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GMTI Direction of Arrival Measurements from Multiple Phase Centers

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GMTI Direction of Arrival Measurements from Multiple Phase Centers

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Abstract

Ground Moving Target Indicator (GMTI) radar attempts to detect and locate targets with unknown motion. Very slow-moving targets are difficult to locate in the presence of surrounding clutter. This necessitates multiple antenna phase centers (or equivalent) to offer independent Direction of Arrival (DOA) measurements. DOA accuracy and precision generally remains dependent on target Signal-to-Noise Ratio (SNR), Clutter-to-Noise Ratio (CNR), scene topography, interfering signals, and a number of antenna parameters. This is true even for adaptive techniques like Space-Time-Adaptive-Processing (STAP) algorithms.

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Contents

Foreword

This report details the results of an academic study. It does not presently exemplify any modes, methodologies, or techniques employed by any operational system known to the author.

Classification

The specific mathematics and algorithms presented herein do not bear any release restrictions or distribution limitations.

This distribution limitations of this report are in accordance with the classification guidance detailed in the memorandum "Classification Guidance Recommendations for Sandia Radar Testbed Research and Development", DRAFT memorandum from Brett Remund (Deputy Director, RF Remote Sensing Systems, Electronic Systems Center) to Randy Bell (US Department of Energy, NA-22), February 23, 2004. Sandia has adopted this guidance where otherwise none has been given.

This report formalizes preexisting informal notes and other documentation on the subject matter herein.

1 Introduction & Background

Fundamental to Intelligence, Surveillance, and Reconnaissance (ISR) is the desire to know, among other things,

1. What is out there?
2. What is moving?
3. Where is it?
4. Where is it going?
5. Where has it been?
6. When is/was it there?

As Albert Einstein said, “Nothing happens until something moves.”

In radar parlance, this is about detecting, locating, and tracking moving targets. The class of radar systems, or at least radar modes, that attempt to do this are collectively termed Moving Target Indicator (MTI) radar systems. More specifically, if we are interested in targets at or near the earth’s surface, we may further designate specialized systems as

1. Ground-MTI (GMTI) radar systems,
2. Dismount [detection] MTI (DMTI) radar systems, and/or
3. Surface-MTI (SMTI) radar systems.

Herein this report, we are interested in these modes operating from an airborne radar system. Achieving good performance for detecting, locating, and tracking targets of interest is particularly problematic for slow-moving targets, especially when their Radar Cross Section (RCS) isn’t particularly greater than the surrounding clutter, e.g. DMTI radar.[1]

Much literature has been generated on the detection of slow-moving targets of interest. Likewise, the literature is rich concerning tracking targets. By comparison, however, the literature on precisely locating slow-moving targets is substantially less. It is this function that we address in this report.

In particular, we wish to locate a slow-moving target in the presence of radar clutter. We shall assume that the target echo’s Doppler is such that it coincides with the Doppler from otherwise stationary clutter. This is an endo-clutter target situation. We will analyze the ability to separate moving target from clutter with same range and Doppler via direct Direction of Arrival (DOA) measurements. This will necessitate multiple antenna subaperture phase-centers or equivalent. The problem can then be cast as finding the frequencies of multiple superimposed sinusoids.

We shall generally treat this problem as a null-steering problem, to minimize some quality function. We note that we can null two simultaneous frequencies (DOA directions) with three or more samples (phase centers). The null locations will indicate the frequencies present, or at least that's the plan.

To facilitate the subsequent analysis, we will make several simplifying assumptions, including

1. We shall presume target locations in the far-field of the antenna, so that we may treat the wave-fronts as planar.
2. We will assume the antenna is a perfect linear array.
3. We will assume perfect antenna performance, with any errors negligible to our analysis.
4. We will assume narrow-band signals, suitably described with a single nominal wavelength.
5. We will assume no interaction of the antenna with any radome or other surrounding structures, i.e. the antenna is performing as if in free space.

We do note that other information might be brought to bear on locating moving targets. Shadows in particular can offer a mechanism to ascertain DOA as they are not shifted in Doppler as are direct returns.[2]

However, in this report we will limit our attention to only the direct radar echo response, foregoing any information that might be derived from shadows or other ancillary data.

Some Comments on Clutter

The simple question "What is clutter?" has all the attributes of a zen kōan (paradoxical story or riddle).

In general, with an MTI radar we are attempting to detect targets, and only targets. A target is something we are interested in detecting, and all other things are non-targets. As a practical matter we can only detect targets with some Probability of Detection (P_D), balanced against some Probability of False Alarm (P_{FA}) or equivalent, that is, the probability of detecting a non-target.

Clutter is the collection of non-targets that could be detected. Clutter is not noise in the sense of random fluctuations in signal energy due to received thermal radiation or equivalent. Rather, clutter is radar echo energy from real objects that happens to be of no particular interest to the radar data exploiter, except to wish it wasn't there to confuse him.[3]

In MTI systems, we are interested in detecting targets that are moving. To first order, clutter is then the collection of objects that aren't moving. But this isn't quite right either. For example, with respect to targets of interest

Moving vehicles probably are,
Moving people probably are,
Moving animals probably are not,
Moving animals with human riders probably are,
Moving animals with loads (pack animals) probably are,
Trees generally are not,
Trees moving (blowing in the wind) also generally not,
Windmills probably are not,
Building turbines sometimes might be,
Objects vibrating due to wind probably are not,
Objects vibrating due to machinery running might be,
etc.

The best we can generally do with radar is set detection thresholds for some minimum reflectivity, and some minimum velocity, and then typically only a line-of-sight velocity component. Consequently, for the purposes of this report, we will define clutter as truly stationary objects or distributed radar echo reflection fields, and everything that is moving as a potential target.

Within the universe of clutter, we will further distinguish two types of clutter

1. distributed clutter, such as vegetation fields, soil, and other entities described by an average Radar Cross Section (RCS) per unit area, and

2. discrete clutter, that is, specular reflectors described by a specific RCS. Individual items of discrete clutter are often called ‘clutter discretes’.[4]

Distributed clutter impacts MTI systems by raising the threshold levels for deciding whether some potential target is indeed detected or not. Consequently, MTI systems seek architectures and processing to lower, or cancel, distributed clutter levels.

Clutter discretes may also be lowered by the methods that mitigate distributed clutter, but often insufficiently so. Consequently they can too often still pass thresholds for real targets. Consequently, MTI systems will often seek to discriminate these from real moving targets via analysis of the direction of arrival of their echo energy in conjunction with Doppler signatures.

A companion report discusses the ability of a multi-aperture antenna to cancel clutter.[5]

2 Analysis of Three Phase Centers

What follows is an analysis of the problem of finding two DOA angles from data collected from three antenna phase centers. We will phrase this as steering two nulls simultaneously towards two signals with weighted measurements from three antenna phase centers.

2.1 Setting Up The Problem

We begin with some definitions. Accordingly we define three antenna phase centers each with index

$$i = \text{index of phase centers, } i \in \{0,1,2\} \ . \tag{1}$$

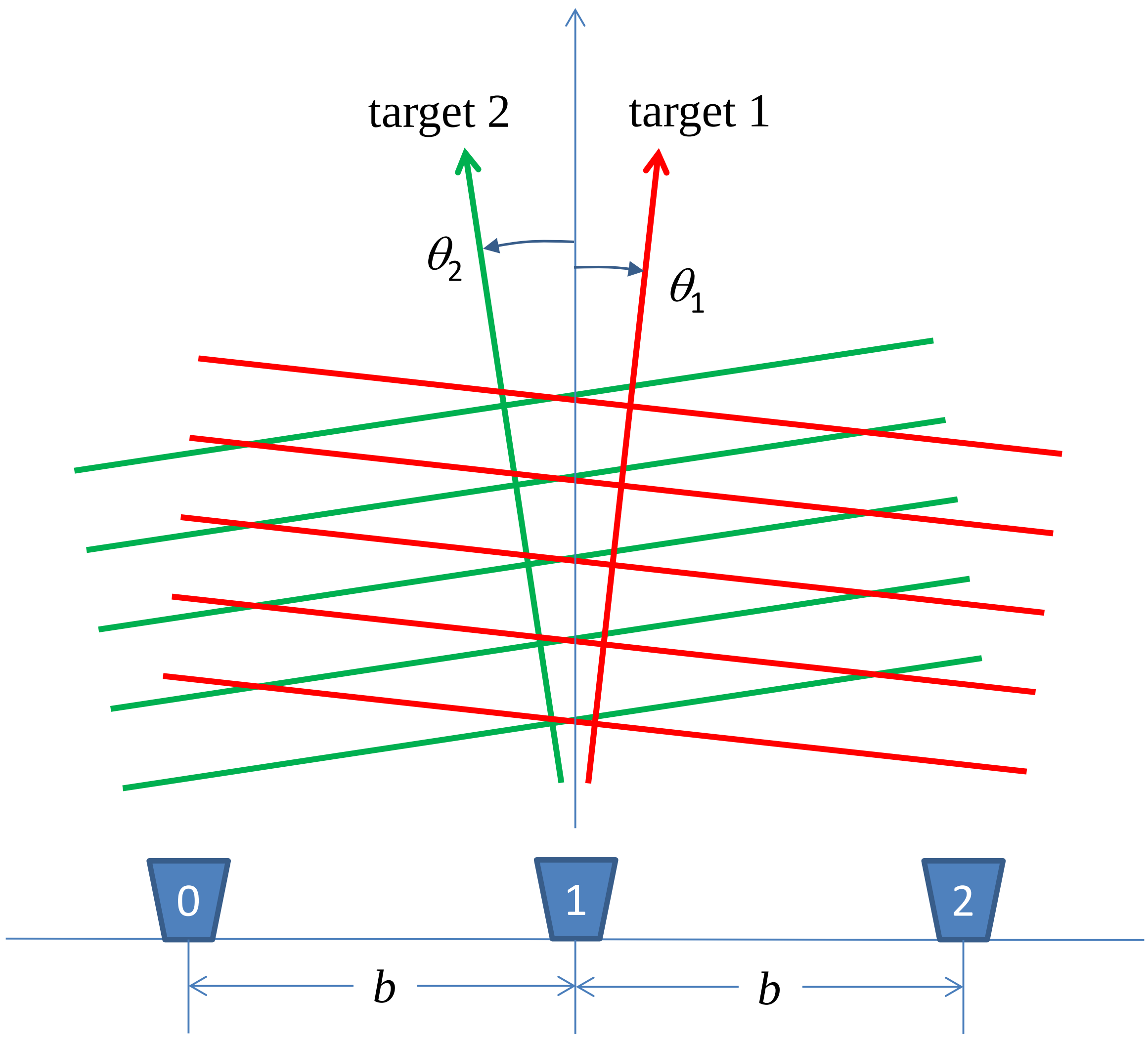

Figure 1. Problem Geometry – two DOA angles with three antenna phase centers.

To keep the development simple, these phase centers are collinear and spaced by equal amounts

$$b = \text{separation of phase centers.} \tag{2}$$

Let two signals be received by each phase center, each from direction

$$\theta_k = \text{direction of target } k,\ k \in \{1,2\}\ . \tag{3}$$

These directions are defined with respect to the normal of the antenna array. We generally allow that they may be the same or different. Each target signal has complex scale of its return given by

$$A_k = \text{complex scale factor for target } k\,. \tag{4}$$

We will assume that this is a constant for any one target and is the same for all antenna phase centers. The tacit assumption here is that the antenna beam pattern for each phase center is identical. This is actually a fairly severe constraint upon the construction of the antenna. Nevertheless, for our purposes we will presume this. Consequently, each phase center *i* receives from target direction *k* a signal of the form

$$m_{i,k} = A_k\, p_{i,k} = \text{signal at } i^{\text{th}} \text{ phase center from target direction } k, \tag{5}$$

where with some malice of forethought we identify the unit vector with only a phase term with

$$p_{i,k} = e^{-j\frac{2\pi b}{\lambda} i \sin\theta_k} = \text{signal's phase term corresponding to } m_{i,k}\,, \tag{6}$$

where

$$\lambda = \text{nominal wavelength of the waveform.} \tag{7}$$

For small angles relative to the antenna boresight we can, and often will, make the approximation

$$\sin\theta_k \approx \theta_k\ . \tag{8}$$

We also note that $p_{i,k}$ is a power series, that is

$$p_{i,k} = p_{1,k}^{i}\ . \tag{9}$$

Nevertheless, each phase center receives a composite signal

$$M_i = \sum_k m_{i,k} \text{ = composite signal at } i^{\text{th}} \text{ phase center.} \tag{10}$$

In practice, any measurement of the composite signal will be corrupted by noise. This is the inescapable bane of all real systems. However, for the immediate discussion we will ignore any noise, and defer discussion of the impact of noise to later in this report. Our task is then to estimate all target directions θ_k from the set of composite signals M_i.

2.2 Steering Nulls to Known Directions

To estimate the target directions, we wish to spatially linear filter the signals across all of the phase centers. We desire that a single set of weights will null signals from both target directions, that is

$$\sum_i M_i w_i^* = 0, \tag{11}$$

where the '*' denotes complex conjugate. A trivial solution would be all weights equal to zero. However, we want a particular nontrivial solution that forces a null for each individual target direction such that

$$\sum_i m_{i,k} w_i^* = 0 \text{ for all signals } k. \tag{12}$$

Clearly, whatever set of weights w_i satisfy this, so too would any linearly scaled set of weights satisfy this. Consequently we may scale the weights by arbitrarily assigning

$$w_0 = 1. \tag{13}$$

We furthermore denote the vector of weights as

$$\mathbf{w} = \begin{bmatrix} w_0 & w_1 & w_2 \end{bmatrix}^T, \tag{14}$$

where the 'T' denotes simple transpose. This lets us create the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 \\ m_{0,1} & m_{1,1} & m_{2,1} \\ m_{0,2} & m_{1,2} & m_{2,2} \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{15}$$

where $\mathbf{w}^*$ is the conjugate of $\mathbf{w}$, containing the conjugate of the individual weights. Observe that for $i = 0$, the phase is such that we may calculate

$$m_{0,k} = A_k \,. \tag{16}$$

Furthermore, we may scale the rows of this matrix equation to the equivalent equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & p_{1,1} & p_{2,1} \\ 1 & p_{1,2} & p_{2,2} \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{17}$$

and making use of Eq. (9) furthermore to

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & p_{1,1}^1 & p_{1,1}^2 \\ 1 & p_{1,2}^1 & p_{1,2}^2 \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{18}$$

Consequently we wish to find constant nontrivial weights that allow us to solve this. With the knowledge that $w_0 = 1$, we can then rewrite the matrix equation as

$$\begin{bmatrix} p_{1,1}^1 & p_{1,1}^2 \\ p_{1,2}^1 & p_{1,2}^2 \end{bmatrix} \begin{bmatrix} w_1^* \\ w_2^* \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \tag{19}$$

Consequently, by finding the unknown weights w_1 and w_2, we can effectively null signals from both target directions simultaneously. The important point here is that this basically shows that weights do in fact exist to do this.

2.2.1 Case 1 – Same Target Directions

For the case $\theta_2 = \theta_1$, the matrix inverse does not exist. Consequently, for this case we will make use of the equation where three unit vectors sum to zero, namely

$$1 + e^{j\left(\frac{2\pi}{3}\right)} + e^{j\left(\frac{4\pi}{3}\right)} = 0 \,. \tag{20}$$

This lets us solve for the unknown weights as

$$\mathbf{w}^* = \begin{bmatrix} 1 & p_{1,1}^{-1} e^{j\left(\frac{2\pi}{3}\right)} & p_{1,1}^{-2} e^{j2\left(\frac{2\pi}{3}\right)} \end{bmatrix}^T . \tag{21}$$

This might also be written as

$$\mathbf{w} = \begin{bmatrix} 1 & p_{1,1}^{-1} e^{j\left(\frac{2\pi}{3}\right)} & p_{1,1}^{-2} e^{j2\left(\frac{2\pi}{3}\right)} \end{bmatrix}^H . \tag{22}$$

where the '*H*' superscript indicates conjugate transpose.

In any case, we may now calculate

$$\begin{bmatrix} 1 & p_{1,1} & p_{2,1} \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 + p_{1,1}^{-1} e^{j\left(\frac{2\pi}{3}\right)} p_{1,1} + p_{1,1}^{-2} e^{j2\left(\frac{2\pi}{3}\right)} p_{2,1} \end{bmatrix} = 0 . \tag{23}$$

This, of course, evaluates to zero, which is what we want. Note that another perfectly fine solution for the $\theta_2 = \theta_1$ case is

$$\mathbf{w}^* = \begin{bmatrix} 1 & p_{1,1}^{-1} e^{j\left(\frac{4\pi}{3}\right)} & p_{1,1}^{-2} e^{j2\left(\frac{4\pi}{3}\right)} \end{bmatrix}^T . \tag{24}$$

This validates the weight solution for this case.

