Maximum Likelihood Estimation of Multiple Frequencies with Constraints to Guarantee Unit Circle Roots

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Abstract—A recently proposed approximate maximum-likelihood estimator (MLE) of multiple exponentials converges the frequency estimation problem into a problem of estimating the coefficients of a \(z\)-polynomial with roots at the desired frequencies. Theoretically, the roots of the estimated polynomial should fall on the unit circle, but MLE, as originally proposed, does not guarantee unit circle roots. This drawback sometimes causes merged frequency estimates, especially at low SNR.

If all the sufficient conditions for the \(z\)-polynomial to have unit circle roots are simultaneously met, the optimization problem becomes too nonlinear and it loses the desirable weighted-quadaratic structure of MLE. In this correspondence, the exact constraints are imposed on each of the first-order factors corresponding to individual frequencies for ensuring unit circle roots. The constraints are applied during optimization alternately for each frequency. In the absence of any merged frequency estimates, the RMS values more closely approach the theoretical Cramer-Rao (CR) bound at low SNR levels.

I. INTRODUCTION

Estimating the underlying parameters of multiple complex exponential signals in noise remains a vigorously researched topic in signal processing [1]-[13]. For a single sinusoid or when the multiple frequencies are well-separated, the periodogram performs reasonably well. However, if the frequencies are closely spaced, which often occurs when the data length is limited or the aperture is too small, the periodogram fails to distinguish the frequencies and produces merged frequency estimates. In order to overcome the periodogram’s resolution limitation, many high-resolution methods have been developed in the past two decades [1]-[13]. In contrast to the periodogram, these methods make effective use of some underlying properties of the true sinusoidal signal model.

Among all the existing high-resolution frequency estimation methods, the MLE appears to provide the most accurate frequency estimates and has the lowest SNR threshold [1]-[4]. Other high-resolution methods rely on signal or noise subspace information, which is extracted from the eigendecomposition of covariance matrix or SVD of data matrix [5], [7]-[11]. On the other hand, the MLE considers the exact model of the exponential signal and attempts to maximize the exact likelihood function to estimate the unknowns. For a single sinusoid, the peak of the periodogram itself corresponds to the ML estimate, but for multiple exponentials, the MLE turns out to be a nonlinear optimization problem [1]-[6], [12], [13].

The MLE approaches developed independently in [1] and [2] estimate the frequencies from the roots of a \(z\)-polynomial. It may be noted here that in literature, these methods are sometimes referred to as KiSS [1], [5], [6] or IQML [2]. In the polynomial domain, the ML optimization problem turns out to be quasi-linear where a weighted-quadratic criterion is minimized iteratively. Although effective to a large extent, MLE is known to possess one fundamental drawback: the optimization procedure in [1], [2] does not impose sufficient theoretical constraints on the polynomial coefficients for the estimated roots to fall on the unit circle. The primary goal of this work is to address this unresolved problem in MLE.

Two conditions must be satisfied for a general \(p\)-th order \(z\)-polynomial to have \(p\) unit circle roots: conjugate symmetry (C1) and a derivative constraint (C2), the details of which are given later. In MLE, only C1 is imposed. The derivative constraint makes the problem highly nonlinear, and hence, C2 cannot be incorporated in the weighted-quadaratic framework of MLE; however, when \(p > 1\), C1 alone is not sufficient for unit circle roots. Furthermore, from the theory of linear-phase FIR filters, it is well-known that the roots of a symmetric \(z\)-polynomial may fall on the unit circle, or they may be in reciprocal pairs falling inside and outside of the unit circle. In fact, it was demonstrated in [1] and [3] that if SNR \(\leq 10\, \text{dB}\) and the frequencies are spaced closely, the roots extracted by MLE sometimes appear in reciprocal pairs. In such cases, two frequencies merge to produce only a single frequency estimate. The alternate approach proposed in this correspondence attempts to alleviate this limitation.

