Probabilistic capacity models and fragility estimates for RC columns retrofitted with FRP composites

Armin Tabandeh *, Paolo Gardoni

A R T I C L E   I N F O

Article history:
Received 15 May 2013
Revised 8 May 2014
Accepted 8 May 2014
Available online 28 May 2014

Keywords:
FRP retrofit
Bridge column
Probability
Markov Chain Monte Carlo (MCMC)
Bayesian approach
Fragility

A B S T R A C T

This paper proposes a probabilistic formulation to assess the effectiveness of the fiber reinforced polymer (FRP) retrofit schemes in enhancing the structural performance of reinforced concrete (RC) bridge columns. Two probabilistic models are proposed to predict the deformation capacities of retrofitted columns. One deformation model corresponds to the flexural failure and the other considers the bond failure in the lap-splice region. A Markov Chain Monte Carlo (MCMC) simulation method is used to estimate unknown model parameters in the context of a Bayesian updating approach. The probabilistic capacity models are used to estimate the fragility curves of three example columns. In this paper, fragility is defined as the conditional probability of failure for given deformation demand. The results compare the column fragilities before and after the application of the retrofit measure. The results from the example columns indicate that the use of FRP composites considerably reduced the fragility for the bond failure mode and is also beneficial but with a moderate impact when considering the flexural failure.

© 2014 Elsevier Ltd. All rights reserved.

1. Introduction

Around the world a number of reinforced concrete (RC) structures have been built with little or no consideration to seismic loading in part because the region was believed to be of low seismicity or due to inadequate design specifications [1]. However, more recent events have shown that regions that were traditionally considered to be of low seismicity might actually experience large earthquakes. For example, the May 2012 Emilia Romagna earthquakes, struck the Northern Italy, with two main shocks, $M_w \approx 5.8$ and $M_w \approx 5.9$, and a sequence of aftershocks in a region that was traditionally believed to be of low seismicity [2]. Likewise, the September 2010, $M_w \approx 7.1$, and the February 2011, $M_w \approx 6.2$, earthquakes near Christchurch, New Zealand, occurred on previously unknown faults in a region historically considered to be of low seismicity [3]. Consequences of past events have raised concerns about the vulnerability of existing structures and the need to undertake seismic retrofit programs.

Deficiencies associated with vulnerable structures include widely spaced transverse reinforcements, inadequate confinement of the core concrete, and short lap splices. In such structures, a premature failure may occur when the column loses its structural integrity before reaching its lateral load carrying capacity and experiencing a significant lateral deformation. Providing additional concrete confinement (e.g., retrofitting with fiber reinforced polymer (FRP) composites or steel jackets) increases the ductility capacity and can avert or delay the formation of undesirable brittle failure modes. Retrofit with FRP composites has become one of the preferred retrofitting schemes because of (1) their favorable properties (such as high strength/weight ratio and corrosion resistance [4–6]); (2) the fact that they increase the ductility and strength, and improve the concrete-reinforcement bond behavior without changing the structural stiffness [5–10]; (3) their ease of installation [5,6,10,11]; and (4) their lower expected cost in comparison to the traditional retrofitting strategies [6]. However, a cost-benefit analysis, that justifies the use of FRP composites as the retrofit measure of existing RC bridges, requires quantification of the reduction in failure probability of the retrofitted bridges. Estimating the reduction in the probability of failure requires a probabilistic formulation that accounts for the relevant uncertainties in the material properties, geometry, and structural behavior. The probabilistic assessment of un-retrofitted RC bridges has already been the subject of numerous studies (e.g., [12–20]). In addition, a few authors have also studied the reliability of flexural members (i.e., beams) retrofitted with FRP composites (e.g., [21–24]). However, the current approaches have four main limitations: (1) the modeling and estimation are structure-specific. Consequently, the result of the assessment for a particular structure cannot be used to assess the performance of another structure; (2) the modeling is carried out at the system level (i.e., the full structure.) Therefore,
the estimates of vulnerability cannot take advantage of experimental test data that are normally available for the structural components (i.e., individual members); (3) the available approaches generally assume an arbitrary function (often a normal or lognormal cumulative distribution function) to express the vulnerability as a function of some intensity measures of the loading and simply estimate the function parameters. As a result the parameters have no direct physical interpretation; (4) these approaches do not properly account for all the uncertainties that are involved.

This paper proposes two novel deformation capacity models for FRP retrofitted RC columns that account for the relevant uncertainties. One deformation model corresponds to the flexural failure and the other considers the bond failure in the lap-splice region. Based on experimental data [5,6,25,26], the flexural failure and the concrete-reinforcement bond failure are found to be the two typical failure modes of RC columns retrofitted with FRP composites under cyclic loading. The proposed probabilistic capacity models are developed by combining information from existing mechanistic models with information about the performance of FRP retrofitted columns from laboratory test data. A Bayesian approach is used to estimate unknown model parameters using the test data. One of the distinctive features of the Bayesian approach, used in this paper, is the ability to update existing models with new information when new data become available. The Bayesian updating is implemented using an efficient Markov Chain Monte Carlo (MCMC) simulation method [27]. The proposed models identify the most influential parameters affecting the capacity of retrofitted columns. As an application of the developed capacity models, the conditional failure probability (fragility) of three example columns, before and after retrofit, are developed for given values of the deformation demand, expressed in terms of drift ratio. The fragility curves objectively quantify the differences in the reliability and show the effectiveness of FRP composites as a retrofitting strategy. As a result, the developed fragility curves provide valuable information that can facilitate the decision-making process for the seismic retrofit of bridge columns.