2.2.2 Case 2 – Different Target Directions

For the case $\theta_2 \neq \theta_1$, the matrix inverse exists and we can solve for the unknown weights as generally

$$\mathbf{w}^* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & p_{1,1}^1 & p_{1,1}^2 \\ 1 & p_{1,2}^1 & p_{1,2}^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{25}$$

With the knowledge that $w_0 = 1$, we can then rewrite the matrix equation as

$$\begin{bmatrix} w_1^* \\ w_2^* \end{bmatrix} = \begin{bmatrix} p_{1,1}^1 & p_{1,1}^2 \\ p_{1,2}^1 & p_{1,2}^2 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} . \tag{26}$$

This clearly identifies the weight solution for this case.

2.3 Calculating Angles from Weights

However we might arrive at the weights, once we have the weights, we can find estimates of the two target direction angles by solving the quadratic equation

$$w_2^* (\hat{p})^2 + w_1^* (\hat{p})^1 + 1 = 0 . \tag{27}$$

where

$$\hat{p} = e^{-j \frac{2\pi b}{\lambda} \sin \hat{\theta}_k} . \tag{28}$$

In general null steering, this polynomial is known as the array factor (AF) and the nulls occur at the zeroes of the polynomial.

We may solve for the angle estimates via the quadratic formula by calculating

$$\hat{p} = e^{-j \frac{2\pi b}{\lambda} \sin \hat{\theta}_k} = \frac{-w_1^* \pm \sqrt{\left(w_1^*\right)^2 - 4 w_2^*}}{2 w_2^*} . \tag{29}$$

More specifically, for now we may select arbitrarily the specific angles

$$\hat{\theta}_1 = \operatorname{asin}\left(\frac{-\lambda}{2\pi b} \operatorname{Arg}\left(\frac{-w_1^* + \sqrt{\left(w_1^*\right)^2 - 4 w_2^*}}{2 w_2^*} \right)\right), \text{ and}$$

$$\hat{\theta}_2 = \operatorname{asin}\left(\frac{-\lambda}{2\pi b} \operatorname{Arg}\left(\frac{-w_1^* - \sqrt{\left(w_1^*\right)^2 - 4 w_2^*}}{2 w_2^*} \right)\right) . \tag{30}$$

The important point here is that if the weights are known, then the angles can be calculated therefrom.

2.4 Calculating Weights from Signals with Unknown DOA

To this point, we know that weights do exist to null two different directions, and once the weights can be found, then we can calculate those directions. The problem is that we desire to find the weights w_i from knowledge of only the composite signals M_i.

Accordingly, we do know that

$$\sum_i w_i^* M_i = 0, \text{ where } w_0 = 1. \tag{31}$$

2.4.1 Case 1 – Same Target Directions

This is the case where $\theta_2 = \theta_1$. We note that a ramification of this is that

$$|M_2| = |M_1| = |M_0|. \tag{32}$$

This might be a good test for this case. Nevertheless, in this case, we use Eq. (20) and then identify the optimal weights as

$$\mathbf{w}^* = \begin{bmatrix} 1 & \frac{M_1}{M_0} e^{j\left(\frac{2\pi}{3}\right)} & \frac{M_2}{M_0} e^{j2\left(\frac{2\pi}{3}\right)} \end{bmatrix}^T. \tag{33}$$

2.4.2 Case 2 – Different Target Directions

This is the case where $\theta_2 \neq \theta_1$. This means that we need to find weights such that

$$\begin{bmatrix} M_0 & M_1 & M_2 \end{bmatrix} \mathbf{w}^* = M_0 + w_1^* M_1 + w_2^* M_2 = 0, \tag{34}$$

stipulating that $w_0 = 1$. A basic problem here is that we have one equation and two unknowns. This means that there is no unique solution to this equation, at least with no other information. While we can find weights to null both directions simultaneously, these weights may not necessarily identify the specific target signal directions. That is, they may not null the different target directions individually. The bottom line is we need more information to find a unique solution. We can gain some more insight by expanding this equation to

$$\begin{bmatrix} \sum_k m_{0,k} & \sum_k m_{1,k} & \sum_k m_{2,k} \end{bmatrix} \mathbf{w}^* = 0. \tag{35}$$

If we independently know angle θ_1, as we might from, say, a Doppler measurement, then we may write a constraint on the weights of

$$\begin{bmatrix} 1 & p_{1,1}^1 & p_{1,1}^2 \end{bmatrix} \mathbf{w}^* = 0 . \qquad (36)$$

Now we can rewrite the constraint and composite signals equation as the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & p_{1,1} & p_{1,1}^2 \\ M_0 & M_1 & M_2 \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} . \qquad (37)$$

If the inverse exists, then we can find the weights as

$$\mathbf{w}^* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & p_{1,1} & p_{1,1}^2 \\ M_0 & M_1 & M_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} . \qquad (38)$$

This inverse will not exist for $\theta_2 = \theta_1$. However, it should exist for $\theta_2 \neq \theta_1$. In fact, we might think about using the rank of the matrix in Eq. (37) to indicate whether a separate target is present, or not.[5]

2.4.3 Comments

In the GMTI problem, the known DOA which we assigned to θ_1 is associated with stationary clutter, whose direction might be calculated independently from Doppler information, or perhaps some other measurements. The relationship between Doppler and DOA of clutter defines the "clutter ridge" in Doppler-angle space. A basic assumption here is that we know the Doppler-angle relationship very well. The popular Space-Time Adaptive Processing (STAP) algorithm attempts to estimate this relationship from the data. More on this later.

However the weights are calculated, a Fourier Transform of the weight vector will clearly show the location of the nulls in angle-space.

2.5 Putting it All Together

From the three phase centers, we identify composite signals $\left(M_0, M_1, M_2\right)$.

From Doppler measurements, we identify the clutter angle, which we now designate θ_c. From this we define the phase term

$$p_c = e^{-j\frac{2\pi b}{\lambda}\sin\theta_c}. \tag{39}$$

Next we form the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & p_c & p_c^2 \\ M_0 & M_1 & M_2 \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{40}$$

If this matrix has no inverse, then there is no target in addition to the clutter. If the matrix has an inverse, then we can calculate weights

$$\mathbf{w}^* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & p_c & p_c^2 \\ M_0 & M_1 & M_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{41}$$

With these weights, we can calculate the two angles

$$\hat{\theta}_1 = \operatorname{asin}\left(\frac{-\lambda}{2\pi b}\operatorname{Arg}\left(\frac{-w_1^* + \sqrt{\left(w_1^*\right)^2 - 4w_2^*}}{2w_2^*}\right)\right), \text{ and}$$

$$\hat{\theta}_2 = \operatorname{asin}\left(\frac{-\lambda}{2\pi b}\operatorname{Arg}\left(\frac{-w_1^* - \sqrt{\left(w_1^*\right)^2 - 4w_2^*}}{2w_2^*}\right)\right). \tag{42}$$

One of these should match the clutter angle, and the other is the angle to the target.

Example

Consider an antenna with three phase centers, and operating with the following parameters.

$$\lambda = 0.02 \text{ m},$$
$$D_{ant} = 0.5 \text{ m} = \text{antenna overall aperture width.} \tag{43}$$

We calculate the phase center spacing as

$$b = D_{ant}/3 , \tag{44}$$

and a reference antenna beamwidth as

$$\theta_{bw} = \lambda/D_{ant} . \tag{45}$$

We define clutter and target angles as

$$\theta_c = 0.1\,\theta_{bw} = \text{clutter angle, and}$$
$$\theta_1 = -0.2\,\theta_{bw} = \text{target angle.} \tag{46}$$

With these parameters, Figure 2 illustrates the Fourier Transform of the weight vector, along with the angles used in the composite signal. A calculation of the null positions would exactly identify the input angles.

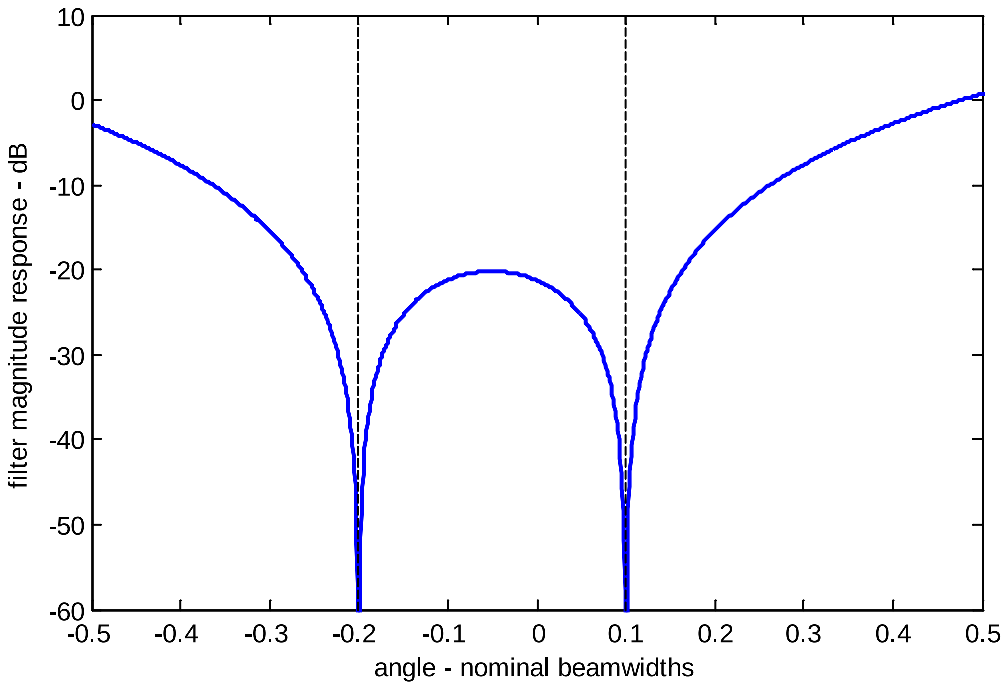

Figure 2. Fourier Transform of weight vector for three antenna phase centers and a noise-free composite signal.

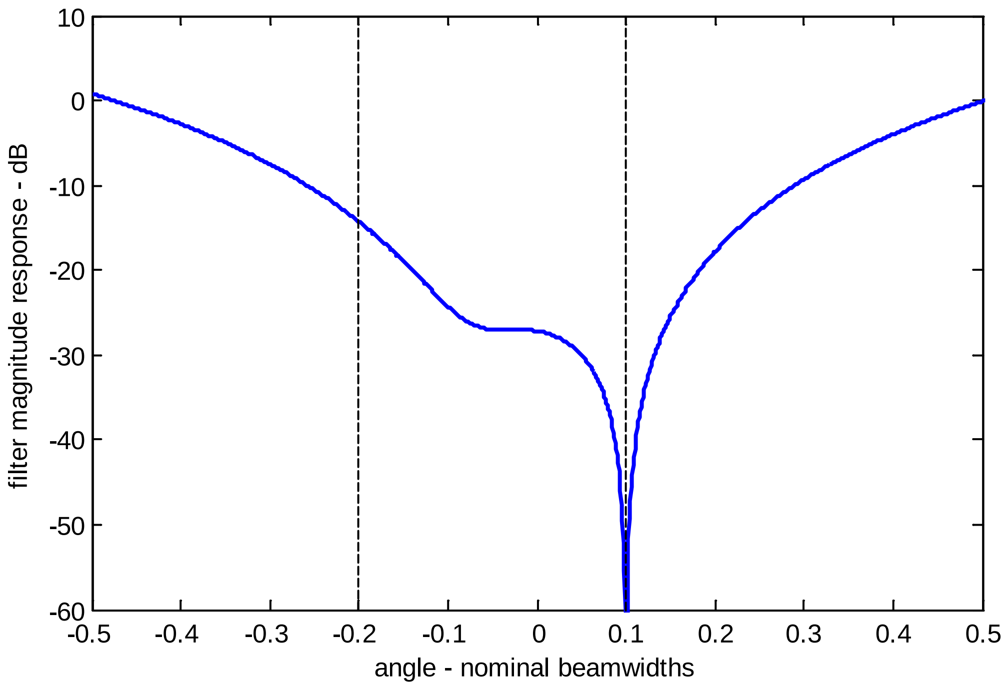

Figure 3. Fourier Transform of weight vector for three antenna phase centers and a noisy measurement of the composite signal, with 20 dB SNR .

2.6 Some Remaining Issues, Like Noise

The foregoing calculations for DOA will provide excellent results provided two conditions are met, namely

1. We have precise knowledge of the clutter direction, and
2. We have perfect measurements of the composite signals.

The problem is that neither of these conditions is really practical.

Precise knowledge of the clutter direction with respect to the antenna phase centers requires precise knowledge of antenna pointing and perhaps the relationship between direction and Doppler measurements. This in turn often requires precise knowledge of target scene terrain. Being off by even a very small fraction of the antenna beamwidth can have a significant degradation on the ability to cancel the clutter. In addition, even stationary clutter will often have some 'width' in angle space.

Precise measurements of the composite signal will be corrupted by measurement noise. Among other things, this guarantees that the matrix in Eq. (40) will be of full rank, even if no target is present. Consequently, rank is not a good indicator of the presence of a target, although the condition number of the matrix might still be somewhat of an indicator in some circumstances, like with very good SNR.

<u>Example</u>

Consider the same conditions and parameters as the previous example, except that any one composite signal is corrupted by Additive White Gaussian Noise (AWGN) with a Signal-to-Noise Ratio (SNR) of 20 dB. Ordinarily this is a quite desirable SNR. However, we observe in this example a significant error in the estimation of the target angle.

Notably, a true target DOA of –0.2 nominal beamwidths is identified from noisy measurements as a DOA of about –0.072 nominal beamwidths. Figure 3 illustrates the Fourier Transform of the calculated weight vector. Immediately obvious is that the target null is not at all well-defined.

3 Arbitrary Number of Phase Centers

What follows is an analysis of the problem of finding at least two Direction of Arrival (DOA) angles from data collected from an arbitrary number of antenna phase centers, with that number being 4 or more. We will phrase this as steering at least two nulls simultaneously towards two signals with weighted composite signals from all antenna phase centers.

We begin again with some definitions. Accordingly we define the antenna phase centers each with index

$$i = \text{index of phase centers, } i \in \{0,1,2,...,(I-1)\} \,. \tag{47}$$

To keep the development simple, these phase centers are collinear and spaced by equal amounts

$$b = \text{separation of phase centers.} \tag{48}$$

Let some number of signals be received by each phase center, each from direction

$$\theta_k = \text{direction of target } k, \; k \in \{1,2,...,K\} \,. \tag{49}$$

These directions are defined with respect to the normal of the antenna array. We generally allow that they may be the same or different. We are most interested for the case $K = 2$, but for now will allow larger K as well. We will stipulate that in all cases

$$K \le I \,. \tag{50}$$

Each target signal has complex scale of its return given by

$$A_k = \text{complex scale factor for target } k \,. \tag{51}$$

We will continue to assume that this is a constant for any one target, that is, the same for all antenna phase centers.

As before, each phase center i receives from target direction k a signal of the form

$$m_{i,k} = A_k \, p_{i,k} = \text{signal at } i^{\text{th}} \text{ phase center from target direction } k, \tag{52}$$

where we identify the phase term with

$$p_{i,k} = e^{-j\frac{2\pi b}{\lambda} i \sin\theta_k} = \text{signal's phase term corresponding to } m_{i,k} \,. \tag{53}$$

Consequently, each phase center receives a composite signal

$$M_i = \sum_k m_{i,k} \text{ = composite signal at } i^{\text{th}} \text{ phase center.} \tag{54}$$

This can be expanded to

$$M_i = \sum_k A_k p_{i,k} = \sum_k A_k p_{1,k}^i \,. \tag{55}$$

We will continue to assume for now that the composite signals are noise-free. Our task is to estimate all target directions θ_k from the set of composite signals M_i.

3.1 Steering Nulls to Known Directions

We desire that a single set of weights will null signals from both target directions, that is

$$\sum_i M_i w_i^* = 0 \,. \tag{56}$$

A trivial solution would be all weights equal to zero. However, we want a particular nontrivial solution that forces a null for each individual target direction such that

$$\sum_i m_{i,k} w_i^* = 0 \text{ for all signals } k. \tag{57}$$

This, of course, means we want the solution where

$$\sum_i p_{1,k}^i w_i^* = 0 \text{ for all signals } k. \tag{58}$$

We furthermore denote the vector of weights as

$$\mathbf{w} = \begin{bmatrix} w_0 & w_1 & w_2 & \dots & w_{I-1} \end{bmatrix}^T . \tag{59}$$

Consequently we wish to find constant nontrivial weights that allow us to equate

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & p_{1,1}^1 & p_{1,1}^2 & \dots & p_{1,1}^{I-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & p_{1,K}^1 & p_{1,K}^2 & \dots & p_{1,K}^{I-1} \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix} . \tag{60}$$

The matrix is in Vandermonde form. Note that we again have scaled the weights by arbitrarily assigning the first weight to have value

$$w_0 = 1. \tag{61}$$

We can then rewrite the matrix equation as

$$\begin{bmatrix} p_{1,1}^1 & p_{1,1}^2 & \cdots & p_{1,1}^{I-1} \\ p_{1,2}^1 & p_{1,2}^2 & \cdots & p_{1,2}^{I-1} \\ \cdots & \cdots & \cdots & \cdots \\ p_{1,K}^1 & p_{1,K}^2 & \cdots & p_{1,K}^{I-1} \end{bmatrix} \begin{bmatrix} w_1^* \\ w_2^* \\ \cdots \\ w_{I-1}^* \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ \cdots \\ -1 \end{bmatrix}. \tag{62}$$

Consequently, by finding the weights w_1 through w_{I-1}, we can effectively null signals from all K target directions simultaneously. This basically shows that weights do in fact exist to do this. This validates the rule-of-thumb that I weights can null $I-1$ sinusoids.

