

\[
= (f - P)(x) - L_n^*(u_n, x) - L_n^*(w_n, x) + \sum_{j \in \mathcal{F}(n)} P(x_j) I_{jn}(x).
\]

By Lemmas 4.3 and 4.5 and as \((1 + |x|)^{-\gamma} \leq 1\) we have

\[
\lim_{n \to \infty} \| (f(x) - L_n^*(f, x)) W(x)(1 + |x|)^{-\gamma} \|_p \\
\leq (1 + C_1) \|(f - P)W\|_p + \lim_{n \to \infty} \| L_n^*(w_n, x)W(x) \|_p,
\]

(4.30)

where \( C_1 \) is independent of \( P \). Since

\[
\lim_{|x| \to \infty} Q(x)/|x| = \infty,
\]

the weighted polynomials \( PW \) are dense in the space \( X = \{ f : fW \in L_p(\mathbb{R}) \} \) with \( \| f \|_X = \| fW \|_p \) (see, for example; [41, Proposition 5.4]). Thus the first term in the right hand side of (4.30) may be made arbitrarily small. Then (4.28) follows if we can show that for every polynomial \( P \),

\[
\lim_{n \to \infty} \| L_n^*(w_n, x)W(x) \|_p = 0.
\]

(4.31)

Since \( w_n(x) = 0 \) for \( |x| \leq Ep_n/4 \), we see that

\[
L_n^*(w_n, x) = L_n(w_n, x) \quad \text{for } n \text{ large enough.}
\]

By the Gauss–Jacobi quadrature formula,

\[
\| L_n(w_n, x)W(x) \|_2^2 = \int_{-\infty}^{\infty} L_n^2(w_n, x)W^2(x) \, dx = \sum_{|x_j| = Ep_n/4} \lambda_{jn} (f - P)^2(x_{jn})
\]

\[
= o\left( \sum_{|x_j| = Ep_n/4} \lambda_{jn} W^{-2}(x_{jn}) \psi^2(q_n) \right) = o(q_n \psi^2(q_n))
\]

by (4.24), Theorem 6, Lemma 3.1(vi) and as \( \psi(u) \sim \psi(q_n) \), for \( Ep_n/4 \leq |u| \leq x_{1n} \).

Suppose now \( p \leq 2 \). Since \( f \) satisfies (3.34) in Lemma 3.4, it is easy to see that we may assume that \( a \leq -1/p \) and \( b \leq -1/p \) in the definition (4.25) of \( \psi \)—for if (4.26) holds for a given \( \psi \), it also holds for any smaller \( \psi \). Then the weighted Nikolskii inequalities in Lemma 4.4 show that

\[
\| L_n(w_n, x)W(x) \|_p \leq C_2 q_n^{-1/p - 1/2} o(q_n^{1/2} \psi(q_n)) = o(1),
\]

as \( a, b \leq -1/p \). Thus (4.31) holds for \( p < 2 \). Next, suppose \( p > 2 \). By Lemma 4.4,

\[
\| L_n(w_n, x)W(x) \|_p \leq C_3 (n/q_n)^{-1/2 - 1/p} o(q_n^{1/2} \psi(q_n)) = o(1),
\]

by (4.27). Thus (4.31) still holds. \( \square \)
Proof of Theorem 3. We see firstly that $f$ satisfies the hypotheses of Lemma 3.4. We must choose a suitable $\psi$ for which (4.24) and (4.26) hold with $\gamma = 0$. Let $0 < \delta < \epsilon$ and for large $|x|$, let $\psi$ be given by (4.25), where

$$
a = b = c = \cdots = -1/p \quad \text{and} \quad d = -1/p - \delta.
$$

Then (4.24) follows from (1.20), and by (4.23),

$$
\left( \int_0^{q_n} \psi(u) \, du \right) q_n^{-1/2} \| p_n W \|_p = o(q_n^{1-1/p}) q_n^{-1/2} O(q_n^{1/p-1/2}) = o(1).
$$

Thus (4.26) holds with $\gamma = 0$. \qed

Proof of Theorem 4. We must show that $\psi(x) = |x|^{-\beta}$ satisfies (4.24), (4.26) and (4.27) with the given value of $\gamma$, and that $f$ satisfies the hypotheses of Lemma 3.4. Firstly, we may rewrite (1.24) in the form

$$
\limsup_{n \to \infty} \left( n/q_n \right)^{1/2-1/p} q_n^{1/2-\beta} < \infty.
$$

(4.32)

Since $q_n = o(n)$ (by Lemma 3.1(viii)), it follows that $\beta > \frac{1}{2}$. Thus $f$ fulfills the requirement (3.34) of Lemma 3.4. Next, (1.23) trivially implies (4.24), while (4.32) is equivalent to (4.27) and (1.25) is equivalent to (4.26). \qed

Proof of Theorem 5. Let $q$ satisfy $1/p + 1/q = 1$. By (1.36),

$$
\left| \int_{-\infty}^\infty f(x) k(x) \, dx - I_n^*[k, f] \right| = \left| \int_{-\infty}^\infty \{ f(x) - L_n^*(f, x) \} k(x) \, dx \right|.
$$

$$
= \left\| (f(x) - L_n^*(f, x)) W(x)(1 + |x|)^{-\gamma} \right\|_p
$$

$$
\times \left\| k(x) W^{-1}(x)(1 + |x|)^\gamma \right\|_q
$$

by Hölder's inequality. Here if $1 < p \leq 2$, we choose $\gamma = 0$ and otherwise take $\gamma$ to be the value in (1.38). Then Theorems 3 and 4 yield the result. \qed

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