Following this introduction, the next section discusses the experimental data used for the Bayesian updating. Next, we present the proposed probabilistic capacity models, including a discussion about the general formulation, variance stabilizing transformations, model calibration and model selection. Finally, the paper presents the assessment of the structural fragility of RC columns retrofitted with FRP composites and, as an illustration, the fragilities of three example columns, before and after retrofit, are compared.

2. Data for constructing probabilistic capacity models

For developing the proposed capacity models, we use data from experimental tests on FRP retrofitted columns available in the literature [5,6,25,26]. The data include the hysteretic responses of 41 FRP retrofitted columns with circular cross-sections, out of which 29 columns failed in flexure (8 columns with a lap-splice at the column base and 21 with no lap-splice), 11 columns experienced bond failure in the lap-splice region, and 1 test on a column with no lap-splice was stopped before failure due to the limitation of the test rig displacement. For each column, the data also include the applied axial load, the column geometry and material properties. Table 1 gives the range of the relevant parameters, where of the test rig displacement. For each column, the data also include the hysteretic responses of example columns, before and after retrofit, are compared.

We first need to estimate the structure's capacity. Gardoni et al. [12] proposed a general formulation for developing probabilistic and unbiased capacity models. In order to facilitate its application in practice and to account for the current available knowledge, the models are developed starting from accepted deterministic expressions. Correction terms are then introduced to correct for the inherent bias and a model error term is added to capture the remaining variability in the residuals due to, for example, inaccuracy of the model form, missing variables, and statistical uncertainties. The general formulation of the univariate probabilistic capacity model is as follows:

\[
C_k(x, \Theta_k) = c_k(x) + \gamma_k(x, \Theta_k) + \sigma_k \theta_k \quad k = f, b
\]

[12] proposed a general formulation for developing probabilistic and unbiased capacity models. In order to facilitate its application in practice and to account for the current available knowledge, the models are developed starting from accepted deterministic expressions. Correction terms are then introduced to correct for the inherent bias and a model error term is added to capture the remaining variability in the residuals due to, for example, inaccuracy of the model form, missing variables, and statistical uncertainties. The general formulation of the univariate probabilistic capacity model is as follows:

\[
c_k(x, \Theta_k) = c_k(x) + \gamma_k(x, \Theta_k) + \sigma_k \theta_k \quad k = f, b
\]

In this expression, \(c_k(x, \Theta_k)\) is the capacity measure of interest; the index \(k\) denotes the failure mode of interest, (i.e., \(k = f\) for the flexural failure and \(k = b\) for the bond failure); \(x\) is a set of measurable variables including material properties, member dimensions, and imposed boundary conditions; \(\Theta_k = (\theta_k, \sigma_k)\) is the set of unknown model parameters that need to be estimated; \(c_k(x)\) is an existing deterministic capacity model; \(\gamma_k(x, \Theta_k)\) is the correction term for the bias inherent in the deterministic model; and \(\sigma_k \theta_k\) is the model error, in which \(\sigma_k\) is the standard deviation of the model error which is assumed to be independent of \(x\) (homoskedasticity assumption) and \(\theta_k\) is a standard normal random variable (normality assumption). The normality and homoskedasticity assumptions and the assumption of an additive form of Eq. (1) typically can be satisfied using an appropriate variance stabilizing transformation. For this purpose, [28] suggested a parameterized family of transformations where a positive variable \(\delta\) (i.e., the drift ratio capacity) transforms to \(C_k = \delta^2\). This family of transformations is controlled by the parameter \(\lambda\). To satisfy the homoskedasticity, normality and additive assumptions, we use a logarithmic transformation to produce the variables \(C_k(x, \Theta_k)\) and \(c_k(x)\). The suitability of the logarithmic transformation (i.e., \(\lambda = 0\)) is also verified by means of diagnostic plots (e.g., [29]).

According to Gardoni et al. [12], the following functional form, linear in \(\theta_k\), is typically sufficiently flexible for writing \(\gamma_k(x, \theta_k)\):

\[
\gamma_k(x, \theta_k) = \sum_{i=1}^{p} \theta_{ki} h_{ki}(x)
\]

where \(h_{ki}(x)\)'s are a set of explanatory functions obtained using an appropriate Box–Cox transformation of basis functions of \(x\). \(h_{ki}(x)\) (i.e., \(h_{ki} = h_{ki}(x)\) in which \(\Lambda = (\lambda_1, \ldots, \lambda_P)\) is a vector of unknown exponents), and \(\theta_{ki}\)'s are the components of the vector \(\theta_k\).

Further, we construct a bivariate model based on the univariate models in Eq. (1) that also accounts for the correlation coefficient, \(\rho\), between \(\delta_f\) and \(\delta_b\) as

\[
\begin{align*}
C_f(x, \Theta_f, \rho, \sigma_f) &= C_f(x) + \gamma_f(x, \Theta_f) + \sigma_f \delta_f \\
C_b(x, \Theta_b, \sigma_b, \rho) &= C_b(x) + \gamma_b(x, \Theta_b) + \sigma_b \delta_b \\
\end{align*}
\]

where \(\Theta = (\Theta_f, \Theta_b, \rho)\) and \(\Lambda = (\lambda_f, \lambda_b)\) are the vectors of all unknown parameters.