We now make some observations.

- We can only find exactly I independent weights if we have exactly $K = I-1$ independent sinusoids. This is the only case for which the matrix is invertible.
- For $K < I-1$, the matrix is not of full rank, which means it is not invertible, which means that a unique set of weights does not exist. That is, an infinite set of weights, suitably constrained, will work.
- Of course if we set $I-1-K$ weights to zero, or some other constant, then the problem can be reframed to an inversion of a lesser-sized matrix.

So the real problem becomes, which set of weights from the infinite set do we choose. We note that in a noise-free environment it doesn't matter, because the directions can be perfectly estimated even by zeroing some weights. The real issue is then, how do we calculate weights that allow us to 'best' estimate the DOAs from noisy measurements?

We set answering this question aside for now.

3.2 Calculating Angles from Weights

However we might arrive at the weights, once we have the weights, we can find estimates of the target direction angles by solving

$$\begin{bmatrix} 1 & \hat{p}^1 & \hat{p}^2 & \ldots & \hat{p}^{I-1} \end{bmatrix} \mathbf{w}^* = 0, \tag{63}$$

where we have stipulated that $w_0 = 1$, and identify

$$\hat{p} = e^{-j\frac{2\pi b}{\lambda}\sin\hat{\theta}_k}. \tag{64}$$

Once again, in general null steering, this polynomial is known as the array factor (AF) and the nulls occur at the zeroes of the polynomial. For every root $\hat{p}_k$ of this polynomial, we may find the corresponding angle

$$\hat{\theta}_k = \text{asin}\left(\frac{j\lambda}{2\pi b} \text{Arg}\left(\hat{p}_k \right) \right). \tag{65}$$

These are the DOA for all significant signals. But first we have to find the weights.

3.3 Calculating Weights from Signals with Unknown DOA

The problem is that we desire to find the weights w_i from knowledge of only the composite signals M_i, with the exception that from perhaps Doppler measurements we may at least be able to identify the clutter angle, which we continue to designate θ_c. As before, we further define the phase term of the clutter with

$$p_c = e^{-j\frac{2\pi b}{\lambda}\sin\theta_c}. \tag{66}$$

Our matrix equation then becomes

$$\begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 1 & p_c & p_c^2 & ... & p_c^{I-1} \\ M_0 & M_1 & M_2 & ... & M_{I-1} \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{67}$$

Since the matrix has only three rows, and we are considering here the case where $I \geq 4$, the matrix is guaranteed to have no inverse. Consequently, we make the following observations.

- There is no unique solution to the weights w_i.
- In the absence of any other criteria, we could just set weights $w_i = 0$ for $I \geq 3$, and achieve perfect results, at least for $K = 2$.
- In order to find a unique set of weights, we need ‘additional criteria’.

To this point we have ignored the fact that our actual measurements of the composite signals will be perturbed by noise. We will find our ‘additional criteria’ in minimizing the effects of noise.

3.4 Adding Additional Designated Nulls

The previous development assumed that only the single direction of clutter was known a priori, and needed a designated null. Now let us presume to designate another fixed angle θ_f as requiring a guaranteed null. We may then define the phase term associated with this fixed null as

$$p_f = e^{-j\frac{2\pi b}{\lambda}\sin\theta_f} . \tag{68}$$

Such a fixed null angle might be to mitigate a known interference source. Nevertheless, our matrix equation with the two fixed nulls then becomes

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & p_c & p_c^2 & \dots & p_c^{I-1} \\ 1 & p_f & p_f^2 & \dots & p_f^{I-1} \\ M_0 & M_1 & M_2 & \dots & M_{I-1} \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{69}$$

We observe that for the cases $I \geq 5$, we have the same problem as the previous section, namely there is no unique solution for the weights.

The solution to this is the same as put forth in the previous section, namely that since any real measurements will include measurement noise, we will adopt criteria that minimizes the impact of the noise on DOA estimation.

We note also that this is readily extensible to even more fixed null directions.

3.5 Steering Nulls in the Presence of Noise

We return to the problem of finding the weights w_i from knowledge of only the composite signals M_i and a single clutter angle that might be inferred from Doppler measurements. Our noise-free matrix equation from the previous development was given in Eq. (67), and is recalled here as

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & p_c & p_c^2 & \ldots & p_c^{I-1} \\ M_0 & M_1 & M_2 & \ldots & M_{I-1} \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{70}$$

For the subsequent development, we define some convenient column vectors as

$$\mathbf{M} = \begin{bmatrix} M_0 & M_1 & \ldots & M_{I-1} \end{bmatrix}^T = \text{composite signal vector, and}$$
$$\mathbf{n} = \begin{bmatrix} n_0 & n_1 & \ldots & n_{I-1} \end{bmatrix}^T = \text{noise vector.} \tag{71}$$

The measurement available to us is then the linear sum of these, namely

$$\mathbf{x} = \mathbf{M} + \mathbf{n} = \begin{bmatrix} x_0 & x_1 & \ldots & x_{I-1} \end{bmatrix}^T = \text{measurement vector.} \tag{72}$$

We identify the clutter angle via a priori knowledge as θ_c, and accordingly define the phase term

$$p_c = e^{-j\frac{2\pi b}{\lambda}\sin\theta_c}. \tag{73}$$

We then identify the clutter steering vector as

$$\mathbf{v}_c = \begin{bmatrix} 1 & p_c & \ldots & p_c^{I-1} \end{bmatrix}^T = \text{clutter steering vector.} \tag{74}$$

We define a vector to constrain the scale of the weights arbitrarily with

$$\mathbf{v}_s = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T = \text{scaling constraint vector.} \tag{75}$$

Our constraints to a solution for an optimal weight vector are then

$$\mathbf{v}_c{}^T \mathbf{w}^* = 0 = \text{the null response towards the clutter, and}$$
$$\mathbf{v}_s{}^T \mathbf{w}^* = 1 = \text{the scale constraint for the weights.} \tag{76}$$

We desire to select weights such that

$$\mathbf{x}^T \mathbf{w}^* = 0, \text{ or at least minimum in the statistical sense,} \tag{77}$$

but subject to the aforementioned constraints. This is embodied in the matrix equation

$$\begin{bmatrix} \mathbf{v}_S{}^T \\ \mathbf{v}_C{}^T \\ \mathbf{x}^T \end{bmatrix} \mathbf{w}^* = \begin{bmatrix} 1 \\ 0 \\ \varepsilon \end{bmatrix}, \tag{78}$$

which may also be written as

$$\mathbf{w}^H \begin{bmatrix} \mathbf{v}_S & \mathbf{v}_C & \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \varepsilon \end{bmatrix}, \tag{79}$$

where we define the error term

$$\varepsilon = \mathbf{w}^H \mathbf{x} = \text{error that we wish to minimize in the statistical sense.} \tag{80}$$

We stipulate that the optimal weights occur when $|\varepsilon|^2$ is minimized in the statistical sense. To proceed, we recognize that

$$E\left\langle \left|\mathbf{w}^H \mathbf{x}\right|^2 \right\rangle = \mathbf{w}^H E\left\langle \mathbf{x}\mathbf{x}^H \right\rangle \mathbf{w} = \mathbf{w}^H \mathbf{R}_{\mathbf{xx}} \mathbf{w}, \tag{81}$$

where

$$E\langle y \rangle = \text{is the expected value of } y\text{, and}$$
$$\mathbf{R}_{\mathbf{xx}} = E\left\langle \mathbf{x}\,\mathbf{x}^H \right\rangle = \text{the covariance matrix of } \mathbf{x}. \tag{82}$$

Taking a cue from the development of optimal filters in Appendix A, we employ the method of Lagrange multipliers and identify the Lagrange function with two constraints as

$$\Lambda = \mathbf{w}^H \mathbf{R}_{\mathbf{xx}} \mathbf{w} + \alpha_c \left(\mathbf{w}^H \mathbf{v}_c - 0\right) + \alpha_c{}^* \left(\mathbf{v}_c^H \mathbf{w} - 0\right) + \alpha_s \left(\mathbf{w}^H \mathbf{v}_s - 1\right) + \alpha_s{}^* \left(\mathbf{v}_s^H \mathbf{w} - 1\right), \tag{83}$$

where

$$\alpha_c = \text{Lagrange multiplier associated with the clutter steering vector, and}$$
$$\alpha_s = \text{Lagrange multiplier associated with the scaling constraint.} \tag{84}$$

We may then find the optimum weight vector by taking the derivative of the Lagrange function with respect to $\mathbf{w}^H$ and setting it to zero. Doing so yields

$$\mathbf{R_{xx}}\mathbf{w}_{opt} - \alpha_c \mathbf{v}_c - \alpha_s \mathbf{v}_s = 0 . \tag{85}$$

This may be solved for the optimal weights as

$$\mathbf{w}_{opt} = \mathbf{R_{xx}}^{-1}\left(\alpha_c \mathbf{v}_c + \alpha_s \mathbf{v}_s\right) = \alpha_c \mathbf{R_{xx}}^{-1}\mathbf{v}_c + \alpha_s \mathbf{R_{xx}}^{-1}\mathbf{v}_s . \tag{86}$$

The earlier constraints may now be employed to derive the equations

$$\begin{aligned} &\mathbf{v}_c{}^H \left(\alpha_c \mathbf{R_{xx}}^{-1}\mathbf{v}_c + \alpha_s \mathbf{R_{xx}}^{-1}\mathbf{v}_s\right) = 0 \text{, and} \\ &\mathbf{v}_s{}^H \left(\alpha_c \mathbf{R_{xx}}^{-1}\mathbf{v}_c + \alpha_s \mathbf{R_{xx}}^{-1}\mathbf{v}_s\right) = 1 , \end{aligned} \tag{87}$$

which can be expanded to

$$\begin{aligned} &\alpha_c \mathbf{v}_c{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_c + \alpha_s \mathbf{v}_c{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_s = 0 \text{, and} \\ &\alpha_c \mathbf{v}_s{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_c + \alpha_s \mathbf{v}_s{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_s = 1 . \end{aligned} \tag{88}$$

This can then be written as the matrix equation

$$\begin{bmatrix} \left(\mathbf{v}_c{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_c\right) & \left(\mathbf{v}_c{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_s\right) \\ \left(\mathbf{v}_s{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_c\right) & \left(\mathbf{v}_s{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_s\right) \end{bmatrix} \begin{bmatrix} \alpha_c \\ \alpha_s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{89}$$

The solution for the Lagrange multipliers can then be found by solving as

$$\begin{bmatrix} \alpha_c \\ \alpha_s \end{bmatrix} = \begin{bmatrix} \left(\mathbf{v}_c{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_c\right) & \left(\mathbf{v}_c{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_s\right) \\ \left(\mathbf{v}_s{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_c\right) & \left(\mathbf{v}_s{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_s\right) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{90}$$

This vector solution may be expanded to

$$\begin{bmatrix} \alpha_c \\ \alpha_s \end{bmatrix} = \frac{\begin{bmatrix} \left(\mathbf{v}_s{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_s\right) & -\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_c\right) \\ -\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_s\right) & \left(\mathbf{v}_c{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_c\right) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_c\right)\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_s\right) - \left(\mathbf{v}_s{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_c\right)\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}^{-1}\mathbf{v}_s\right)} . \tag{91}$$

From this, the individual scalars can be calculated and identified as

$$\alpha_c = \frac{-\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)}{\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_s\right)-\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_s\right)}, \text{ and}$$

$$\alpha_s = \frac{\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)}{\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_s\right)-\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_s\right)}. \tag{92}$$

We will ignore for now just how exactly we will identify the covariance matrix $\mathbf{R_{xx}}$.

3.6 Putting It All Together

Given measurements, we identify constraints on the optimal weights as

$$\mathbf{v}_c = \begin{bmatrix} 1 & p_c & \ldots & p_c^{I-1} \end{bmatrix}^T = \text{clutter steering vector, and}$$

$$\mathbf{v}_s = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T = \text{scaling constraint vector.} \tag{93}$$

The assumption is of course that we know the clutter DOA precisely.

We next identify the covariance matrix $\mathbf{R_{xx}}$, via some as yet unspecified measurement or calculation. For now we will simply presume that we know this a priori.

Using these, we calculate Lagrange multipliers as

$$\alpha_c = \frac{-\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)}{\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_s\right)-\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_s\right)}, \text{ and}$$

$$\alpha_s = \frac{\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)}{\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_s\right)-\left(\mathbf{v}_s{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c\right)\left(\mathbf{v}_c{}^H \mathbf{R_{xx}}{}^{-1}\mathbf{v}_s\right)}. \tag{94}$$

Optimal weights are then calculated with

$$\mathbf{w}_{opt} = \mathbf{R_{xx}}{}^{-1}\left(\alpha_c \mathbf{v}_c + \alpha_s \mathbf{v}_s\right) = \alpha_c \mathbf{R_{xx}}{}^{-1}\mathbf{v}_c + \alpha_s \mathbf{R_{xx}}{}^{-1}\mathbf{v}_s. \tag{95}$$

Once we have the weights, we can find estimates of the target direction angles by solving for the roots of

$$\begin{bmatrix} 1 & \hat{p}^1 & \hat{p}^2 & \ldots & \hat{p}^{I-1} \end{bmatrix} \mathbf{w}_{opt}^* = 0, \tag{96}$$

where we identify specific roots as

$$\hat{p}_k = e^{-j\frac{2\pi b}{\lambda}\sin\hat{\theta}_k}. \tag{97}$$

The nulls occur at the zeroes of the polynomial, and for every root $\hat{p}_k$ of this polynomial, we may find the corresponding angle that is the DOA for a significant signal as

$$\hat{\theta}_k = \operatorname{asin}\left(\frac{j\lambda}{2\pi b}\operatorname{Arg}\left(\hat{p}_k\right)\right). \tag{98}$$

One of these null directions will coincide with the clutter direction θ_c.

Otherwise, we may simply take the Fourier Transform of the optimal weight vector, perhaps wish substantial oversampling of the output. Nulls should be clearly observable. The deepest non-clutter null will identify the DOA of the unknown target signal.

Example

Consider the parameters of the previous examples, except that the antenna now has four phase centers, still operating with the following parameters.

$\lambda = 0.02$ m,
$D_{ant} = 0.5$ m = antenna overall aperture width. (99)

We again define clutter and target angles as

$\theta_c = 0.1\,\theta_{bw}$ = clutter angle,
$\theta_1 = -0.2\,\theta_{bw}$ = target angle, and
$SNR = 20$ dB for an individual phase center (with AWGN). (100)

With these parameters, Figure 4 illustrates the Fourier Transform of the weight vector, along with the angles used in the composite signal. A calculation of the target's null position gives approximately -0.203 nominal beamwidths.

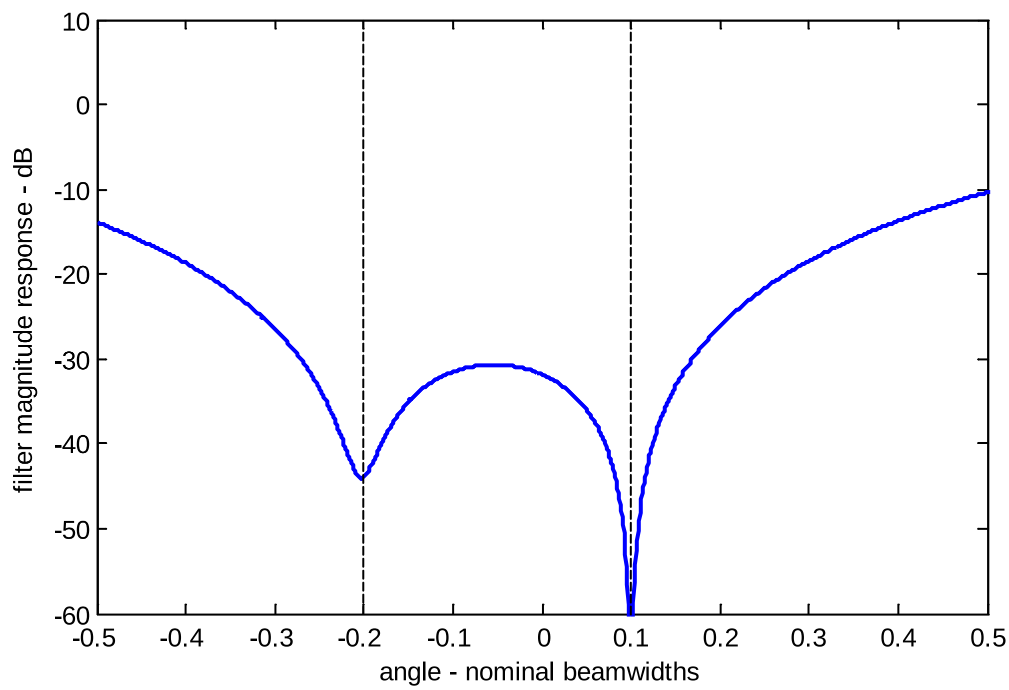

Figure 4. Fourier Transform of weight vector for four antenna phase centers based on the covariance matrix of the target signal in noise.