There is one exception to the two conditions stated above: For \(p = 1\), the conjugate symmetry constraint C1 alone is sufficient for the single root to fall on the unit circle. This is the main idea that will be utilized in developing the proposed constrained-MLE (C-MLE) algorithm. Specifically, C1 will be imposed on each of the first-order factors of the \(p\)-th order \(z\)-polynomial, such that each individual root falls on the unit circle. This process need not be applied to all the frequencies at all SNRs. The constraints are imposed only on those first-order factors that produce merged frequency estimates at convergence of MLE. The factors for which the roots are already on the unit circle are held fixed. The proposed algorithm may be considered to be a polynomial-domain counterpart of the "alternating projection" approach [13] where the ML criterion is minimized w.r.t. one frequency at a time while the other frequencies are held at the previously estimated values. Our work appears to be the first attempt to guarantee unit circle roots on the polynomial coefficients for maximum-likelihood frequency estimation. The constraints are primarily effective at low SNR levels when there is a higher possibility for MLE to produce merged frequency estimates. In simulations, the RMS values of the frequency estimates using C-MLE were found to be closer to the theoretical CR bounds than those of the original MLE algorithm.

This correspondence is arranged as follows: In Section II, the MLE problem is stated, the original MLE algorithm is briefly discussed, and the conditions needed for unit circle roots are stated. In Section III, the proposed constrained version of MLE is introduced. Simulation results are given in Section IV to verify the performance of C-MLE.

II. THE MAXIMUM LIKELIHOOD PROBLEM AND A BRIEF OVERVIEW OF MLE

The observed samples of a complex multiple exponential signal can be represented as

\[
x(n) = \sum_{k=1}^{p} c_k e^{j\omega_k n} + \phi_k + z(n) \quad n = 0, 1, \ldots, N-1
\]

where \(\omega_k\), \(c_k\), and \(\phi_k\) denote the unknown angular frequency, amplitude, and phase, respectively, of the \(k\)th sinusoid; \(p\) is the assumed number of sinusoids; and \(z(n)\) represents i.i.d. \(N(0, \sigma^2)\) Gaussian noise samples. For this signal model, the MLE corresponds to optimization of the following error criterion [1]-[4]:

\[
\min_{\omega_1, \ldots, \omega_p, a_1, \ldots, a_p} ||x - Ta||_2^2
\]

where

\[
\begin{align*}
\min_{\omega_1, \ldots, \omega_p, a_1, \ldots, a_p} ||x - Ta||_2^2
\end{align*}
\]
where

\[ x = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix} \triangleq \mathbf{Ta} \]

\[ \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix} \]

\[ \mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N) \end{pmatrix} \]

\[ \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix} \]

\[ a_k \triangleq c_k e^{j\omega_k} \text{ for } k = 1, 2, \ldots, p, \] respectively, are the complex amplitudes. The MLE problem stated in (2) is a nonlinear optimization problem with respect to the angular frequencies. Instead, MLE forms an alternative but equivalent error criterion in the polynomial coefficient domain, which has a quasi-linear structure that is well-suited for iterative optimization. A brief summary of the MLE criterion is in order.

Let \( B(z) \triangleq b_0 + b_1 z^{-1} + \ldots + b_p z^{-p} \) be a \( p \)th degree \( z \)-polynomial with \( p \) roots at \( e^{j\omega_1}, e^{j\omega_2}, \ldots, e^{j\omega_p} \), respectively, and \( \mathbf{b} \triangleq (b_0 b_1 b_2 \cdots b_p)^T \) be the coefficient vector. The MLE criterion for estimating \( \mathbf{b} \) in [1]–[4]:

\[ \min_{\mathbf{b}} E(\mathbf{b}) = \mathbf{b}^H \mathbf{X}^H (\mathbf{BB}^H)^{-1} \mathbf{X} \mathbf{b} \]

(4)

where

\[ \mathbf{B} \triangleq \begin{pmatrix} b_p & \ldots & b_1 & b_0 \\ 0 & \ldots & b_p & b_0 \\ \vdots & \ddots & \ldots & \vdots \\ 0 & \ldots & b_0 & b_p \end{pmatrix} \]

\[ \mathbf{X} \triangleq \begin{pmatrix} x(p) & \ldots & x(0) \\ x(p+1) & \ldots & x(1) \\ \vdots & \ddots & \vdots \\ x(N-1) & \ldots & x(N-p-1) \end{pmatrix} \]

(5)

The criterion in (4) appears to be quadratic in \( \mathbf{b} \), except that the weight matrix itself depends on the unknown coefficients. Hence, this criterion is minimized iteratively. At the \((k-1)\)th iteration

\[ \min_{\mathbf{b}} \mathbf{b}^H \mathbf{X}^H (\mathbf{BB}^H)^{-1} \mathbf{X} \mathbf{b} \]

(6)

is optimized, where the weight matrix \((\mathbf{BB}^H)\) is formed using the estimate of \( \mathbf{b} \) found at the previous iteration. At convergence of these iterations, the frequencies are found from the roots of the estimated polynomial \( \hat{B}(z) \). Unfortunately, direct optimization of the criterion in (4) does not guarantee that the roots of \( \hat{B}(z) \) will indeed fall on the unit circle, and it was recognized in [1], [3] that two conditions must be satisfied to guarantee unit circle roots.