### Table 1

Ranges of variables from database.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compressive strength of concrete (MPa)</td>
<td>(f_c)</td>
<td>16.7–44.8</td>
</tr>
<tr>
<td>Yield stress of longitudinal reinforcement (MPa)</td>
<td>(f_y)</td>
<td>300–481</td>
</tr>
<tr>
<td>Yield stress of transverse reinforcement (MPa)</td>
<td>(f_y^{tr})</td>
<td>210–492</td>
</tr>
<tr>
<td>FRP tensile strength (MPa)</td>
<td>(f_{FRP})</td>
<td>425–4433</td>
</tr>
<tr>
<td>Lap-splice length (mm)</td>
<td>(l_s)</td>
<td>305–760</td>
</tr>
<tr>
<td>Longitudinal reinforcement ratio (%)</td>
<td>(\rho_1)</td>
<td>1.23–5.79</td>
</tr>
<tr>
<td>FRP volumetric ratio (%)</td>
<td>(\rho_2)</td>
<td>0.03–10.49</td>
</tr>
<tr>
<td>Slenderness ratio</td>
<td>(H/D_x)</td>
<td>1.5–6</td>
</tr>
<tr>
<td>Axial load ratio</td>
<td>(4P/(\pi D_x^2))</td>
<td>0.05–0.34</td>
</tr>
</tbody>
</table>
3.1. Calibration of Box–Cox transformation, Bayesian updating, and model selection

We can find $\Lambda_k$ and $\Theta_k$ simultaneously with a nonlinear regression, treating $\Lambda_k$ and $\Theta_k$ in the same way. In this case $\Lambda_k$ and $\Theta_k$ are re-estimated at each step in the model selection process (discussed later in this section). However, providing accurate estimation of $\Lambda_k$ and $\Theta_k$ simultaneously can be computationally challenging. Furthermore, the increased number of unknown parameters (considering $\Lambda_k$ in addition to $\Theta_k$) might deteriorate the accuracy of the estimations (e.g., [30]). To circumvent such issues, we propose to find $\Lambda_k$ first, to satisfy the normality assumption, and then use a point estimate, at the closest rounded value in the 95% confidence interval, when re-estimating $\Theta_k$ in the model selection process. This approach is faster than, and the estimates are comparable to, those obtained using the nonlinear regression.

To estimate $\Lambda_k$, we use the multivariate generalization of the Box–Cox method to transform all $\gamma_i(x, \theta)$'s simultaneously to jointly normal functions $h_i(x)$'s. Following Velilla [31], we use the maximum likelihood criterion in this paper to make inferences about the parameters $\Lambda_k$. The likelihood function in this method can be written, following Weisberg [32], as

$$L(\Lambda_k, \mathbf{M}_k, \mathbf{V}_k) \propto \prod_{q=1}^{n} \left\{ \frac{1}{2} \ln [h_i(x_q) - \mathbf{M}_k] \right\}^{1/2}$$

$$\times \exp \left\{ -\frac{1}{2} \left[ h_i(x_q) - \mathbf{M}_k \right] \mathbf{V}_k \left[ h_i(x_q) - \mathbf{M}_k \right] \right\}$$

(4)

where $\mathbf{M}_k$ and $\mathbf{V}_k$ are the mean vector and covariance matrix of the random functions $h_i(x) = [\eta_{k1}^a(x), \ldots, \eta_{kb}^a(x)]^T$. Estimates of $\mathbf{M}_k$ and $\mathbf{V}_k$ can be obtained by maximizing $L(\Lambda_k, \mathbf{M}_k, \mathbf{V}_k)$ and have the following forms [32]:

$$\mathbf{M}_k(\Lambda_k) = \frac{1}{n} \sum_{q=1}^{n} h_i(x_q)$$

$$\mathbf{V}_k(\Lambda_k) = \frac{1}{n} \sum_{q=1}^{n} \left[ h_i(x_q) - \mathbf{M}_k(\Lambda_k) \right] \left[ h_i(x_q) - \mathbf{M}_k(\Lambda_k) \right]$$

(5)

where $\mathbf{M}_k(\Lambda_k)$ and $\mathbf{V}_k(\Lambda_k)$ are the sample mean and the sample covariance matrix, respectively. Substituting these estimates into the Eq. (4) and rearranging terms, we find the log-likelihood as

$$\ln L(\Lambda_k) \propto -\frac{n}{2} \ln (2\pi) - \frac{n}{2} \ln \left( \mathbf{V}_k(\Lambda_k) \right) - \frac{n}{2}$$

(6)

This equation can be maximized to obtain the estimates of $\Lambda_k$ by minimizing the determinant of $\mathbf{V}_k(\Lambda_k)$ over values of $\Lambda_k$.

Once we found the $\Lambda_k$, we can transform the explanatory functions and move forward to estimate $\Theta_k$. In the Bayesian approach, the unknown model parameters, $\Theta_k$, are estimated using the following updating rule [30]:

$$f(\Theta_k) = \xi(\Theta_k) p(\Theta_k)$$

(7)

where $f(\Theta_k)$ is the posterior distribution that reflects the updated state of information about $\Theta_k$: $L(\Theta_k)$ is the likelihood function that captures the information from the data; $p(\Theta_k)$ is the prior distribution that represents the information available before collecting the data; and $\xi = [f(\Theta_k)p(\Theta_k)d\Theta_k]^{-1}$ is a normalizing factor. The Bayesian inference relies on the feasibility of computing the posterior statistics. If $L(\Theta_k)p(\Theta_k)$ not being proportional to a familiar probability density function (PDF), computing $\xi$ can be challenging in the case of multidimensional problems. In this paper, we use an adaptive delayed rejection MCMC simulation method, also known as the DRAM method [27], to obtain the posterior statistics of the unknown model parameters.