3.7 Some Remaining Issues

For the calculation of the null position of the target, we have relied on two pieces of a priori knowledge, namely

1. The DOA of the clutter, and
2. The covariance matrix of the target signal in noise.

As previously indicated, precise knowledge of the clutter direction with respect to the antenna phase centers requires precise knowledge of antenna pointing and perhaps the relationship between direction and Doppler measurements. This in turn requires perhaps precise knowledge of target scene terrain. Being off by even a very small fraction of the antenna beamwidth can have a significant degradation on the ability to cancel the clutter.

Although we now have the ability to make fairly precise calculations of DOA in noise, in order to do so we need a fairly good estimate of the statistics of the target signal in noise, in the form of the covariance matrix. The 'noise' part of the covariance matrix is pretty straightforward to at least estimate, perhaps from radar equation calculations, but the target 'signal' part of the covariance matrix is another matter. Furthermore, in typical GMTI processing, we have but a single range-Doppler cell that contains the target, so any averaging of data to find the necessary covariance matrix is not really possible.

"It is the direction and not the magnitude which is to be taken into consideration."
-- Thomas Paine

4 Beamforming in the Presence of Noise

In this section we depart somewhat from the previous developments. We will examine a slightly different problem; that of passing a desired signal while minimizing the manifestations of undesired signals like clutter, interference, and noise. Accordingly, we identify the following constituents of a received signal

$$\mathbf{m}_t = \begin{bmatrix} m_{t,0} & m_{t,1} & \dots & m_{t,I-1} \end{bmatrix}^T = \text{target signal vector,}$$
$$\mathbf{m}_c = \begin{bmatrix} m_{c,0} & m_{c,1} & \dots & m_{c,I-1} \end{bmatrix}^T = \text{clutter signal vector,}$$
$$\mathbf{n} = \begin{bmatrix} n_0 & n_1 & \dots & n_{I-1} \end{bmatrix}^T = \text{AWGN noise vector.} \quad (101)$$

We shall for this development ignore any other interfering signals, or any additional targets. Consequently, the received signal that includes target, clutter, and noise is still

$$\mathbf{x} = \mathbf{m}_t + \mathbf{m}_c + \mathbf{n} = \begin{bmatrix} x_0 & x_1 & \dots & x_{I-1} \end{bmatrix}^T = \text{measurement vector,} \quad (102)$$

except that we will now parse these constituents into target and interference+noise, that is

$$\mathbf{x} = \mathbf{m}_t + \boldsymbol{\eta} = \text{measurement vector,} \quad (103)$$

where

$$\boldsymbol{\eta} = \mathbf{m}_c + \mathbf{n} = \begin{bmatrix} \eta_0 & \eta_1 & \dots & \eta_{I-1} \end{bmatrix}^T = \text{interference+noise vector.} \quad (104)$$

Out of pure laziness, we will refer to the interference+noise vector as the "interference" vector, and this entity more generally as "interference." This might also be characterized as "colored" noise. More generally, this includes everything that isn't specifically the target of interest.

In general, we seek to find a weight vector

$$\mathbf{w} = \begin{bmatrix} w_0 & w_1 & \dots & w_{I-1} \end{bmatrix}^T = \text{weight vector,} \quad (105)$$

which allows us to calculate the inner-product,

$$y = \mathbf{w}^H \mathbf{x}, \quad (106)$$

with some optimal properties, even in the presence of interference, like the ability to perhaps detect and measure DOA of the target signal.

The problem for which we now pursue a solution is examining a collection of DOAs from some same data set to find a DOA with peak response of a desired target signal. This is often termed "beamforming" with some specific calculations identified as a "beamformer." We will examine two such beamformer techniques.

The first beamformer is the well-known standard beamformer. The second beamformer is the somewhat less well-known adaptive beamformer. The standard beamformer is related to Fourier power spectral analysis of temporal signals. The adaptive beamformer is related to Wiener filtering.

4.1 Standard Beamformer

The standard beamformer involves a Fourier-like relationship between the illumination of the antenna and the angle to the target. We define this standard beamformer filter for a uniform linear array (ULA) as:

$$\mathbf{w}_{std} = \frac{\mathbf{v}_d}{I} \tag{107}$$

where:

$\mathbf{v}_d$ = the steering vector (in the desired signal direction), and
I = the number of phase centers used in the beamformer. (108)

This "standard" beamformer is sometimes also referred to as a "conventional" beamformer, or "delay-and-sum" beamformer. The steering vector is given as the general form

$$\mathbf{v}_d = \begin{bmatrix} 1 & e^{-j\frac{2\pi}{\lambda}b\sin\theta_d} & e^{-j\frac{2\pi}{\lambda}2b\sin\theta_d} & \cdots \end{bmatrix}^T . \tag{109}$$

If the desired steering vector, $\mathbf{v}_d$, is "scanned", the peak return will occur when this filter is "steered" in the direction of a signal, which we hope is the target signal. This is the spatial equivalent of the matched filter, which is known to be optimal for AWGN.

We might use a priori knowledge of clutter DOA to rule out a peak due to clutter (see more about clutter elimination in the discussion on the adaptive beamformer below). In this sense, the standard beamformer acts like a Discrete Fourier Transform (DFT) over the antenna elements. As such, it is non-parametric and is not data-dependent. However, it tends to have limited flexibility in null placement and the resolution is limited by the reciprocal of the antenna dimension. Both resolution and null placement are very important for maintaining low minimum detectable velocity (MDV) and for estimating the location of slow moving objects.

4.2 Adaptive Beamformer

The adaptive beamformer is also a non-parametric technique, but it is data-dependent. It allows for more flexible null placement and greater resolution in many circumstances.[6] However, the adaptive beamformer adds some complication to the processing and is dependent upon signal-to-noise and data quality. Specifically, it depends upon knowing the covariance matrix of the interference. Consequently, data quality issues arise when an estimate of the covariance matrix from the measured interference data is required. Since the standard beamformer is quite well-known, we will focus on how the adaptive beamformer extends the previous discussion to the signal perturbed by interference case.

As in the prior section, we want to avoid the trivial solution for any weight vector; therefore, we will constrain the weight vector $\mathbf{w}$, and choose

$$\mathbf{w}^H \mathbf{v}_d = 1 \tag{110}$$

where $\mathbf{v}_d$ is still the steering vector in the desired direction. Eq. (110) is the so-called distortionless constraint.

Essentially, we want to pass any signal in the direction of θ_d, provided this is not the direction of an undesired signal like clutter, but minimize the effects of any interference in the data that are a distraction to us, as embodied in the interference plus noise vector $\boldsymbol{\eta}$. We furthermore identify

$$\mathbf{R}_{\boldsymbol{\eta\eta}} = E\left\langle \boldsymbol{\eta\eta}^H \right\rangle = \text{the covariance matrix of } \boldsymbol{\eta}, \tag{111}$$

where

$$E\left\langle y \right\rangle = \text{is the expected value of } y. \tag{112}$$

The general minimization solution results in the vector form of the Wiener filter. This is sometimes simply referred to as "optimum filtering" with details provided in Appendix A. Nevertheless, the optimal weights are calculated to be

$$\mathbf{w}_{opt} = \alpha \mathbf{R}_{\boldsymbol{\eta\eta}}^{-1} \mathbf{v}_d, \tag{113}$$

where the constraint in equation (110) sets the scalar α as described in Appendix A to

$$\alpha = \left(\mathbf{v}_d^H \mathbf{R}_{\boldsymbol{\eta\eta}}^{-1} \mathbf{v}_d \right)^{-1}. \tag{114}$$

At this point, we will effectively scan across various DOAs by scanning θ_d. At each scan angle we will calculate $\mathbf{w}_{opt}$, and then apply it to our measurement vector $\mathbf{x}$ to calculate y. A peak in this response as we scan will identify the best estimate of a target's DOA, or at least that's the desired outcome.

Recall that the Wiener filter is an "optimal filter" in the sense that it maximizes the Signal-to-Interference-plus-Noise Ratio (SINR). The term "Interference" is used to show that the noise may be colored and not just white noise. The Wiener filter uses knowledge of the spectrum of the signal and the interference plus noise to optimally filter. Note that the optimal filter in the presence of white noise, in the Wiener sense, is the standard matched filter. If the signal is a spatial sinusoid, then this becomes the standard beamformer mentioned above.

The resulting filter goes by many names, but a common one that we will use is the Minimum Variance Distortionless Response (MVDR) beamformer. This beamformer can be shown to lead to the Maximum Likelihood Estimator (MLE) under certain conditions. These conditions include Gaussian noise, and known signal locations. The most interesting condition is that it requires that the covariance matrix $\mathbf{R}_{\boldsymbol{\eta\eta}}$ be known. This latter condition plays an important role in filtering, especially STAP, since the covariance matrix is rarely known in practice. In STAP, the assumption that we know the covariance matrix is the so-called "clairvoyant" assumption. Estimation of the covariance matrix and the errors in its estimation have been the subject of substantial amounts of research. This is the hard part of STAP.

We discuss the estimation of the covariance matrix $\mathbf{R}_{\boldsymbol{\eta\eta}}$ in some more detail in Appendix C.

We will now give an example to drive home some of the more subtle but very powerful implications of the above filtering process and show the relationship with the previous section and with the standard beamformer. These examples will use one and two spatial sinusoids. The case of one or two sinusoids with distinct directions can be treated in the manner above much like one or two jammers.

Single Interfering Sinusoid (e.g. Clutter)

We examine the case of a single sinusoid. This represents the idealization of the case where the interference is just clutter and AWGN. The assumption here is that we have a desired sinusoid signal *plus* a single *interfering* sinusoid which we term clutter, and all additionally perturbed by AWGN. In this respect, this is much like the two sinusoid non-noise case. Note that the assumption here for the moment is that we know the direction of the clutter, but not the target signal.

The starting point for this discussion is the covariance matrix. In this case the covariance matrix is given by

$$\mathbf{R}_{\boldsymbol{\eta\eta},1} = \sigma_c^2 \mathbf{v}_c \mathbf{v}_c^H + \sigma_n^2 \mathbf{I} \tag{115}$$

where, we recall

$$\mathbf{v}_c = \begin{bmatrix} 1 & p_c & \ldots & p_c^{I-1} \end{bmatrix}^T \text{ = clutter steering vector, with}$$

$$p_c = e^{-j\frac{2\pi b}{\lambda}\sin\theta_c}, \tag{116}$$

and we introduce

σ_c^2 = the power of the clutter signal at the i^{th} phase center, and

σ_n^2 = the power in the noise at the i^{th} phase center. (117)

The inverse of the covariance matrix is easily found by applying the well-known matrix inversion lemma (with a straight-forward twist):

$$\mathbf{R}_{\boldsymbol{\eta\eta},1}^{-1} = \frac{1}{\sigma_n^2}\left[\mathbf{I} - \left(\frac{I\beta_c}{I\beta_c + 1}\right)\frac{\mathbf{v}_c \mathbf{v}_c^H}{I}\right] \tag{118}$$

where

$\beta_c = \dfrac{\sigma_c^2}{\sigma_n^2}$ = the SNR for the clutter, i.e. the Clutter-to-Noise Ratio (CNR), and

I = the number of phase centers. (119)

This leads to a calculation of the optimal weights as

$$\mathbf{w}_{opt}^H = \frac{\alpha I}{\sigma_n^2}\left[\frac{\mathbf{v}_d^H}{I} - \rho_{dc}\rho_{INR}\frac{\mathbf{v}_c^H}{I}\right], \tag{120}$$

where

$\rho_{dc} = \dfrac{\mathbf{v}_d^H \mathbf{v}_c}{I}$, and

$\rho_{INR} = \dfrac{I\beta_c}{I\beta_c + 1}$. (121)

Equation (120) is packed with important revelations about the behavior of MVDR and how it relates to, and is different from, the standard beamformer. We will continue the discussion with emphasis on these scale factors to further illuminate the optimal filter behavior.

An obvious implication of equation (120), is that for the specific case where the CNR is zero (i.e., $\beta_c = 0$), then the optimal filter is merely the standard beamformer.

In the general case, the optimal filter is simply a linear combination of two standard beamformers. The first standard beamformer is steered to the desired signal, whereas the second beamformer is steered at the interfering clutter. A key point is that the interference steering vector information derives from the covariance matrix. The cancellation term comes from this second beamformer. The cancellation is controlled by the scalars, ρ_{dc} and ρ_{INR}. The term ρ_{dc} is factor that accounts for the correlation between the desired signal and the interfering clutter steering vectors. The second scale factor, ρ_{INR}, is a function of the CNR. We have labeled this as ρ_{INR} to show that this term behaves very much like a correlation coefficient.

Let us now for the moment assume that the interference to noise ratio is nearly infinite, which means that $\rho_{INR} = 1$. The emphasis in this case is then on the scale factor ρ_{dc}. This term behaves like a correlation coefficient between the desired signal steering vector $\mathbf{v}_d$, and the interfering clutter steering vector $\mathbf{v}_c$. Literally, it is the projection of the interference steering vector in the direction of the desired signal steering vector. If $\rho_{INR} = 1$, then the optimal filter given in equation (120) is the projection of the desired steering vector orthogonal to the interference. The variation of this projection function with angular separation between the desired and interference steering vectors is related to the array factor. In the limit, as the target moves into the sidelobes , ρ_{dc} approaches zero and this leads to the standard beamformer again This is what we would expect for exo-clutter GMTI.

More generally, as the SNR of the interfering sinusoid term (i.e. the clutter) increases, the filter passes through the signal in the desired direction minus a scaled projection of the interfering sinusoid in the direction of the desired signal.

To summarize, we note that in general the amount of cancellation from the second term in equation (120) is not infinite, but rather is controlled by is controlled by the CNR, as well as the null width relative to the target and clutter angular separation.

To complete this section, we present the scale factor, α, for this case from equation (114)

$$\alpha = \left\{ \frac{I}{\sigma_n^2} \left[1 - \rho_{INR} \left| \rho_{dc} \right|^2 \right] \right\}^{-1} . \tag{121}$$

Example:

Consider the parameters of the previous examples, where the antenna has four phase centers, still operating with the following parameters.

$$\lambda = 0.02 \text{ m},$$
$$D_{ant} = 0.5 \text{ m} = \text{antenna overall aperture width.} \tag{122}$$

We again define clutter and target angles as

$$\theta_c = 0.1\,\theta_{bw} = \text{clutter angle},$$
$$\theta_1 = -0.2\,\theta_{bw} = \text{target angle},$$
$$CNR = 20 \text{ dB } (\rho_{INR} \approx 0.9975) \text{ for an individual phase center (with AWGN), and}$$
$$SNR = 20 \text{ dB for an individual phase center (with AWGN).} \tag{123}$$

Figure 5 illustrates the corresponding response to a scanned steering vector, along with the relevant angles. We note that although the optimal weights when steered to the target DOA provide 0 dB gain as desired, there is however no clear indication of a peak in the target direction, even with perfect knowledge of the covariance matrix and a relatively good SNR. We note that in order to keep the constraint $\mathbf{w}^H \mathbf{v}_d = 1$ as we approach the clutter DOA, the filter gain must increase to overcome the clutter null's sidewall attenuation. Absent clutter, a broad peak near the target angle would present itself.

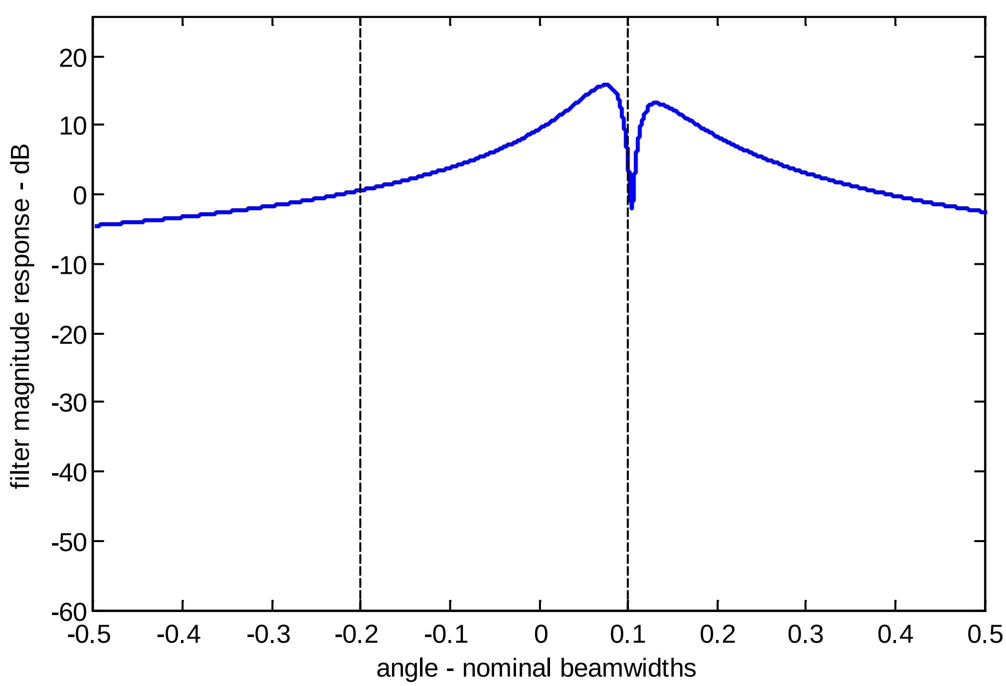

Figure 5. Response to a scanned steering vector for four antenna phase centers based on the covariance matrix of the clutter signal in noise.