C1: The coefficients possess conjugate symmetry

\[ b_k = b_{p-k}, \text{ for } k = 0, 1, \ldots, p \]

(7)

and

C2: For \( p > 1 \), the derivative of \( B(z) \), i.e.

\[ B'(z) \triangleq \frac{\partial B(z)}{\partial z} \]

(8)

must have zeros either inside or on the unit circle.

The polynomial domain MLE, as originally proposed, imposes the conjugate symmetry constraint only [1], [2]. C2 makes the optimization problem highly nonlinear, and the weighted-quadratic structure of (4) is lost if C2 is incorporated in the algorithm. Hence, no attempt was made in [1]–[4] to include C2 in the algorithm. But if \( p > 1 \), C1 is not a sufficient condition for unit circle roots. The same condition may, in fact, lead to roots in reciprocal pairs that can and do occur in MLE, especially at low SNR. In such cases, two closely spaced frequencies are estimated as a single frequency only [1], [3].

**Important Observation:** For \( p = 1 \), the conjugate symmetry alone is a sufficient condition to ensure unit-circle root. Hence, we propose to impose C1 sequentially on each first-order factor of \( B(z) \) during the optimization of (4). In that case, the optimization at each step will be with respect to only a first-order factor of \( B(z) \), and hence, there will be no need to satisfy C2.

### III. CONSTRAINED MLE (C-MLE)

The \( p \)-th order polynomial \( B(z) \) can be expressed in factored form as

\[ B(z) = B^{(p-1)}(z) B^{(1)}(z) \]

(9)

where \( B^{(p-1)}(z) \Delta \cong b_0^{(p-1)} + b_1^{(p-1)} z^{-1} + \ldots + b_p^{(p-1)} z^{-p+1} \) and \( B^{(1)}(z) \cong b_0^{(1)} + b_1^{(1)} z^{-1} \) are the \((p-1)\)th-order and first-order factors, respectively. If conjugate symmetry is imposed on the first-order factor, then \( B^{(1)}(z) = b_0^{(1)} + b_1^{(1)} z^{-1} \). Note that in (9), the coefficients of the polynomial \( B(z) \) are formed as the convolution of the coefficients of \( B^{(p-1)}(z) \) and \( B^{(1)}(z) \). Hence, in matrix-vector notation

\[ \mathbf{b} = \begin{pmatrix} b_0^{(p-1)} \\ b_1^{(p-1)} \\ b_0^{(1)} \\ b_1^{(1)} \end{pmatrix} \]

(10)

\[ \Delta \mathbf{B}_{p-1} = \begin{pmatrix} 1 & j \\ 1 & -j \end{pmatrix} \Delta \mathbf{B}_{p-1} \mathbf{C} \]

where \( \mathbf{B}_{p-1} \) denotes the matrix-factor with the \( r \)th first-order factor removed and \( b_0^{(1)} \cong b_0^{(1)} + j b_1^{(1)} \). Using (10) in (6), each first-order factor of \( B(z) \) is estimated by optimizing

\[ \min_{\mathbf{b}_i} \mathbf{b}_i^H \mathbf{C}_i^H \mathbf{W}_i^{(k-i)} \mathbf{X}_i (\mathbf{B}_i^{(k-i)} \mathbf{B}_i^{(k-i)})^{-1} \mathbf{X}_i \mathbf{b}_i \]

(11)

for \( i = 1, 2, \ldots, p \). This is a weighted-quadratic criterion of the form

\[ \mathbf{b}_i^H \mathbf{W}_i^{(k-i)} \mathbf{b}_i \]