In writing $L(\Theta_k)$, data need to be divided into equality data when the capacity of interest is observed and lower bound data (or censored data) when lower values are observed and not the actual capacity of interest. For instance, when a column cannot be pushed up to the failure point, due to the limitation of the stroke of the actuator, what we observe in the test is a lower bound capacity of the column and not the actual capacity. Additional detail about the treatment of censored data can be found in [12]. For the flexural model, there are 29 equality data and 12 lower bound data. Similarly, for the bond model, there are 11 equality data and 8 lower bound data. Under the assumption of statistically independent observations, $L(\Theta_k)$ can be written, following Gardoni et al. [12], as

$$L(\Theta_k) \propto \prod_{q=1}^{n} \left\{ \frac{1}{2} \ln [r_{kq}(\Theta_k)] \right\}$$

$$\times \prod_{q=1}^{n} \left\{ \frac{1}{2} \ln [\sigma_k] \right\}$$

(8)

where, $r_{kq}(\Theta_k) = c_{kq} - \gamma_k(x_q)$ is the prediction’s residual for the observation $q$. In the case of bivariate capacity model, the likelihood function, $L(\Theta_k)$, for each observation takes the form shown in Table 2 (adapted from Gardoni et al. [12]). When there is no prior information about the distribution of unknown model parameters, we use a noninformative prior distribution $p(\Theta_k) \propto 1/\sigma_k$ for univariate models and $p(\Theta_k) \propto 1/(\sigma_1, \sigma_k)$ for the bivariate model [12].

Inclusion of unimportant terms in $\gamma_i(x, \theta)$ might lead to a loss of precision in the estimates, larger values of $\sigma_k$ and over-fitting the data. A parsimonious form of $\gamma_i(x, \theta)$ (with as few explanatory functions as possible) can be developed using a stepwise deletion process. Following Gardoni et al. [12], we begin with the complete model (i.e., all $p$ candidate explanatory functions) and successively eliminate one function $h_i(x)$ at a time on the basis of the posterior statistics (i.e., the posterior coefficient of variation, COV) of the corresponding coefficient $\theta_i$. After each elimination, the remaining explanatory functions are refitted to the data and $\theta_{i,j}$’s are re-estimated.

Since $\sigma_k$ is approximately equal to the COV of the predicted capacity, the accuracy of the model is not expected to improve by including a term that has a COV much larger than $\sigma_k$. However, when the largest COV for a parameter becomes close in magnitude to $\sigma_k$, eliminating an extra term can lead to an unacceptable increase in $\sigma_k$. What constitutes an unacceptable increase in $\sigma_k$ is somewhat subjective and depends on the desired model accuracy, the level of uncertainty in the other variables in the model (e.g., the variability in the model parameters, $\theta_i$), and the desired level of parsimony. Furthermore, we can check for the pairwise linear correlations between the remaining parameters. If two parameters, $\theta_{i,k}$ and $\theta_{j,k}$, are strongly correlated (i.e., $|\rho_{i,j,k}| \geq 0.7$), we can linearly combine them as follows:

$$\hat{\theta}_{i,j,k} = \mu_{i,j,k} + \rho_{i,j,k} \sigma_{i,j,k} \left( \theta_{i,j,k} - \mu_{i,j,k} \right)$$

(9)

where, $\mu_{i,j,k}$ and $\sigma_{i,j,k}$ are the posterior mean and standard deviation of $\theta_{i,j,k}$, respectively.

The interpretation of the individual signs and values of multiple regression coefficients is problematic. Specifically, the coefficients

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Probability terms for bivariate capacity model (adapted from [12]).</strong></td>
</tr>
<tr>
<td><strong>Capacity model l</strong></td>
</tr>
<tr>
<td>Failure datum</td>
</tr>
<tr>
<td>$\Phi \Phi \left[ -\frac{h_k}{\sigma_k} \right]$</td>
</tr>
<tr>
<td>$\Phi \left[ -\frac{h_k}{\sigma_k} \right]$</td>
</tr>
</tbody>
</table>

Note: $\mu_{i,j,k} = \rho(\sigma_{i,j,k}) \left( \theta_{i,j,k} - 0 \right)$.
should not be simply interpreted as a direct relationship between the response and corresponding explanatory function, particularly in the case of high positive or negative correlations between coefficients [33]. Therefore, the physical interpretation of the signs and values of each coefficient needs to be further investigated.

3.2. Deformation capacity for flexural failure

Flexural failure of FRP retrofitted columns is associated with the loss of confinement in the plastic hinge region (e.g., [7]). In this mechanism, concrete crushing occurs along with the failure of FRP composites and the rupture or buckling of the longitudinal reinforcement. Flexural failures typically occur with some residual displacement ductility. Thus, a flexural failure is more desirable than other more brittle failure modes (e.g., [7]). Using Eq. (1), we can write a probabilistic deformation capacity model for the flexural failure as

\[
\ln \left[ \hat{\delta}_f (x, \Theta) \right] = \ln \left[ \hat{\delta}_f (x) \right] + \gamma_f (x, \Theta) + \sigma_f \epsilon_f
\]

(10)

where \( \hat{\delta}_f (x, \Theta) \) = \( \Delta_p / H \) is the drift ratio capacity for the flexural failure; \( \Delta_p \) is the corresponding deformation capacity; \( \hat{\delta}_f (x) \) = \( \Delta_p / H \) is an existing deterministic model for predicting \( \hat{\delta}_f \); and \( \Delta_p \) is the associated deterministic deformation capacity.

3.2.1. Deterministic model

For a bridge pier responding as a cantilever and subject to horizontal forces, the lateral deformation capacity can be estimated as

\[
\hat{\delta}_f (x) = \frac{1}{H} \left( \Delta_p + \Delta_\phi + \Delta_s \right)
\]

(11)

where \( \Delta_p \) is the elastic component due to the onset of yield; \( \Delta_\phi \) is the inelastic component due to the plastic flow; and \( \Delta_s \) is the contribution of the fixed-ended rotation due to the longitudinal reinforcement slipage from the anchorage zone when the end section reaches the ultimate curvature.