"When you have cleared all of your clutter,
you can be of greater service to those around you."
-- *Michael B. Kitson*

5 Steering Nulls Revisited

We now combine some aspects of the techniques previously discussed. In the previous section we scanned a steering vector to place a unit gain in the direction of interest, intending that a peak response would indicate the DOA of a target sinusoid, which proved problematic. Now, instead of scanning a unit gain, we will instead scan a null, intending that a minimum response will indicate the DOA of a target sinusoid.

Accordingly, we identify the following constituents of a received signal as

$$\mathbf{m}_t = \begin{bmatrix} m_{t,0} & m_{t,1} & \cdots & m_{t,I-1} \end{bmatrix}^T = \text{target signal vector,}$$
$$\mathbf{m}_c = \begin{bmatrix} m_{c,0} & m_{c,1} & \cdots & m_{c,I-1} \end{bmatrix}^T = \text{clutter signal vector,}$$
$$\mathbf{n} = \begin{bmatrix} n_0 & n_1 & \cdots & n_{I-1} \end{bmatrix}^T = \text{AWGN noise vector.} \tag{124}$$

We shall for this development ignore any other interfering signals, or any additional targets. Consequently, the received signal that includes target, clutter, and noise is still

$$\mathbf{x} = \mathbf{m}_t + \mathbf{m}_c + \mathbf{n} = \begin{bmatrix} x_0 & x_1 & \cdots & x_{I-1} \end{bmatrix}^T = \text{measurement vector,} \tag{125}$$

except that we will now parse these constituents into target and interference+noise, that is

$$\mathbf{x} = \mathbf{m}_t + \boldsymbol{\eta} = \text{measurement vector,} \tag{126}$$

where

$$\boldsymbol{\eta} = \mathbf{m}_c + \mathbf{n} = \begin{bmatrix} \eta_0 & \eta_1 & \cdots & \eta_{I-1} \end{bmatrix}^T = \text{interference+noise vector.} \tag{127}$$

We will continue to refer to the interference+noise vector as the "interference" vector, and this entity more generally as "interference." This might also be characterized as "colored" noise. More generally, this includes everything that isn't specifically the target of interest.

In general, we seek to find a weight vector

$$\mathbf{w} = \begin{bmatrix} w_0 & w_1 & \cdots & w_{I-1} \end{bmatrix}^T = \text{weight vector,} \tag{128}$$

that allows us to calculate the scalar product,

$$y = \mathbf{w}^H \mathbf{x}, \tag{129}$$

with some optimal properties, even in the presence of interference, like the ability to perhaps detect and measure DOA of the target signal.

These weights will be chosen to statistically minimize the error

$$\varepsilon = \mathbf{w}^H \boldsymbol{\eta} \ . \tag{130}$$

We stipulate that the optimal weights occur when $|\varepsilon|^2$ is minimized in the statistical sense. To proceed, we recognize that

$$E\left\langle \left|\mathbf{w}^H \boldsymbol{\eta}\right|^2 \right\rangle = \mathbf{w}^H E\left\langle \boldsymbol{\eta}\boldsymbol{\eta}^H \right\rangle \mathbf{w} = \mathbf{w}^H \mathbf{R}_{\boldsymbol{\eta}\boldsymbol{\eta}} \mathbf{w} , \tag{131}$$

where

$$\mathbf{R}_{\boldsymbol{\eta}\boldsymbol{\eta}} = E\left\langle \boldsymbol{\eta}\boldsymbol{\eta}^H \right\rangle = \text{the covariance matrix of } \boldsymbol{\eta}, \tag{132}$$

where

$$E\left\langle y \right\rangle = \text{is the expected value of } y . \tag{133}$$

We want to find weights that will minimize the error subject to some constraints. Among them, we will avoid the trivial solution by constraining

$$\mathbf{w}^H \mathbf{v}_s = 1 = \text{the scale constraint for the weights.} \tag{134}$$

5.1 Steering a Single Null

We want to scan a null steering vector and choose weights such that

$$\mathbf{w}^H \mathbf{v}_d = 0 . \tag{135}$$

The idea is to scan the null steering vector $\mathbf{v}_d$ to find a minimum in the filtered data, i.e. in $y = \mathbf{w}^H \mathbf{x}$.

Taking a cue from the development of optimal filters in Appendix A, we employ the method of Lagrange multipliers and identify the Lagrange function with two constraints as

$$\Lambda = \mathbf{w}^H \mathbf{R}_{\boldsymbol{\eta}\boldsymbol{\eta}} \mathbf{w} + \alpha_d \left(\mathbf{w}^H \mathbf{v}_d - 0\right) + \alpha_d^{\,*} \left(\mathbf{v}_d^H \mathbf{w} - 0\right) + \alpha_s \left(\mathbf{w}^H \mathbf{v}_s - 1\right) + \alpha_s^{\,*} \left(\mathbf{v}_s^H \mathbf{w} - 1\right), \tag{136}$$

where

α_d = Lagrange multiplier associated with the null steering vector, and
α_s = Lagrange multiplier associated with the scaling constraint. (137)

We may then find the optimum weight vector by taking the derivative of the Lagrange function with respect to $\mathbf{w}^H$ and setting it to zero. Doing so yields

$$\mathbf{R}_{\boldsymbol{\eta\eta}}\mathbf{w}_{opt} - \alpha_d \mathbf{v}_d - \alpha_s \mathbf{v}_s = 0 \,. \tag{138}$$

This may be solved for the optimal weights as

$$\mathbf{w}_{opt} = \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\left(\alpha_d \mathbf{v}_d + \alpha_s \mathbf{v}_s\right) = \alpha_d \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d + \alpha_s \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_s \,. \tag{139}$$

The earlier constraints may now be written as the matrix equation

$$\begin{bmatrix} \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d\right) & \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_s\right) \\ \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d\right) & \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_s\right) \end{bmatrix} \begin{bmatrix} \alpha_d \\ \alpha_s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{140}$$

The solution for the Lagrange multipliers can then be found by solving as

$$\begin{bmatrix} \alpha_d \\ \alpha_s \end{bmatrix} = \begin{bmatrix} \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d\right) & \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_s\right) \\ \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d\right) & \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_s\right) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{141}$$

From this, the individual scalars can be calculated and identified as

$$\alpha_d = \frac{-\left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d\right)}{\left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d\right)\left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_s\right) - \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d\right)\left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_s\right)}, \text{ and}$$

$$\alpha_s = \frac{\left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d\right)}{\left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d\right)\left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_s\right) - \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d\right)\left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_s\right)} \,. \tag{142}$$

Putting It All Together

Given measurements, we identify constraints on the optimal weights as

$$\mathbf{v}_d = \begin{bmatrix} 1 & p_1 & \cdots & p_1^{I-1} \end{bmatrix}^T \text{ = target steering vector, and}$$

$$\mathbf{v}_s = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T \text{ = scaling constraint vector.} \tag{143}$$

We next identify the covariance matrix $\mathbf{R}_{\boldsymbol{\eta\eta}}$, via appropriate measurement or calculation, perhaps as discussed in Appendix C. For now we will simply presume that we know this a priori.

Using these, we calculate Lagrange multipliers as

$$\begin{bmatrix} \alpha_d \\ \alpha_s \end{bmatrix} = \begin{bmatrix} \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d \right) & \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s \right) \\ \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d \right) & \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s \right) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{144}$$

Optimal weights are then calculated with

$$\mathbf{w}_{opt} = \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \left(\alpha_d \mathbf{v}_d + \alpha_s \mathbf{v}_s \right) = \alpha_d \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d + \alpha_s \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s . \tag{145}$$

These weights are then applied to the measured data via the multiplication

$$y = \mathbf{w}_{opt}^H \mathbf{x} . \tag{146}$$

This is repeated for all steering angles of interest. The angle that provides a minimum magnitude for $y = \mathbf{w}_{opt}^H \mathbf{x}$ is the DOA for the target.

Example

Consider the parameters of the previous examples, where the antenna has four phase centers, still operating with the following parameters.

$\lambda = 0.02$ m,
$D_{ant} = 0.5$ m = antenna overall aperture width. (147)

We again define clutter and target angles as

$\theta_c = 0.1\,\theta_{bw}$ = clutter angle,
$\theta_1 = -0.2\,\theta_{bw}$ = target angle, and
$CNR = 20$ dB for an individual phase center (with AWGN), and
$SNR = 20$ dB for an individual phase center (with AWGN). (148)

We will assume perfect knowledge of the covariance matrix $\mathbf{R}_{\boldsymbol{\eta\eta}}$.

Figure 6 illustrates the corresponding response to a scanned steering vector null, along with the relevant angles. Figure 7 is the same plot zoomed out and rendered with the target and clutter contributions to the composite signal, which is also plotted. We make several observations.

- The fundamental problem is that target and clutter signals are highly overlapped, so that the composite signal exhibits a single peak. This is precisely what makes effective use of a standard beamformer problematic.
- A well-defined null is almost always present, usually within 0.2 nominal beamwidths of the target angle for the parameters given.
- Accuracy improves dramatically for greater SNR target signals.
- Accuracy also improves as CNR decreases.
- There is a noticeable peak in scanned response in the vicinity of the clutter DOA in spite of the scanned null traversing the clutter signal. This is because in the vicinity of the clutter DOA, the matrix in Eq. (141) becomes increasingly ill-conditioned and the optimal weight calculation loses sensitivity to the scanned null. Stuff blows up.

We next examine placing a dedicated null onto the clutter DOA, mainly to lose sensitivity to the peak.

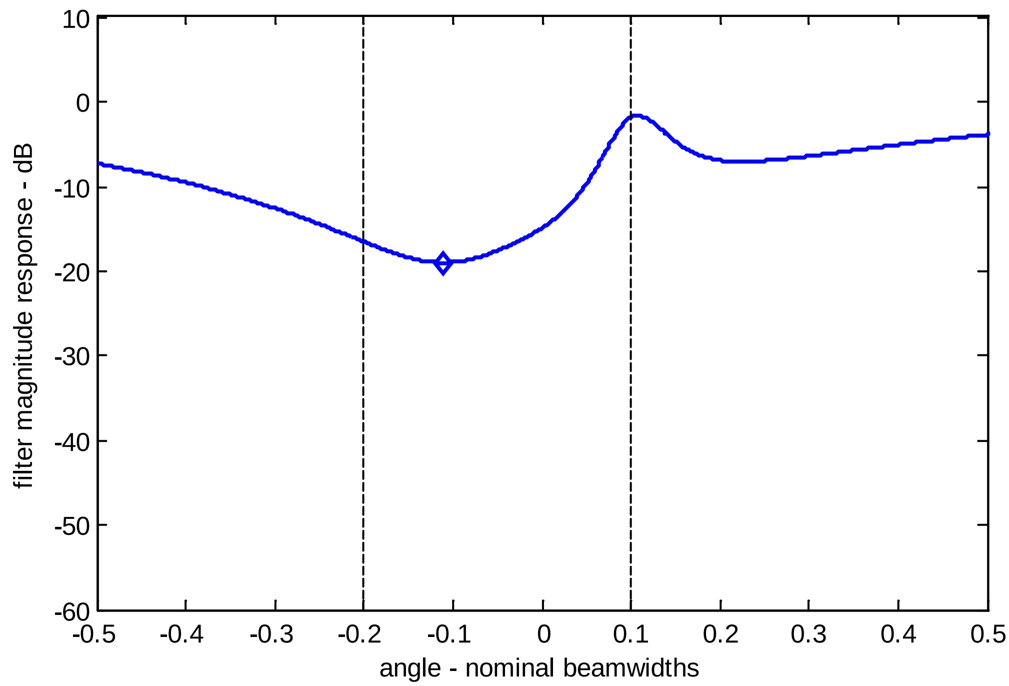

Figure 6. Response to a scanned steering vector null for four antenna phase centers based on the covariance matrix of the clutter signal in noise. The calculated minimum is located at approximately -0.11 nominal beamwidths, denoted with the marker.

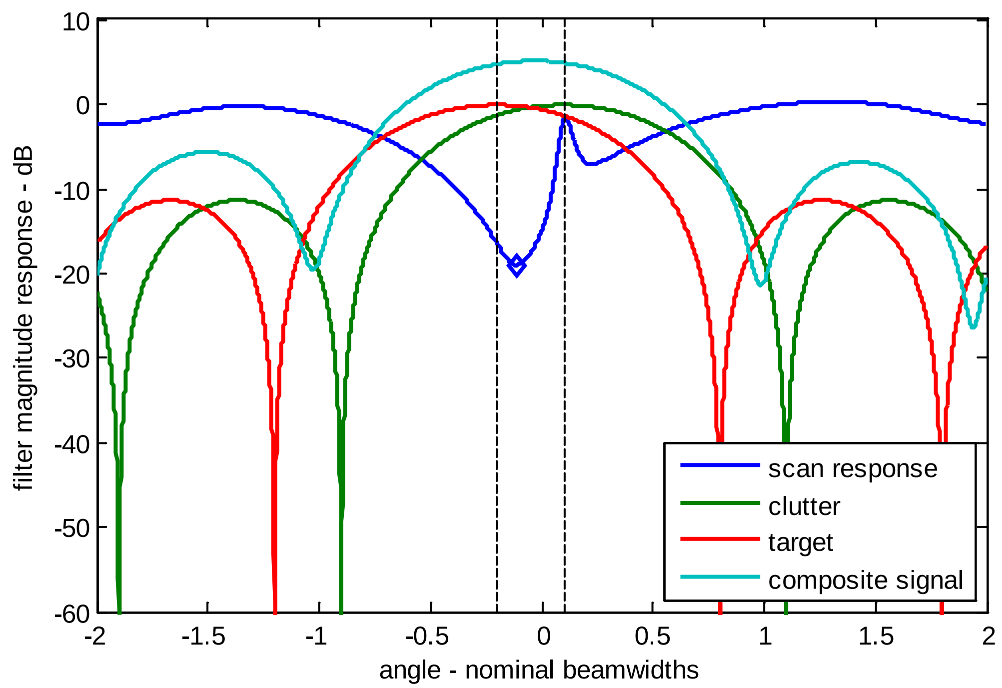

Figure 7. The same plot as Figure 6, but with additional renderings of target, clutter, and composite signal with noise.

5.2 Steering Two Nulls

For this development, we will continue to scan a null steering vector such that

$$\mathbf{w}^H \mathbf{v}_d = 0 . \tag{149}$$

However, we will also force a null in the direction of the clutter, so that

$$\mathbf{w}^H \mathbf{v}_c = 0 . \tag{150}$$

The idea continues to be to scan the null steering vector $\mathbf{v}_d$ to find a minimum in in the filtered data $y = \mathbf{w}^H \mathbf{x}$.

In this case, the Lagrange function with three constraints is identified as

$$\Lambda = \mathbf{w}^H \mathbf{R}_{\boldsymbol{\eta\eta}} \mathbf{w} + \begin{bmatrix} \alpha_d \left(\mathbf{w}^H \mathbf{v}_d - 0 \right) + \alpha_d{}^* \left(\mathbf{v}_d^H \mathbf{w} - 0 \right) \\ +\alpha_c \left(\mathbf{w}^H \mathbf{v}_c - 0 \right) + \alpha_c{}^* \left(\mathbf{v}_c^H \mathbf{w} - 0 \right) \\ +\alpha_s \left(\mathbf{w}^H \mathbf{v}_s - 1 \right) + \alpha_s{}^* \left(\mathbf{v}_s^H \mathbf{w} - 1 \right) \end{bmatrix}, \tag{151}$$

where

α_d = Lagrange multiplier associated with the null steering vector,
α_c = Lagrange multiplier associated with the clutter null steering vector, and
α_s = Lagrange multiplier associated with the scaling constraint. (152)

We may then find the optimum weight vector by taking the derivative of the Lagrange function with respect to $\mathbf{w}^H$ and setting it to zero. Doing so yields

$$\mathbf{R}_{\boldsymbol{\eta\eta}} \mathbf{w}_{opt} - \alpha_d \mathbf{v}_d - \alpha_c \mathbf{v}_c - \alpha_s \mathbf{v}_s = 0 . \tag{153}$$

This may be solved for the optimal weights as

$$\mathbf{w}_{opt} = \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \left(\alpha_d \mathbf{v}_d + \alpha_c \mathbf{v}_c + \alpha_s \mathbf{v}_s \right) = \begin{bmatrix} \alpha_d \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d \\ +\alpha_c \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_c \\ +\alpha_s \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s \end{bmatrix}. \tag{154}$$

The earlier constraints may now be written as the matrix equation

$$\begin{bmatrix} \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s\right) \\ \left(\mathbf{v}_c{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_c{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_c{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s\right) \\ \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s\right) \end{bmatrix} \begin{bmatrix} \alpha_d \\ \alpha_c \\ \alpha_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (155)$$

The solution for the Lagrange multipliers can then be found by solving as

$$\begin{bmatrix} \alpha_d \\ \alpha_c \\ \alpha_s \end{bmatrix} = \begin{bmatrix} \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s\right) \\ \left(\mathbf{v}_c{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_c{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_c{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s\right) \\ \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s\right) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (156)$$

The individual scalars can be identified accordingly.