(12a)

where

\[ \mathbf{W}_i^{(k-i)} \cong \mathbf{C}_i^H \mathbf{B}_i^{(k-i)} \mathbf{X}_i (\mathbf{B}_i^{(k-i)} \mathbf{B}_i^{(k-i)})^{-1} \mathbf{X}_i \mathbf{C}_i \]

and

\[ \mathbf{b}_i \cong \begin{pmatrix} b_0^{(1)} \\ b_1^{(1)} \end{pmatrix} \]

(12b)

Note that the weight matrix \( \mathbf{W}_i^{(k-i)} \) is formed with the estimates found at \((k-i)\)th iteration step when the unconstrained MLE algorithm is assumed to have converged. The criterion in (11) can be optimized sequentially or concurrently for each first-order factor. At each iteration, \( \mathbf{b}_i \) is estimated as the eigenvector corresponding to the minimum eigenvalue of \( \mathbf{W}_i^{(k-i)} \) \( \cong \mathbf{R}^{2r} \) [3], [4]. The advantage of using (12a) instead of (6) is that since each \( B^{(1)}(z) \) is a first-order \( z \)-polynomial, the conjugate symmetry constraint is sufficient to guarantee the root of \( B^{(1)}(z) \) to fall on the unit circle. In practice, the alternate optimization procedure in (11) need not be carried out for all the \( p \) factors of \( B(z) \). It needs to be invoked only in those
IV. SIMULATION RESULTS

The algorithm described in this paper has been tested with the same simulated data set used in [1] and [2]. The following formula was used to generate the data:

\[ x(n) = a_1 e^{j\omega_1 n} + a_2 e^{j\omega_2 n} + z(n) \]  
\[ n = 0, 1, \ldots, 24 \]

where \( \omega_1 = 2\pi f_1, \omega_2 = 2\pi f_2 \), with \( f_1 \) and \( f_2 \) being 0.52 and 0.50, respectively; \( a_1 = 1; a_2 = e^{j\frac{\pi}{6}} \); and \( z(n) \) is a computer-generated white zero-mean, complex Gaussianly distributed noise sequence with variance \( \sigma^2 \), i.e., \( \frac{\sigma^2}{2} \) is the variance of the real and the imaginary parts of \( z(n) \). SNR is defined as \( 10\log_{10}(\frac{\sigma_1^2}{\sigma_2^2}) \). Two hundred data sets with independent noise epochs were used.

Fig. 1(a) and (b) shows the estimated roots for 200 independent trials of MLE for SNR = 5 and 10 dB, respectively. Fig. 1(d) and (e) shows the corresponding results with C-MLE. For the 10-dB case, Fig. 1(c) and (f) shows only the merged cases before and after applying the exact constraints. Notice that the unit circle roots in Fig. 1(f) have a wider spread than the corresponding merged frequency estimates in Fig. 1(c). Fig. 2 compares the performance of MLE
the proposed method outperforms the AP method for this example, especially at low SNR. The performance of C-MLE has also been compared with that of the AP method [13], and the results are displayed in Fig. 3. Clearly, the proposed method outperforms the AP method for this example, especially at low SNR.

![Comparison of Performance](image1)

Fig. 2. Performance comparison of MLE and C-MLE with the theoretical CR-bound; 200 independent trials were used.

![Comparison of Performance](image2)

Fig. 3. Performance comparison of C-MLE and AP methods with the theoretical CR-bound; 200 independent trials were used.

and C-MLE with the theoretical CR bound. The results verify that C-MLE performs better than the original MLE at low SNR range. The performance of C-MLE has also been compared with that of the AP method [13], and the results are displayed in Fig. 3. Clearly, the proposed method outperforms the AP method for this example, especially at low SNR.

**REFERENCES**


**Asymptotic Analysis of the Cumulant-Based MUSIC Method in the Presence of Sample Cumulant Errors**

X. Fan and N. H. Younan

Abstract—This correspondence presents an asymptotical analysis of the cumulant-based MUSIC for harmonic retrieval. Assuming that measurement errors are zero mean and uncorrelated, it is shown that the harmonic estimates obtained by the cumulant-based MUSIC are asymptotically unbiased and statistically efficient. Furthermore, the asymptotical distribution of the estimates is also investigated for non-Gaussian measurement errors.

I. INTRODUCTION

It is well known that the cumulant-based MUSIC is commonly used for harmonic retrieval and its equivalent problem of direct-

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