The term \( \Delta_\phi \) can be written as

\[
\Delta_\phi = \Delta_{\phi_{bl}} + \Delta_{\phi_{sh}}
\]

(12)

where \( \Delta_{\phi_{bl}} \) is the flexural component based on a linear curvature distribution along the equivalent column height; and \( \Delta_{\phi_{sh}} \) is the contribution of the shear deformation to the yield displacement. Given the yield curvature, \( \phi_y \), the flexural component of the yield displacement can be written as

\[
\Delta_{\phi_{bl}} = \frac{1}{3} \phi_y l_p^2
\]

(13)

where \( l_p = H + YP \) is the effective column length, in which \( YP \) is the length of yield penetration estimated as \( YP = 0.022 \delta_p \). [34]. The shear component, \( \Delta_{\phi_{sh}} \), for columns with FRP confinement is obtained from Biskinis and Fardis [35] empirical expression, valid for circular columns.

\[
\Delta_{\phi_{sh}} = 0.0027H \left[ 1 - \left( 2 \frac{H}{15 D_p} \right)^2 \right]
\]

(14)

The second term in Eq. (11), \( \Delta_p \), can be computed based on its relation to the sectional curvature. The plastic curvature, \( \phi_p \), is defined as \( \phi_p = \phi_u - \phi_y \); where \( \phi_u \) is the ultimate curvature. It is assumed that the plastic curvature occurs along the plastic hinge length \( l_p \) estimated as \( l_p = 0.07H + 8.16d_\phi \) for columns with FRP confinement [11], where \( d_\phi \) is the diameter of the reinforcement. Based on a section analysis, Binici [10] proposed the following closed-form expression for the ultimate curvature of FRP confined circular sections:

\[
\phi_u = \frac{0.0034 \left( 4.44 \delta_p^2 + 0.65 + 3.84 \delta_p (1 + 0.59) \eta + 0.56l + 0.33 \right)}{D_p \left( 0.44n + 0.32l + 0.04n \eta + 0.02 \right)}
\]

(15)

where \( \delta = (\varepsilon_{frp}/0.002)^{0.45} \) in which \( \varepsilon_{frp} \) is the rupture strain of the FRP composite; \( \eta = (2E_{frp}d_{frp}t_{frp}) / (D_p f_{yc}) \) in which \( E_{frp} \) and \( t_{frp} \) are the tensile modulus and thickness of the FRP composite, respectively; \( l = (f_y / f_{yc}) \); and \( n = 4p / (\pi D_p f_{yc}) \). Given the ultimate curvature, from Eq. (15), the plastic deformation, \( \Delta_p \), can be written as

\[
\Delta_p = \phi_u l_p (H - 0.5l_p) = (\phi_u - \phi_y) l_p (H - 0.5l_p)
\]

(16)

Finally, \( \Delta_p \) for FRP confined members at the ultimate curvature can be computed according to Biskinis and Fardis [35] as

\[
\Delta_p = \frac{\phi_u d_{frp} H}{10 \sqrt{f_{yc}}}
\]

(17)

3.2.2. Model correction

As initial candidate explanatory functions, we select \( \eta_{frp} (x) = 1 \) to capture the potential bias in the model, independent from \( x \); \( \eta_{frp} (x) = \Delta_p / H, \eta_{frp} (x) = \Delta_p / H, \eta_{frp} (x) = \Delta_p / H, \) and \( \eta_{frp} (x) = \Delta_p / H \) to correct a possible bias in each of the deterministic terms; \( \eta_{frp} (x) = D_p / H \) to account for the possible dependence of the bias on the slenderness of the column; \( \eta_{frp} (x) = (f_y / f_{yc}) \) and \( \eta_{frp} (x) = (f_y / f_{yc}) \) to capture the possible effects of longitudinal reinforcement and the strength of the composite material; \( \eta_{frp} (x) = (2E_{frp} / D_p) \) to account for the possible influence of the confinement; and \( \eta_{frp} (x) = \epsilon_{uc} \) to capture the possible dependence of the bias on the concrete properties, where \( \epsilon_{uc} \) is the ultimate strain of the FRP confined concrete. The explanatory functions \( \eta_{frp} (x) \)'s are then obtained as described earlier in the paper. To compare the results obtained using the proposed approach and a simultaneous nonlinear regression, we also performed the time-consuming simultaneous estimation of \( \Delta_p \) and \( \Theta \). The results confirmed that the values of \( \Delta_p \) do not change significantly over the deletion process. Table 3 summarizes the transformed functions, \( \eta_{frp} (x) \)'s.

3.2.3. Parameter estimation and model selection

The unknown parameters, \( \Theta \), are estimated using Eq. (7). Choe et al. [18] developed a probabilistic model for the flexural capacity of un-retrofitted columns. Since the explanatory functions considered in this paper are different, in order to account for the effects of the FRP composites, we consider a noninformative prior for \( \Theta \). However, because the model in Choe et al. [18] and the proposed capacity model are predicting the same response variable, the quality of the prediction in Choe et al. [18], captured by the model standard deviation, can provide some indications of the expected accuracy of the proposed model before the model is actually calibrated, using the experimental data. In fact, the model in Choe et al. [18] is a special case of the model proposed in this paper.