Putting It All Together

Here we presume that the clutter DOA is known or calculated. Given measurements, we identify constraints on the optimal weights as

$$\begin{aligned} \mathbf{v}_d &= \begin{bmatrix} 1 & p_1 & \dots & p_1^{I-1} \end{bmatrix}^T = \text{target steering vector,} \\ \mathbf{v}_c &= \begin{bmatrix} 1 & p_c & \dots & p_c^{I-1} \end{bmatrix}^T = \text{clutter steering vector, and} \\ \mathbf{v}_s &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T = \text{scaling constraint vector.} \end{aligned} \quad (157)$$

We next identify the covariance matrix $\mathbf{R}_{\boldsymbol{\eta\eta}}$, via appropriate measurement or calculation, perhaps as discussed in Appendix C. For now we will simply presume that we know this a priori.

Using these, we calculate Lagrange multipliers as

$$\begin{bmatrix} \alpha_d \\ \alpha_c \\ \alpha_s \end{bmatrix} = \begin{bmatrix} \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_d{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s\right) \\ \left(\mathbf{v}_c{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_c{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_c{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s\right) \\ \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_s{}^H \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1} \mathbf{v}_s\right) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (158)$$

Optimal weights are then calculated with

$$\mathbf{w}_{opt} = \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\left(\alpha_d \mathbf{v}_d + \alpha_c \mathbf{v}_c + \alpha_s \mathbf{v}_s\right) = \begin{bmatrix} \alpha_d \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_d \\ +\alpha_c \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_c \\ +\alpha_s \mathbf{R}_{\boldsymbol{\eta\eta}}{}^{-1}\mathbf{v}_s \end{bmatrix}. \tag{159}$$

These weights are then applied to the measured data via the multiplication

$$y = \mathbf{w}_{opt}^H \mathbf{x}. \tag{160}$$

This is repeated for all steering angles of interest. The non-clutter angle that provides a minimum magnitude for $y = \mathbf{w}_{opt}^H \mathbf{x}$ is the DOA for the target.

<u>Example</u>

Consider the parameters of the previous examples, where the antenna has four phase centers, still operating with the following parameters.

$$\begin{aligned} &\lambda = 0.02 \text{ m}, \\ &D_{ant} = 0.5 \text{ m} = \text{antenna overall aperture width.} \end{aligned} \tag{161}$$

We again define clutter and target angles as

$$\begin{aligned} &\theta_c = 0.1\,\theta_{bw} = \text{clutter angle,} \\ &\theta_1 = -0.2\,\theta_{bw} = \text{target angle, and} \\ &CNR = 20 \text{ dB for an individual phase center (with AWGN), and} \\ &SNR = 20 \text{ dB for an individual phase center (with AWGN).} \end{aligned} \tag{162}$$

We will assume perfect knowledge of the covariance matrix $\mathbf{R}_{\boldsymbol{\eta\eta}}$.

Figure 8 illustrates the corresponding response to a scanned steering vector, along with the relevant angles. Figure 9 is the same plot zoomed out and rendered with the target and clutter contributions to the composite signal, which is also plotted. We make several observations.

- A well-defined null is almost always present, usually within 0.2 nominal beamwidths of the target angle for the parameters given.
- Accuracy improves dramatically for greater SNR target signals.
- The noticeable peak in Figure 6 is no longer present, having been mitigated by the forced peak at the clutter DOA.
- Not readily obvious, but nevertheless present, is an undefined point, or ‘hole’ at the clutter DOA angle. This results from the inverse in Eq. (159) not existing.

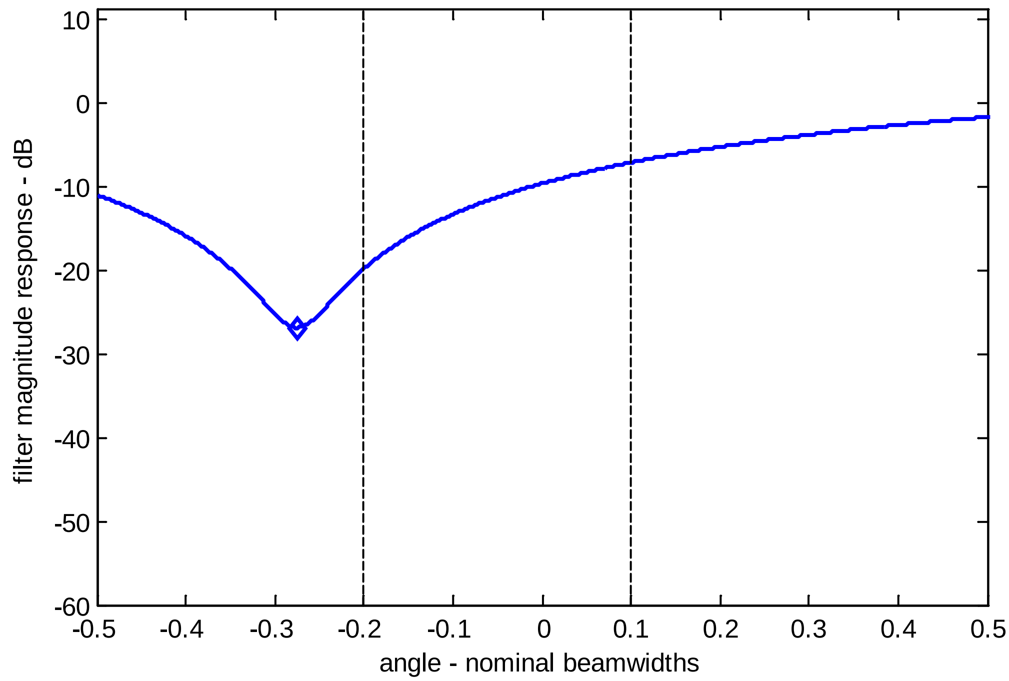

Figure 8. Response to a scanned steering vector null for four antenna phase centers based on the covariance matrix of the clutter signal in noise. The calculated minimum is located at approximately -0.274 nominal beamwidths, denoted with the marker.

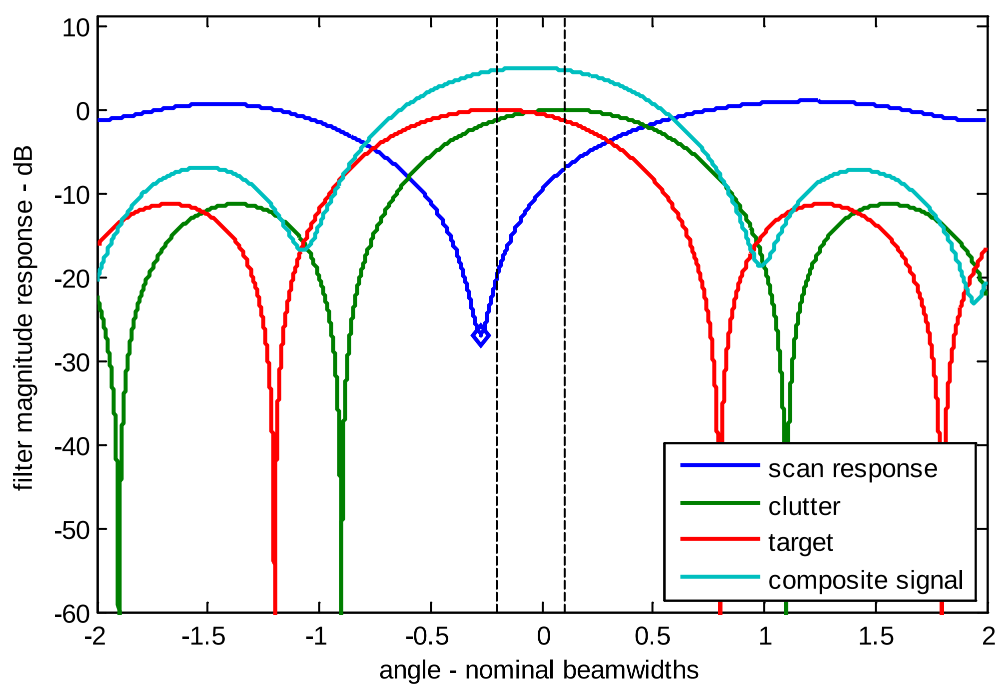

Figure 9. The same plot as Figure 8 but with additional renderings of target, clutter, and composite signal with noise.

5.3 Steering Two Nulls – Simplified Noise Model

In the development of the last section, we found weights based on both the covariance matrix with the clutter signal as well as forcing a null on top of the clutter DOA. In some sense this might be overkill. Here we assume a simplified covariance matrix that includes only the noise. That is, we require

$$\mathbf{R_{nn}} = E\left\langle \mathbf{nn}^H \right\rangle = \text{the covariance matrix of the AWGN vector } \mathbf{n}. \tag{163}$$

Using this, we now calculate Lagrange multipliers as

$$\begin{bmatrix} \alpha_d \\ \alpha_c \\ \alpha_s \end{bmatrix} = \begin{bmatrix} \left(\mathbf{v}_d{}^H \mathbf{R_{nn}}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_d{}^H \mathbf{R_{nn}}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_d{}^H \mathbf{R_{nn}}^{-1} \mathbf{v}_s\right) \\ \left(\mathbf{v}_c{}^H \mathbf{R_{nn}}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_c{}^H \mathbf{R_{nn}}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_c{}^H \mathbf{R_{nn}}^{-1} \mathbf{v}_s\right) \\ \left(\mathbf{v}_s{}^H \mathbf{R_{nn}}^{-1} \mathbf{v}_d\right) & \left(\mathbf{v}_s{}^H \mathbf{R_{nn}}^{-1} \mathbf{v}_c\right) & \left(\mathbf{v}_s{}^H \mathbf{R_{nn}}^{-1} \mathbf{v}_s\right) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{164}$$

Optimal weights are then calculated with

$$\mathbf{w}_{opt} = \mathbf{R_{nn}}^{-1}\left(\alpha_d \mathbf{v}_d + \alpha_c \mathbf{v}_c + \alpha_s \mathbf{v}_s\right) = \begin{bmatrix} \alpha_d \mathbf{R_{nn}}^{-1} \mathbf{v}_d \\ +\alpha_c \mathbf{R_{nn}}^{-1} \mathbf{v}_c \\ +\alpha_s \mathbf{R_{nn}}^{-1} \mathbf{v}_s \end{bmatrix}. \tag{165}$$

These weights are then applied to the measured data via the multiplication

$$y = \mathbf{w}_{opt}^H \mathbf{x}. \tag{166}$$

This is repeated for all steering angles of interest. The non-clutter angle that provides a minimum magnitude for $y = \mathbf{w}_{opt}^H \mathbf{x}$ is the DOA for the target.

Example

The example of the previous section is again revisited. Figure 10 shows a sample result with the covariance matrix $\mathbf{R_{nn}}$, which is for the AWGN only. Observe that it exhibits characteristics very much like the previous section where the covariance matrix $\mathbf{R}_{\boldsymbol{\eta\eta}}$ was used.

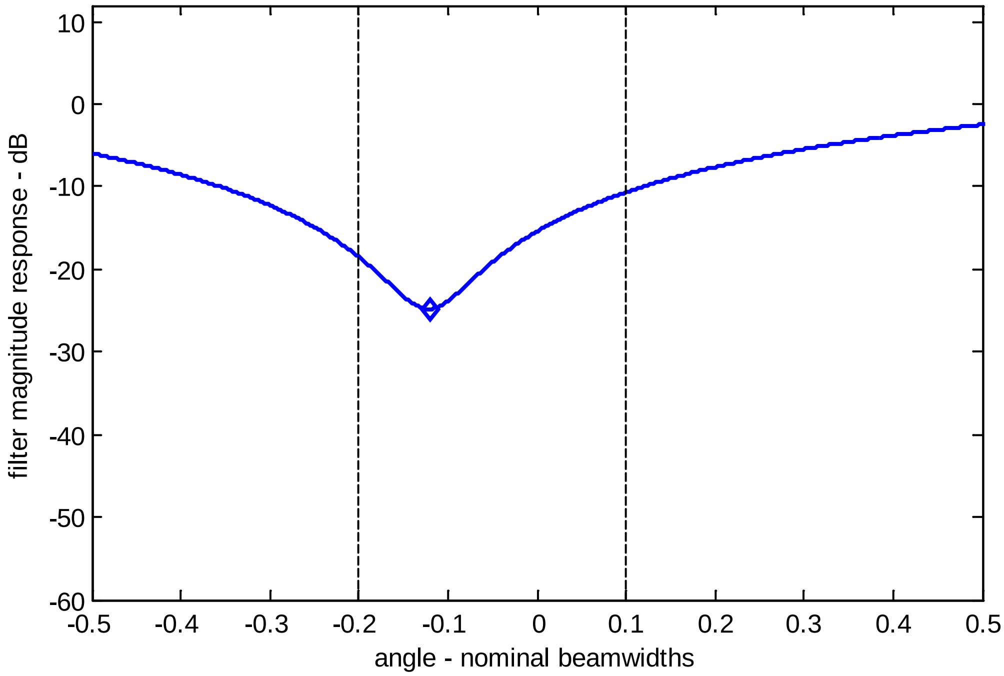

Figure 10. Response to a scanned steering vector null for four antenna phase centers based on the covariance matrix of the noise only. The calculated minimum is located at approximately -0.121 nominal beamwidths, denoted with the marker.

The attractiveness of this technique is that we might estimate the noise-only covariance matrix in parametric non-data-driven manner. The only data driven quantity required is an estimate of the clutter DOA.

6 Space-Time Adaptive Processing (STAP)

Space-Time Adaptive Processing (STAP) has emerged as a favorite processing technique primarily for detecting moving targets otherwise obscured by clutter and interference. STAP is a rich area of study with corresponding literature. An excellent book on the topic is written by Guerci.[6]

We cannot do justice to the topic in this report, so a detailed discussion of STAP is beyond the scope we will entertain here. However, since it has a strong relationship to the previous sections, we will make some high-level observations here to place the work herein in context with the STAP literature.

STAP is all about adapting the filter nulls to the undesired signals based on the data itself. It is an adaptive filter. This is what the 'A' in STAP stands for. Conventionally, the adaption part is to properly estimate the 'noise' covariance matrix from the data itself. Noise in this case includes background noise, clutter signals, and any other undesired interfering signals. The assumption that we adequately know the covariance matrix is referred to as the "clairvoyant" assumption in STAP. This estimation is normally facilitated by some crucial presumptions. Typically these include

1. The clutter must be homogeneous,
2. The clutter field must be stationary (across secondary data, e.g. range), and
3. All data must be independent and identically distributed.

Different variants of STAP tend to make different presumptions. Nevertheless, the hard part of STAP is typically estimating the covariance matrix.

A problem manifests in that these assumptions are rarely explicitly true, especially over the large number of observations required to properly estimate the covariance matrix. The consequence is that typical STAP implementations are notorious for allowing excessive false alarms, simply because the filter so generated is not after all optimal for the real data. The usual treatment is to then employ a variety of heuristic False Alarm Mitigation (FAM) techniques, with varying degrees of success, but often at the expense of a reduction in the detection probability, and detection performance overall. Another problem is nulling too much and thereby losing target detections; the so-called "over-nulling" problem.

These problems led to the development of techniques to constrain the solution so that it required fewer data samples to estimate the necessary covariance matrix, where the reduced set of observations more likely did meet the typical assumptions listed above. These techniques are generally referred to as "rank reduction" techniques, and there is much literature on various ways of doing this. Kang, et al.,[7] discuss some of these variants.

Most STAP algorithms are very “covariance-centric” in their calculations. However, we might remember that the end goal of STAP is to find a filter that passes the target signal while suppressing non-target energy, and that the covariance matrix is merely a means to that end, and not the end itself. More recently there have emerged adaptation techniques that don’t even require explicit calculation of a covariance estimate. These include techniques that have a “cascaded canceller” or Multi-Stage Weiner Filter architecture.[8,9] Some of these additionally promise improved convergence characteristics.