Table 3

<table>
<thead>
<tr>
<th>Initial explanatory function</th>
<th>Estimated</th>
<th>Transformed explanatory function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_{frp} (x) ) = ( D_p / H )</td>
<td>1.00</td>
<td>( \eta_{frp} (x) = \Delta_p / H )</td>
</tr>
<tr>
<td>( \eta_{frp} (x) ) = ( \Delta_p / H )</td>
<td>0.44</td>
<td>( \eta_{frp} (x) = \Delta_p / H )</td>
</tr>
<tr>
<td>( \eta_{frp} (x) ) = ( \Delta_p / H )</td>
<td>0.50</td>
<td>( \eta_{frp} (x) = \Delta_p / H ) )</td>
</tr>
<tr>
<td>( \eta_{frp} (x) ) = ( \Delta_p / H )</td>
<td>0.25</td>
<td>( \eta_{frp} (x) = \Delta_p / H ) )</td>
</tr>
<tr>
<td>( \eta_{frp} (x) ) = ( D_p / H )</td>
<td>1.45</td>
<td>( \eta_{frp} (x) = \Delta_p / H ) )</td>
</tr>
<tr>
<td>( \eta_{frp} (x) = (f_y / f_{yc}) )</td>
<td>1.00</td>
<td>( \eta_{frp} (x) = (f_y / f_{yc}) )</td>
</tr>
<tr>
<td>( \eta_{frp} (x) = \epsilon_{uc} )</td>
<td>0.59</td>
<td>( \eta_{frp} (x) = \epsilon_{uc} )</td>
</tr>
</tbody>
</table>
where the thickness of the FRP composite is set to zero (i.e., \( t_{FRP} = 0 \)). Therefore, we use the posterior estimate for the standard deviation in [18] as a prior in the Bayesian updating. We also note that because there is a limited number of data with FRP confinement, if we use a noninformative prior, the standard deviation of the model error will be artificially small. Using an informative prior, obtained from a larger dataset, helps obtain an estimate of the model standard deviation that is more realistic and likely closer to the one that we would have if more data were available and could be used for the model calibration. In the complete model with all candidate explanatory functions, \( \theta_{2} \) has the largest COV (=2.07); hence, to simplify the model, we eliminate the term \( \theta_{2}h_{2}(x) \). Next, we assess the reduced model. Eventually after nine steps, the model selection process identifies \( \theta_{1}h_{1}(x) \) and \( \theta_{9}h_{9}(x) \) as the explanatory functions needed in an accurate but parsimonious model. Fig. 1 summarizes this stepwise deletion process. For each step, the figure shows the COV of the model parameters (solid dots) and the posterior mean of the model standard deviation (open square). In addition, analysis of the posterior statistics of the reduced model reveals that \( \theta_{1} \) and \( \theta_{9} \) are correlated \((\rho_{1,9} = -0.80)\). As a further simplification, these two terms are linearly combined, using Eq. (9), giving the following expression for \( \gamma_{f}(x, \theta) \):

\[
\gamma_{f}(x, \theta) = \theta_{1} + (0.109 - 2.406\theta_{1}) \left( \frac{2t_{FRP}f_{FRP}}{D_{e}c} \right)
\]

(18)

Table 4 lists the posterior statistics of the parameters \( \Theta = (\theta_{1}, \sigma_{f}) \). The following observation is noteworthy: The positive mean of \( \theta_{1} \) indicates that, independent from the variables \( x \), the deterministic model, \( \hat{\beta}(x) \), tends to underestimate the deformation capacity of a column with FRP confinement. In addition, the mean of \( \theta_{9} \) is \(-2.69\), which indicates that \( \hat{\beta}(x) \) tends to overestimate the contribution of FRP composites on the confinement. Fig. 2 shows a comparison between the measured and predicted values of the drift ratio capacities for the test columns based on the deterministic (left plot) and probabilistic (right plot) models. For a perfect model, the failure data would line up along the 1:1 solid line and the censored data would lie above it. Visual inspection of the data shows the bias in \( \hat{\beta}(x) \) and that the proposed probabilistic model effectively corrects this bias. For the probabilistic model, the failure data points are evenly distributed within the one standard deviation limits and almost all of the censored data are above the 1:1 line.

For the sake of comparison, we have also provided the estimates of the unknown parameters with a nonlinear regression (i.e., estimating \( \hat{\beta}_{9} \) simultaneously with \( \theta_{1} \) and \( \theta_{9} \)) in Table 5. In addition, Fig. 3 compares the predicted capacities based on the proposed approach and the nonlinear regression. We can see that even though the individual estimates of the unknown parameters are different (see Tables 4 and 5), the values of the predicted capacities are in close agreement. We should also note that due to the strong correlation between \( \hat{\beta}_{9} \) and \( \theta_{1} \) and \( \theta_{9} \), obtained with the nonlinear regression, comparing individual estimates between the two models is problematic. The high correlation coefficients mask the actual impact of individual terms on the capacity and do not let the values speak for themselves.

3.3. Deformation capacity for bond failure

The combined action of the reinforcing steel and the neighboring concrete requires a load transfer mechanism that is referred to as a bond and is typically idealized as a continuous stress field that develops in the vicinity of the steel-concrete interface [36]. The

![Fig. 1. Stepwise deletion process of the deformation capacity for the flexural failure.](image)

![Fig. 2. Comparison between the measured and predicted drift ratio capacities based on deterministic (left) and probabilistic (right) models for the flexural failure.](image)

### Table 4

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Correlation coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_{1} )</td>
<td>1.072</td>
<td>0.106</td>
<td>1</td>
</tr>
<tr>
<td>( \sigma_{f} )</td>
<td>0.337</td>
<td>0.036</td>
<td>-0.057</td>
</tr>
</tbody>
</table>

### Table 5

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Correlation coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_{1} )</td>
<td>1.607</td>
<td>0.290</td>
<td>1</td>
</tr>
<tr>
<td>( \theta_{9} )</td>
<td>-2.821</td>
<td>0.329</td>
<td>-0.732</td>
</tr>
<tr>
<td>( \hat{\beta}_{9} )</td>
<td>0.581</td>
<td>0.141</td>
<td>-0.813</td>
</tr>
<tr>
<td>( \sigma_{f} )</td>
<td>0.320</td>
<td>0.039</td>
<td>-0.155</td>
</tr>
</tbody>
</table>
After computing $f_s$, the corresponding strain, $\varepsilon_s$, can be obtained from the steel stress–strain constitutive relation.