As a consequence of a general nagging discontent with STAP performance, at this time STAP remains an active area of research.

7 Conclusions

We summarize herein the following.

- The bulk of endo-clutter GMTI and DMTI literature concerns itself with the ‘detection’ of targets.

- While related, the ‘location’ of targets is a different problem from the detection of the same targets. A principal location parameter is the DOA of the radar target echo.

- The DOA problem can be cast as a “null steering” problem with respect to multiple antenna phase centers (or equivalent).

- N phase centers can place at most $(N-1)$ nulls. This is from basic array mathematics.

- One null (at least) is reserved for stationary clutter.

- If DOA is required for a target in the endo-clutter Doppler region, then at least three antenna phase centers (or equivalent) are required.

- Additionally nulling other interfering signals will require additional antenna phase centers (or equivalent).

- Data-driven null placement requires good knowledge of the noise+interference covariance matrix. This is hard to get.

- Good target DOA estimation also requires good SINR.

“Out of clutter, find simplicity.”
— *Albert Einstein*

Appendix A - Optimal filter

The derivation of the optimal filter can be found in numerous text books and articles. Here we present a very simplified version.

We want to find the optimum linear filter vector, $\mathbf{w}$, applied to a noisy signal vector, $\mathbf{x}$. Optimality requires a choice, for which the standard choice is to minimize the mean squared error. This optimization is typically performed subject to some constraint on the solution. There are many possible constraints that lead to a variety of interesting results. In our case, we will use the so-called "minimum distortion constraint" which leads to the following development. Accordingly, we define several relevant parameters as follows.

$$\mathbf{x} = \begin{bmatrix} x_0 & x_1 & \ldots & x_{I-1} \end{bmatrix}^T \text{ = noisy measurement vector, and}$$
$$\mathbf{w} = \begin{bmatrix} w_0 & w_1 & \ldots & w_{I-1} \end{bmatrix}^T \text{ = weight vector which combines the measurements.} \tag{167}$$

We identify the superscript '*' as the conjugate, the superscript 'T' as the transpose, and the superscript 'H' as the conjugate transpose.

We define a steering vector which identifies the direction of desired peak response as a vector of unit modulus complex exponentials, namely

$$\mathbf{v}_d \text{ = steering vector towards desired response.} \tag{168}$$

We desire a weight vector that combines the measurements in a manner to best estimate information in the direction of the steering vector. To force a non-trivial solution, our constraint is then

$$\mathbf{w}^H \mathbf{v}_d = 1 . \tag{169}$$

This best estimate occurs when $\left|\mathbf{w}^H \mathbf{x}\right|^2$ is minimized in the statistical sense. To proceed, we recognize that

$$E\left\langle \left|\mathbf{w}^H \mathbf{x}\right|^2 \right\rangle = \mathbf{w}^H E\left\langle \mathbf{x}\mathbf{x}^H \right\rangle \mathbf{w} = \mathbf{w}^H \mathbf{R}_{\mathbf{xx}} \mathbf{w} , \tag{170}$$

where

$$E\langle y \rangle \text{ = is the expected value of } y \text{, and}$$
$$\mathbf{R}_{\mathbf{xx}} = E\left\langle \mathbf{x}\,\mathbf{x}^H \right\rangle \text{ = the covariance matrix of } \mathbf{x} . \tag{171}$$

Using the method of Lagrange multipliers, we can construct the Lagrange function

$$\Lambda = \mathbf{w}^H \mathbf{R}_{\mathbf{xx}} \mathbf{w} + \lambda \left(\mathbf{w}^H \mathbf{v}_d - 1 \right) + \lambda^* \left(\mathbf{v}_d^H \mathbf{w} - 1 \right), \tag{172}$$

where to maintain consistency with the literature, we identify for this development

$$\lambda = \text{the Lagrange multiplier.} \tag{173}$$

We may then find the optimum weight vector by taking the derivative of the Lagrange function with respect to $\mathbf{w}^H$ and setting it to zero. Doing so yields

$$\mathbf{R}_{\mathbf{xx}} \mathbf{w}_{opt} - \lambda \mathbf{v}_d = 0, \tag{174}$$

where we identify the specific solution

$$\mathbf{w}_{opt} = \text{the optimum weight vector} \tag{175}$$

which may be further rearranged to solve for

$$\mathbf{w}_{opt} = \lambda \mathbf{R}_{\mathbf{xx}}^{-1} \mathbf{v}_d . \tag{176}$$

Plugging into the constraint equation yields

$$\mathbf{w}_{opt}^H \mathbf{v}_d = 1 = \lambda^* \mathbf{v}_d^H \mathbf{R}_{\mathbf{xx}}^{-1} \mathbf{v}_d . \tag{177}$$

We may then solve for the Lagrange multiplier as

$$\lambda^* = \lambda = \left(\mathbf{v}_d^H \mathbf{R}_{\mathbf{xx}}^{-1} \mathbf{v}_d \right)^{-1} . \tag{178}$$

We observe that λ is a real scalar (constant). Finally, we arrive at the result

$$\mathbf{w}_{opt}^H = \left(\mathbf{v}_d^H \mathbf{R}_{\mathbf{xx}}^{-1} \mathbf{v}_d \right)^{-1} \mathbf{v}_d^H \mathbf{R}_{\mathbf{xx}}^{-1} \tag{179}$$

Note that we have made use of several identities from complex calculus and properties of the covariance matrix, including

$$\frac{\mathrm{d}\mathbf{w}}{d\mathbf{w}^H} \text{ is undefined,}$$
$$\mathbf{R}_{\mathbf{xx}}^H = \mathbf{R}_{\mathbf{xx}}, \text{ and}$$
$$\left(\mathbf{R}_{\mathbf{xx}}^{-1} \right)^H = \mathbf{R}_{\mathbf{xx}}^{-1} . \tag{180}$$

Appendix B – Phase/Frequency Measurements in Noise

We here analyze the accuracy and precision with which we may make phase and/or frequency measurements of a single sinusoid in noise.

Consider a signal of the form

$$s_i = A_0 e^{\omega_0 i} = \text{signal samples} \tag{181}$$

where

$$\begin{aligned} &A_0 = \text{complex-valued constant,} \\ &\omega_0 = \text{sample frequency in units of radians per sample, and} \\ &i = \text{sample integer index value } (0 \le i < I). \end{aligned} \tag{182}$$

The signal is corrupted by zero-mean Additive White Gaussian Noise (AWGN), which we designate as

$$n_i = \text{noise sample.} \tag{183}$$

The noise samples are complex, and can be written as

$$n_i = n_{r,i} + j\, n_{q,i}\,, \tag{184}$$

where the constituent components $n_{r,i}$ and $n_{q,i}$ are themselves independent zero-mean Gaussian random variables, albeit individually real-valued. The respective variances are given as

$$\mathrm{Var}\left(n_{r,i}\right) = \mathrm{Var}\left(n_{q,i}\right) = \sigma^2 . \tag{185}$$

The measured values available for processing are

$$x_i = s_i + n_i\,, \tag{186}$$

We calculate the expected Signal-to-Noise Ratio (SNR) as

$$SNR = \frac{\left|A_0\right|^2}{2\sigma^2} . \tag{187}$$

For the subsequent analysis, we lose no generality by assuming

$$\begin{aligned} &A_0 = 1\text{, and} \\ &\omega_0 = 0 . \end{aligned} \tag{188}$$

We will also assume generally good SNR, that is $SNR >> 1$. Furthermore, we will be interested in the utility of linearly combining multiple samples, specifically

$$y = \sum_{i=0}^{I-1} x_i \ . \tag{189}$$

Sum of Individual Phase Measurements

Our individual measurements will exhibit phase

$$\phi_i = \operatorname{atan}\left(\frac{n_{q,i}}{1+n_{r,i}} \right) \approx n_{q,i} \ . \tag{190}$$

We may then calculate the variance in the individual samples as

$$\sigma_{\phi_i}^2 = \operatorname{Var}(\phi_i) \approx \sigma^2 . \tag{191}$$

Recall that we have *I* independent samples of phase. We calculate the sample sum of the individual phase measurements as

$$\phi_\Sigma = \sum_{i=0}^{I-1} \phi_i \approx \sum_{i=0}^{I-1} n_{q,i} \ . \tag{192}$$

The variance in the sample phase sum is then calculated as

$$\operatorname{Var}(\phi_\Sigma) \approx I\sigma^2 . \tag{193}$$

Note that this can be written as

$$\operatorname{Var}(\phi_\Sigma) \approx \frac{I}{2\ SNR} . \tag{194}$$

We observe that for a two-channel interferometer $I = 2$, and this reduces to the familiar

$$\operatorname{Var}(\phi_\Sigma) \approx (SNR)^{-1} \quad \text{for a two-channel interferometer.} \tag{195}$$

If instead of the sample phase sum, we were interested in the sample phase mean value, then the variance in the sample phase mean is given by

$$\operatorname{Var}\left(\frac{\phi_\Sigma}{I} \right) \approx \frac{\sigma^2}{I} = \frac{1}{I\ 2\ SNR} . \tag{196}$$

Phase of Sum of Samples

We expand the linear sum of sample values to

$$y = \sum_{i=0}^{I-1} \left(1 + n_{r,i}\right) + j \sum_{i=0}^{I-1} n_{q,i} \,. \tag{197}$$

The phase of this sum is given as

$$\phi_y = \operatorname{atan}\left(\frac{\sum_{i=0}^{I-1} n_{q,i}}{\sum_{i=0}^{I-1} \left(1 + n_{r,i}\right)} \right) \approx \frac{1}{I} \sum_{i=0}^{I-1} n_{q,i} \,. \tag{198}$$

The variance of this is calculated as

$$\operatorname{Var}\left(\phi_y\right) \approx \frac{\sigma^2}{I} = \frac{1}{I \; 2 \, SNR} \,. \tag{199}$$

From this, we observe that the variance in the sample sum is equal to the variance in the mean of the phase of the samples.

Frequency of Sequence of Phase Samples

A single constant frequency results in a linear change in phase across linearly spaced samples. That is, frequency is a phase slope with spaced samples. Given a set of linearly-spaced phase measurements, we may find the best slope to those samples using linear regression analysis. Given a set of linearly spaced phase measurements ϕ_i, we may calculate the best fit phase slope (frequency) as

$$\hat{\omega}_0 = \frac{\sum_{i=0}^{I-1} \left(i - \mu_i\right)\left(\phi_i - \mu_\phi\right)}{\sum_{i=0}^{I-1} \left(i - \mu_i\right)^2} \text{ radians per sample,} \tag{200}$$

where

$$\mu_i = \frac{1}{I}\sum_{i=0}^{I-1} i = \frac{I-1}{2} \text{ = mean of sample positions, and}$$

$$\mu_\phi = \frac{1}{I}\sum_{i=0}^{I-1} \phi_i \text{ = mean of phase measurements.} \tag{201}$$

We stipulate that the phase measurements need to be unwrapped prior to these calculations. We note that frequency is the derivative of phase, whose existence requires the phase to be continuous.

Note that for a two-channel interferometer where $I = 2$, this reduces to the anticipated phase measurement difference

$$\hat{\omega}_0 = \phi_1 - \phi_0 . \tag{202}$$

A frequency error will result from a net linear component to the phase perturbations due to noise with sample index number. Consequently, to determine the frequency error, we wish to find the slope of a best-fit line to phase perturbations with sample position, in this case index number.

From linear regression analysis, since we will conveniently assume without loss of generality that $\omega_0 = 0$, and that for now the noise is with respect to $A_0 = 1$, we recognize that the best fit line to the noisy phase samples has a slope calculated by

$$\omega_\varepsilon = \frac{\sum_{i=0}^{I-1}(i-\mu_i)(n_{q,i})}{\sum_{i=0}^{I-1}(i-\mu_i)^2} \text{ radians per sample.} \tag{203}$$

Recall that the noise samples are zero-mean. This slope represents the apparent frequency perturbation of the signal due to noise.

To proceed, we identify the series sum

$$\sum_{i=0}^{I-1}(i-\mu_i)^2 = \frac{I(I-1)(I+1)}{12} = \frac{I\left(I^2-1\right)}{12} . \tag{204}$$

The denominator in Eq. (204) is a constant, which we calculate and use to simplify the error slope to the expression

$$\omega_\varepsilon = \frac{12}{I\left(I^2-1\right)}\sum_{i=0}^{I-1}\left(i-\mu_i\right)\left(n_{q,i}\right). \tag{205}$$

We are interested in the variance of the error slope, and calculate it as

$$\mathrm{Var}\left(\omega_\varepsilon\right) = \mathrm{Var}\left(\frac{12}{I\left(I^2-1\right)}\sum_{i=0}^{I-1}\left(i-\mu_i\right)\left(n_{q,i}\right)\right). \tag{206}$$

For independent noise samples, this can be calculated as

$$\mathrm{Var}\left(\omega_\varepsilon\right) = \left(\frac{12}{I\left(I^2-1\right)}\right)\sigma^2. \tag{207}$$

Since $A_0 = 1$, this can be written in terms of the sample SNR as

$$\mathrm{Var}\left(\omega_\varepsilon\right) = \left(\frac{6}{I\left(I^2-1\right)}\right)\frac{1}{SNR}. \tag{208}$$

We reiterate that this is the variance in the frequency estimate from the noisy data. For example, this characterizes how much the peak of a Fourier Transform of the data would jump around from one group of *I* samples to the next.

Note that for a two-channel interferometer $I = 2$, and this again reduces to the familiar

$$\mathrm{Var}\left(\omega_\varepsilon\right) = \mathrm{Var}\left(\phi_1-\phi_0\right) = \left(SNR\right)^{-1}. \tag{209}$$

This equivalence was exploited in an earlier paper by Doerry and Bickel.[10]

We may compare the uncertainty in Eq. (209) to the nominal resolution of the data, which we calculate as

$$\rho_\omega = \frac{2\pi}{I} \text{ radians per sample.} \tag{210}$$

The variance may then be written as

$$\mathrm{Var}\left(\omega_\varepsilon\right) = \left(\frac{6I}{I^2-1}\right)\frac{{\rho_\omega}^2}{\left(2\pi\right)^2 SNR}. \tag{211}$$

Perhaps more interestingly, the standard deviation may be written as

$$\operatorname{Std}(\omega_\varepsilon) = \left(\frac{\sqrt{\frac{6I}{I^2 - 1}}}{2\pi\sqrt{SNR}} \right) \rho_\omega . \tag{212}$$

We observe that the quantity in the parenthesis that multiplies ρ_ω describes the statistical perturbation in a spectral peak in units of nominal resolution. This is the RMS error of the spectral peak.

These results are consistent with those reported by Rife and Boorstyn.[11]

Vectorized Calculations

We revisit the calculations for our frequency estimates and recognize that these may be written in a more generalized vector form including overt signal amplitude as

$$\hat{\omega}_0 = \left(\mathbf{i}^T \mathbf{i}\right)^{-1} \left(\mathbf{i}^T \mathbf{\Phi}\right), \text{ and}$$
$$\omega_\varepsilon = \left(\mathbf{i}^T \mathbf{i}\right)^{-1} \left(\frac{\mathbf{i}^T \mathbf{n}_q}{|A_0|} \right), \tag{213}$$

where we define column vectors

$$\mathbf{i} = \begin{bmatrix} i_0 & i_1 & \cdots & i_{(I-1)} \end{bmatrix}^T - \mu_i = \text{mean adjusted sample positions,}$$
$$\mathbf{\Phi} = \begin{bmatrix} \phi_0 & \phi_1 & \cdots & \phi_{(I-1)} \end{bmatrix}^T - \mu_\phi = \text{mean-adjusted phase samples, and}$$
$$\mathbf{n}_q = \begin{bmatrix} n_{q,0} & n_{q,1} & \cdots & n_{q,(I-1)} \end{bmatrix}^T = \text{sample quadrature noise samples.} \tag{214}$$

In this development, the various elements of vector $\mathbf{i}$ need not necessarily even be integer values. However, frequencies $\hat{\omega}_0$ and ω_ε are still the phase-shift per unit increase in elements of $\mathbf{i}$.

All entries are real-valued. We calculate the variance in the error frequency as

$$\operatorname{Var}(\omega_\varepsilon) = \frac{1}{|A_0|^2} E\left\langle \left(\mathbf{i}^T \mathbf{i}\right)^{-2} \left(\mathbf{i}^T \mathbf{n}_q\right) \left(\mathbf{n}_q{}^T \mathbf{i}\right) \right\rangle . \tag{215}$$

This may be expanded to

$$\mathrm{Var}(\omega_\varepsilon) = \frac{1}{|A_0|^2}\left(\mathbf{i}^T\mathbf{i}\right)^{-2}\mathbf{i}^T\mathbf{R}_{\mathbf{n}_q\mathbf{n}_q}\mathbf{i}\,, \tag{216}$$

where

$$\mathbf{R}_{\mathbf{n}_q\mathbf{n}_q} = E\left\langle \mathbf{n}_q\mathbf{n}_q{}^T\right\rangle . \tag{217}$$

Under the specific assumptions that

$$\mathbf{R}_{\mathbf{n}_q\mathbf{n}_q} = \mathbf{I}\sigma^2 \text{, and}$$
$$\mathbf{i} = \begin{bmatrix}0 & 1 & \ldots & (I-1)\end{bmatrix}^T - \mu_i\,, \tag{218}$$

we may calculate

$$\mathrm{Var}(\omega_\varepsilon) = \left(\mathbf{i}^T\mathbf{i}\right)^{-1}\frac{\sigma^2}{|A_0|^2} = \frac{\left(\mathbf{i}^T\mathbf{i}\right)^{-1}}{2SNR}\,. \tag{219}$$

As in the earlier development, this may be written as

$$\mathrm{Var}(\omega_\varepsilon) = \left(\frac{I^2}{2\left(\mathbf{i}^T\mathbf{i}\right)}\right)\frac{\rho_\omega{}^2}{(2\pi)^2\,SNR}\,. \tag{220}$$

For the stated linear increments in the elements of vector $\mathbf{i}$, this reduces to Eq. (212).