To find $\phi_a$, we first use the section analysis, as described in Binici [10], to find the depth of the compression zone, $c$, and then relate $\phi_a$ to $\varepsilon_a$, using rules of mechanics, as

$$\phi_a = \frac{\varepsilon_a}{W - c}$$

where $W = D_s - cover$. Given $\varepsilon_a$ and $c$, the ultimate curvature can be written as

$$\phi_a = \frac{\varepsilon_a}{W - D_s + 1.7(-0.38 + 0.44\psi)}$$

where, $\psi = (n + l + 0.47)/0.321 + 0.37l + 0.19$ is the angle between the cross-sectional radii that delimit the compression zone, in which $n$, $\eta$, and $I$ are the same as defined in Eq. (15).

3.3.2. Model correction

As initial candidate explanatory functions, we select $\eta(1)$ to capture a potential bias in the model that is independent of $x$:

$$\eta_1(x) = \Delta_g/H.$$  

$$\eta_2(x) = D_s/H.$$  

$$\eta_3(x) = \Delta_s/H.$$  

$$\eta_4(x) = \Delta_b/H.$$  

$$\eta_5(x) = \Delta_b/H.$$  

To examine the potential bias associated with the individual components of $\partial_1(x)$; $\eta_6(x) = (21/297/D_4)(f'_{FRP}/f_s)$ to account for the possible influences on the bias of the FRP confinement; $\eta_7(x) = d_s/I_s$ to explore the contribution on the bias from the splice length; and $\eta_8(x) = \varepsilon_0$ to correct for the effects on the bias of the material properties. Before moving on to the model selection, the transformed explanatory functions, $h_i(x)$, are obtained as described earlier in the paper. Table 6 summarizes the transformed functions $h_i(x)$’s.

3.3.3. Parameter estimation and model selection

Given that no prior information is available about $\Theta_0$, we consider a noninformative prior distribution in Eq. (7). In this section we shorten the list of $h_i(x)$’s to obtain a parsimonious model. Figure 4 shows the posterior COV of $h_i$’s (as dots) and mean of $\sigma_i$ (as an open square) at each step in the deletion process. The deletion process is stopped at the 7th step, keeping $\theta_1, \theta_2, \theta_3$, and $\theta_4$ in the model. The examination of the correlation coefficient indicates a high dependence ($\rho_{12} = -0.98$) between $\theta_1$ and $\theta_2$. Therefore, we combine them, using Eq. (9), and $\gamma_b$ is left with only one model parameter.

$$\gamma_b(x, \theta_1) = \theta_{1\theta} + (128.33 - 107.14\theta_{1\theta})(\Delta_b/H)$$

Table 7 lists the posterior statistics of the parameters $\Theta_0 = (\theta_{1\theta}, \sigma_1)$. The positive mean of $\theta_{1\theta}$ suggests, independent from the variables $x$, the deterministic model, $\partial_1(x)$, tends to underestimate the deformation capacity. Furthermore, the negative mean of $\theta_2 = -58.95$ indicates that $\partial_2(x)$ tends to overestimate the contribution of the flexural deformation. There is a
possible physical interpretation of this latter observation. The yield curvature, \( y \), in RC columns are often ill-defined because of crack- ing of concrete and sequential yielding of reinforcement bars [40]. In the literature, there are various definitions for the yield curvature and each works well under particular circumstances. There is no unique definition that works well everywhere. While additional work and in particular an experimental investigation would be needed to better understand this issue, we believe that the simple model used in this paper, following [10], overestimates \( y \) for a column that fails in bond. This bias is corrected in the proposed model. However, as noted earlier, the interpretation of the individual signs and values of the multiple regression coefficients is problematic. Therefore, the above interpretation would need to be further investigated.

Fig. 5 shows a comparison between the measured and predicted values of drift ratio capacities for the test columns based on the deterministic and probabilistic models. Fig. 6 compares the predicted capacities based on the proposed approach and the nonlinear regression. The predicted capacities with the two models are in close agreement. As discussed before, due to the strong correlation between \( \lambda_2 \) and \( \theta_0 \) in the nonlinear regression method, we should avoid comparing individual estimates of the unknown parameters in the two models.

Table 8 provides the estimates of the unknown parameters with a nonlinear regression (i.e., estimating \( \lambda_2 \) simultaneously with \( \theta_0, \theta_1 \) and \( \theta_2 \)) and Fig. 6 compares the predicted capacities based on the proposed approach and the nonlinear regression. The predicted capacities with the two models are in close agreement. As discussed before, due to the strong correlation between \( \lambda_2 \) and \( \theta_0, \theta_1 \) and \( \theta_2 \) in the nonlinear regression method, we should avoid comparing individual estimates of the unknown parameters in the two models.

### 3.4. Bivariate flexure-bond deformation capacity model

All 41 data points can be used for the flexural model but out of them only 19 can be used for the bond model. Therefore, we use the 22 data points that cannot be used for the bond model to develop an informative prior distribution for \( \Theta_b \), a noninformative prior is used for \( \Theta_f \) and the remaining 19 data points are used to develop the bivariate likelihood function according to Table 2. The bivariate model uses the reduced model forms in Eqs. (18) and (23) and \( \Lambda = ( \lambda_f, \lambda_b ) \) are the same as those in the univariate models. Table 9 lists the posterior statistics of the parameters \( \Theta \). As expected, the estimates of \( \Theta_f, \Theta_b \) and \( \Theta \) are nearly identical to the corresponding estimates for the univariate models. In addition, the bivariate model provides an estimate of the correlation coefficient, \( \rho \), between \( \lambda_f, \lambda_b \).

### 4. Fragility estimates of the FRP-retrofitted columns

The developed probabilistic capacity models can be used to compute the fragility of FRP-retrofitted columns. Fragility functions...
express the conditional probability of meeting or exceeding a pre-
scribed limit state for a given value of the demand measure. Evalu-
ating the fragility of columns before and after retrofit provides
valuable insights on quantifying the effectiveness of the retrofit
measure in improving the performance of the retrofitted columns
and mitigating their vulnerabilities.

Following Ditlevsen and Madsen [41], we define a limit state
function $g_k(x, H_k)$ such that the event $g_k(x, H_k) ≤ C_k$
indicates the failure in the $k$th failure mode. By partitioning $x$ as
$x = (r, s)$, where $r$ is a vector of material and geometrical variables
and $s$ is a vector of demand variables such as boundary forces or
deformations, the limit state function can be written as

$$g_k(r, s, Θ_k) = C_k(r, s) - D_k(r, s)$$

where $C_k$ is computed using Eqs. (10) and (19) and $D_k$ denotes
the given demand for the $k$th failure mode. The column fragility
can then be written as

$$F(s, Θ) = P \left\{ g_k(r, s, Θ_k) ≤ C_k \right\}$$

where $P[A|B]$ denotes the conditional probability of the event $A$ for
the given event $B$. As discussed in Gardoni et al. [12], the predictive
fragility estimate, $\tilde{F}(s)$, that incorporates the uncertainties in
$Θ$, can be computed as the expected value of $F(s, Θ)$ over the posterior dis-
tribution of $Θ$ as $\tilde{F}(s) = \int F(s, Θ) f(Θ) dΘ$. Further, to show the effect
of the epistemic uncertainty in $Θ$, we compute the confidence
bounds of $F(s, Θ)$. Following Gardoni et al. [12], the confidence
bounds can be written as $\{Φ[−\tilde{β}(s) − σ_β(s)], Φ[−\tilde{β}(s) + σ_β(s)]\}$

Table 9
Posterior statistics of the bivariate deformation capacity model.

<table>
<thead>
<tr>
<th>$θ_{f1}$</th>
<th>$σ_f$</th>
<th>$θ_{b1}$</th>
<th>$σ_b$</th>
<th>$ρ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.078</td>
<td>0.331</td>
<td>0.656</td>
<td>0.146</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.105</td>
<td>0.038</td>
<td>0.201</td>
<td>0.035</td>
</tr>
</tbody>
</table>

Correlation coefficient

| $θ_{f1}$ | 1   |
| $σ_f$     | -0.26 |
| $θ_{b1}$ | 0.05 |
| $σ_b$     | -0.00 |
| $ρ$       | -0.00 |

Fig. 7. Fragility estimate for (a) example Column 1, (b) example Column 2, and (c) example Column 3, before and after retrofit.
In the last expression, \( \phi_s(\cdot) \) denotes the posterior covariance matrix. The bounds approximately correspond to 0.15 and 0.85 quantiles.

The sensitivities of the response to the design values, \( f(b) \), are similar to those already presented for Column 1. However, because of the smaller ratio demand for the flexural failure mode than the bond failure capacity (i.e., the 0.5 probability of failure occurs at a lower drift ratio), the retrofit is more effective in increasing the mean value of the bond capacity than of the flexural capacity (as shown also in the right plot). The retrofit is also shows the confidence band created by the confidence bounds. The thin lines correspond to the as-built condition and the thick lines correspond to the retrofitted condition. The right plot shows that for the as-built column the bond mode response to \( d_r \) before and after they are retrofitted. The fragilities of the fragility index, \( \beta(b, \Theta) \), obtained using a first-order Taylor expansion around the posterior mean of model parameters, \( \mu_\theta \). In the last expression, \( \nabla_\alpha(\cdot) \) is the gradient row vector of \( \beta(b, \Theta) \) at the mean point \( \mu_\theta \), and \( \Sigma_{\text{cov}} \) denotes the posterior covariance matrix. The bounds approximately correspond to 0.15 and 0.85 quantiles.

Table 10 shows a summary of the geometry and the material properties of the example columns and, when appropriate, their conservative bias. The correction terms are constructed starting from candidate explanatory functions developed using the Box–Cox transformation. A stepwise deletion process is then used to eliminate the least informative terms and simplify the models, while maintaining an acceptable level of accuracy. As an application of the probabilistic models, fragility curves are developed for non-seismically designed columns. Though the thickness of FRP composites in the three example columns differs, they are all determined following a consistent design approach. Furthermore, sensitivity analysis, which is not provided in the paper, shows that further increase of the FRP composite thickness, beyond the designed values, does not change the fragilities significantly.

5. Conclusion

Two probabilistic models are proposed to predict the deformation capacity of FRP retrofitted RC columns. One capacity model predicts the deformation capacity corresponding to a flexural failure and the other model predicts the deformation capacity for bond failure. To develop the probabilistic models, simple correction terms are added to deterministic models to compensate for their conservative bias. The correction terms are constructed starting from candidate explanatory functions developed using the Box–Cox transformation. A stepwise deletion process is then used to eliminate the least informative terms and simplify the models, while maintaining an acceptable level of accuracy. As an application of the probabilistic models, fragility curves are developed for three example columns. Two columns are representative of non-seismically designed columns. Though the thickness of FRP composites in the three example columns differs, they are all determined following a consistent design approach. Furthermore, sensitivity analysis, which is not provided in the paper, shows that further increase of the FRP composite thickness, beyond the designed values, does not change the fragilities significantly.

The right plot shows that the as-built column the bond mode of failure dominates. In addition, the retrofit measure significantly reduces the fragility for each mode of failure and, as a result, also the fragility that considers either mode of failure. Again, the results clearly quantify the effect of FRP retrofit measure in reducing the probability of failure of RC columns. For the considered levels of FRP thickness, the results indicate that the FRP retrofitting measure considerably reduces the fragility for bond failure.