“How many things are there which I do not want.”
– Socrates

Appendix C – Estimating Covariance Matrix

We stipulate that a data vector can be conceptually parsed into the target response, and everything else. The ‘everything else’ includes clutter, other interfering signals, and AWGN.

In the absence of knowledge of the covariance matrix, which is usually the case, an estimate of the covariance matrix from the data is usually substituted. Volumes have been written on this topic alone, and it continues to be an area of active research. Here we address some relatively simple techniques that illustrate the diversity of techniques, and refer the reader to the published literature for the many variations and enhancements that might or might not offer some improved performance.

Direct Estimation

The covariance matrix is often estimated from the data as

$$\hat{\mathbf{R}}_{\boldsymbol{\eta\eta}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{\eta}_n \boldsymbol{\eta}_n^H \text{ = estimate of the covariance matrix of } \boldsymbol{\eta}. \tag{221}$$

Equation (222) is referred to as the “sample” covariance matrix. The average is performed over independent samples, *n*, all presumed to be absent any target signals. At this point, we could easily get bogged down in this discussion, and will not say much more about equation (222), other than it can be shown to be the MLE of the covariance matrix for Gaussian signals and noise, and that its characteristics are significant with respect to any derived filter.[6]

Reed, et al.,[12] show that if the data is Wide-Sense Stationary (WSS) Gaussian, then to obtain a filter that yields an SINR to be within 3 dB of optimum, we require

$$N \geq 2\, I\, J\,, \tag{222}$$

where

I = number of antenna phase centers, and
J = number of Doppler resolution cells. (223)

More is better. However, even this number can be quite a lot of samples.

Parametric Estimation

We identify that our principal interest in this report is the case of a clutter signal in AWGN. If we presume that this may be modelled as a single sinusoid in the presence of AWGN, the covariance matrix may be given by

$$\mathbf{R}_{\boldsymbol{\eta\eta},1} = \sigma_c^2 \mathbf{v}_c \mathbf{v}_c^H + \sigma_n^2 \mathbf{I} \tag{224}$$

where, we recall

$$\mathbf{v}_c = \begin{bmatrix} 1 & p_c & \dots & p_c^{I-1} \end{bmatrix}^T \text{ = clutter steering vector, with}$$

$$p_c = e^{-j\frac{2\pi b}{\lambda}\sin\theta_c}, \tag{225}$$

and corresponding power levels are

$$\sigma_c^2 = \text{ the power of the clutter signal at the } i^{\text{th}} \text{ phase center, and}$$
$$\sigma_n^2 = \text{the power in the noise at the } i^{\text{th}} \text{ phase center.} \tag{226}$$

The noise power can often be relatively accurately estimated from radar parameters.

The clutter power can be estimated from a statistical calculation of the clutter neighborhood.

The clutter DOA may be estimated by any of a variety of techniques, but all assume that the bulk of the clutter neighborhood is not contaminated by other energy, including interference, moving targets, etc. These techniques might include

1. Direct calculations of clutter DOA from Doppler and depression angle information as discussed in Appendix D, and
2. Relative DOA calculations from known (or presumed known) DOA estimates in the clutter neighborhood. The DOA estimates might be calculated as discussed in Appendix B, with relative offsets as might be calculated based on relationships discussed in Appendix D.

Any data measures will benefit from statistical calculations over the clutter neighborhood.

Appendix D – Relating Doppler to DOA

In a range-Doppler map, or image, the Doppler measurement itself will provide an estimate of DOA, quite separate from the DOA estimated from multi-aperture antenna data. We now investigate the relationship of Doppler angle and DOA angle. Bickel provides a detailed examination of this relationship in a report and accompanying paper.[13,14] Our intent here is to discuss predicting the multi-aperture antenna DOA estimate from a Doppler angle measurement.

Accordingly, we define several convenient unit vectors, namely

$\mathbf{u}_v$ = the velocity direction of the radar,
$\mathbf{u}_b$ = the baseline orientation of the multiple antenna apertures,
$\mathbf{u}_r$ = the direction of a target location, and
$\mathbf{u}_z$ = the vertical (up) direction. (227)

Each vector is a triple coordinate in 3-dimensional space. It will also be convenient to define additional direction unit vectors

$\mathbf{u}_x$ = a horizontal direction, and
$\mathbf{u}_y$ = an orthogonal horizontal direction, (228)

such that they satisfy a right-hand coordinate frame via the vector cross-product

$$\mathbf{u}_z = \mathbf{u}_x \times \mathbf{u}_y . \quad (229)$$

Where necessary, we shall define coordinates of the vectors such that

$$\begin{bmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \end{bmatrix} = \mathbf{I} . \quad (230)$$

We stipulate that neither the velocity vector nor the baseline vector is parallel to the target direction or parallel to the vertical direction.

We identify the Doppler angle as

$$\theta_{Doppler} = \operatorname{acos}\left(\mathbf{u}_r \cdot \mathbf{u}_v\right) \text{ = the Doppler angle.} \quad (231)$$

Note that the set of all target directions with the same constant Doppler angle describes a cone about the velocity vector.

We identify the multi-aperture antenna DOA angle as

$$\theta_{DOA} = \operatorname{acos}\left(\mathbf{u}_r \cdot \mathbf{u}_b\right) \text{ = the DOA angle.} \quad (232)$$

Note that the set of all target directions with the same constant DOA angle describes a cone about the baseline vector.

It will also be convenient later to identify a depression angle, which we identify as with respect to zenith as

$$\theta_z = \mathrm{acos}\left(\mathbf{u}_r \cdot \mathbf{u}_z\right) \text{ = the depression angle with respect to zenith.} \tag{233}$$

<u>Case 1. Baseline Oriented with Velocity</u>

In this case, the baseline and velocity are aligned.

If we assume that the unit vectors are coincident, that is

$$\mathbf{u}_b = \mathbf{u}_v , \tag{234}$$

then the DOA and Doppler angles are the same, that is

$$\theta_{DOA} = \theta_{Doppler} . \tag{235}$$

Furthermore, this is true for all target location vector directions and all ranges.

A corollary to this is the case where the unit vectors are in antipodal directions, that is

$$\mathbf{u}_b = -\mathbf{u}_v , \tag{236}$$

In which case the DOA and Doppler angles are supplementary, that is

$$\theta_{DOA} + \theta_{Doppler} = \pi . \tag{237}$$

This is also true for all target location vector directions and all ranges.

Either of these conditions can be succinctly described by

$$\left|\mathbf{u}_b \cdot \mathbf{u}_v\right| = 1 . \tag{238}$$

Nevertheless, with these geometries we can relate θ_{DOA} directly to $\theta_{Doppler}$ without explicitly knowing the target direction $\mathbf{u}_r$, or even the depression angle θ_z.

This alignment of the baseline with the velocity vector is common for many Active Electronically Steered Array (AESA) antenna systems that are flown on larger aircraft. This is certainly an attractive feature of AESA systems, although we acknowledge that AESA technology is not without its own issues.

Case 2. Baseline Oriented Differently from Velocity

This is the case where the baseline is oriented in a direction differently from the velocity vector with at least some orthogonal component, for which we observe

$$\left|\mathbf{u}_b \cdot \mathbf{u}_v\right| < 1. \tag{239}$$

We seek now to know how θ_{DOA} is related to $\theta_{Doppler}$, and what influences this relationship.

We may reason that if we knew $\mathbf{u}_r$ then we could calculate either θ_{DOA} or $\theta_{Doppler}$ exactly. We will presume to know $\theta_{Doppler}$, and are interested in predicting θ_{DOA}. Consequently, we will wish to learn $\mathbf{u}_r$ from knowledge of $\theta_{Doppler}$. However, to calculate a direction in 3-dimensional space, we need at least two non-coplanar angles. We stipulate that the second angle for which we presume knowledge is the depression angle θ_z. We defer for the moment how we know this angle. Consequently, we presume to know the following sufficient information

$$\begin{aligned} &\mathbf{u}_v^T \mathbf{u}_r = \cos\theta_{Doppler}, \\ &\mathbf{u}_z^T \mathbf{u}_r = \cos\theta_z, \text{ and} \\ &\mathbf{u}_r^T \mathbf{u}_r = 1. \end{aligned} \tag{240}$$

We note that this is not a straightforward linear set of equations. We may nevertheless solve this system of equations by any of several techniques. Here, we choose to first identify vector components as

$$\begin{aligned} &\mathbf{u}_v = \begin{bmatrix} u_{v,x} & u_{v,y} & u_{v,z} \end{bmatrix}^T, \\ &\mathbf{u}_r = \begin{bmatrix} u_{r,x} & u_{r,y} & u_{r,z} \end{bmatrix}^T, \text{ and} \\ &\mathbf{u}_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T. \end{aligned} \tag{241}$$

We will refer to the individual components in order as the *x*-, *y*-, and *z*- components, and the respective component directions as the *x*-, *y*-, and *z*- directions.

Without loss of generality, but for convenience, we shall presume that the *x*-direction is aligned with the horizontal projection of the velocity vector, that is

$$u_{v,y} = 0. \tag{242}$$

Other coordinate frames may be easily rotated to and from this one.

The constraint equations may then be manipulated to

$$u_{r,z} = \cos\theta_z \,,$$
$$u_{r,x} = \frac{\cos\theta_{Doppler} - u_{v,z}\cos\theta_z}{u_{v,x}} \text{, and}$$
$$u_{r,y}{}^2 = 1 - \cos^2\theta_z - \left(\frac{\cos\theta_{Doppler} - u_{v,z}\cos\theta_z}{u_{v,x}}\right)^2 . \quad (243)$$

The inherent ambiguity in $u_{r,y}$ may be solved by simply identifying whether the target direction is to the left or right of the velocity vector as projected to the horizontal plane. Consequently, the direction vector to the target is then identified as

$$\mathbf{u}_r = \begin{bmatrix} \left(\dfrac{\cos\theta_{Doppler} - u_{v,z}\cos\theta_z}{u_{v,x}}\right) \\ \pm\sqrt{1 - \cos^2\theta_z - \left(\dfrac{\cos\theta_{Doppler} - u_{v,z}\cos\theta_z}{u_{v,x}}\right)^2} \\ \cos\theta_z \end{bmatrix} . \quad (244)$$

We finally identify the multi-aperture antenna DOA angle as

$$\theta_{DOA} = \operatorname{acos}\left(\mathbf{u}_r{}^T \mathbf{u}_b\right) . \quad (245)$$

A remaining issue is estimating the depression angle θ_z, which we recall that we have identified as the angle with respect to zenith. But first we make the following observations.

- The depression angle θ_z generally influences all three components of the target direction vector $\mathbf{u}_r$.
- Even for perfectly horizontal velocity vectors where $u_{v,z} = 0$, we still see the *y*- and *z*- components of $\mathbf{u}_r$ depending on θ_z.

- An error in measuring or otherwise estimating depression angle θ_z will cause an error in the calculation of target direction vector $\mathbf{u}_r$. This is the 'layover' phenomenon in range-Doppler images.
- If we by some means knew a priori the DOA angle θ_{DOA}, then an error in the target direction vector $\mathbf{u}_r$ is not separable from an error in the baseline vector $\mathbf{u}_b$, at least from a single observation.

Finding/Measuring the Depression Angle

One technique to determine the depression angle θ_z might be to measure it directly, as with interferometric techniques. Such techniques are the basis for Interferometric SAR (IFSAR, or InSAR). Depending on the accuracy and precision desired, a system will need to be mindful of

1. Radome refraction effects,
2. Atmospheric refraction effects, and/or
3. Multipath-induced perturbations in elevation DOA measures.

An alternate technique to determine the depression angle θ_z might be to calculate it based on trigonometric principles using knowledge of the target scene topography.

Assuming a flat earth for simplicity, we may calculate

$$\cos\theta_z = \frac{-(h_a - h_t)}{R}, \tag{246}$$

where

$$\begin{aligned} R &= \text{slant range to target,} \\ h_a &= \text{height of radar, and} \\ h_t &= \text{height of target.} \end{aligned} \tag{247}$$

It is straightforward to develop equations for a curved earth. Nevertheless, it is reasonably easy for the radar to know the slant-range to the target, and to know the radar's own height or altitude. The difficulty is typically for the radar to know the target's height or altitude. Of course, for a GMTI application we might reasonably assume that the target is on the ground, and auxiliary data on ground topography can often provide this information. Nevertheless, depending on the accuracy and precision desired, a system will need to be mindful of

1. The accuracy and precision of the terrain elevation data that is available, and/or
2. The assumption that the moving target is on the ground may not always be accurate.

Relative Doppler Related to Relative DOA

An interesting question is "If we know the DOA at some location, how well can we know the DOA at another location nearby?" The term "nearby" implies some small Doppler angle offset, or some small depression angle offset perhaps due to range.

We investigate the Doppler angle sensitivity by calculating

$$\frac{d\theta_{DOA}}{d\theta_{Doppler}} = \left(\frac{u_{b,x}}{u_{v,x}} - \frac{\frac{u_{b,y}}{u_{v,x}}\left(\frac{\cos\theta_{Doppler} - u_{v,z}\cos\theta_z}{u_{v,x}}\right)}{\sqrt{1-\cos^2\theta_z - \left(\frac{\cos\theta_{Doppler} - u_{v,z}\cos\theta_z}{u_{v,x}}\right)^2}} \right) \frac{\sin\theta_{Doppler}}{\sin\theta_{DOA}} . \tag{248}$$

We investigate the depression angle sensitivity by calculating

$$\frac{d\theta_{DOA}}{d\theta_z} = \left(u_{b,z} - \frac{u_{b,x}u_{v,z}}{u_{v,x}} - \frac{u_{b,y}\left(\cos\theta_z - \frac{u_{v,z}}{u_{v,x}}\left(\frac{\cos\theta_{Doppler} - u_{v,z}\cos\theta_z}{u_{v,x}}\right)\right)}{\sqrt{1-\cos^2\theta_z - \left(\frac{\cos\theta_{Doppler} - u_{v,z}\cos\theta_z}{u_{v,x}}\right)^2}} \right) \frac{\sin\theta_z}{\sin\theta_{DOA}} . \tag{249}$$

In typical gimbaled antenna GMTI operation,

$\left|\sin\theta_{DOA}\right| \approx 1$, for antenna beams near broadside to baseline vector,
$u_{b,z} \approx 0$, for nearly horizontal baseline, and
$u_{v,z} \approx 0$, for nearly horizontal radar velocity. (250)

In addition, both $\sin\theta_z$ and $\sin\theta_{Doppler}$ are typically significantly away from zero. With this collection of conditions, the derivatives are smooth and fairly steady, meaning that we may expect fairly linear behavior of the DOA angle in a neighborhood of Doppler angle and depression angle. This suggests that for smooth topography, DOA is highly predictable based on small Doppler and depression angle offsets.

Clutter Ridge

Quite common in GMTI analysis is to plot Doppler angle versus DOA angle for a constant range. This relationship is often called a "clutter ridge."

Example

We illustrate the foregoing principles with an example. We will presume the following parameters.

$$h_a = 5000 \text{ m},$$
$$\mathbf{u}_v = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \text{ and}$$
$$\mathbf{u}_b = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}^T. \tag{251}$$

In addition, we identify a plan view of target locations in Figure 11. We will also assume that the target locations exhibit a topography as illustrated in Figure 12, where peak deviations from nominal are ±100 m. We observe that the radar height corresponds to a range of 10 km and depression angle of 30 degrees to the center of the target scene.

If no allowance is made for the topography, and a uniform target scene height of zero is assumed, then the DOA estimate error is identified and plotted in Figure 13. If the depression angle is in fact corrected for topography, then the DOA estimate error is identified and plotted in Figure 14. These plots show that not accounting for target scene topography will result in a DOA estimate error.

Figure 15 and Figure 16 illustrate how Doppler and depression angles are relatively linear with DOA angle. Slight offsets in Doppler or depression angle allow fairly straightforward prediction of DOA offsets. Typically, with gimbaled antenna systems, we are interested in only a few degrees span centered around a DOA angle of 90 degrees.

Figure 17 illustrates the clutter ridges at various slant ranges.

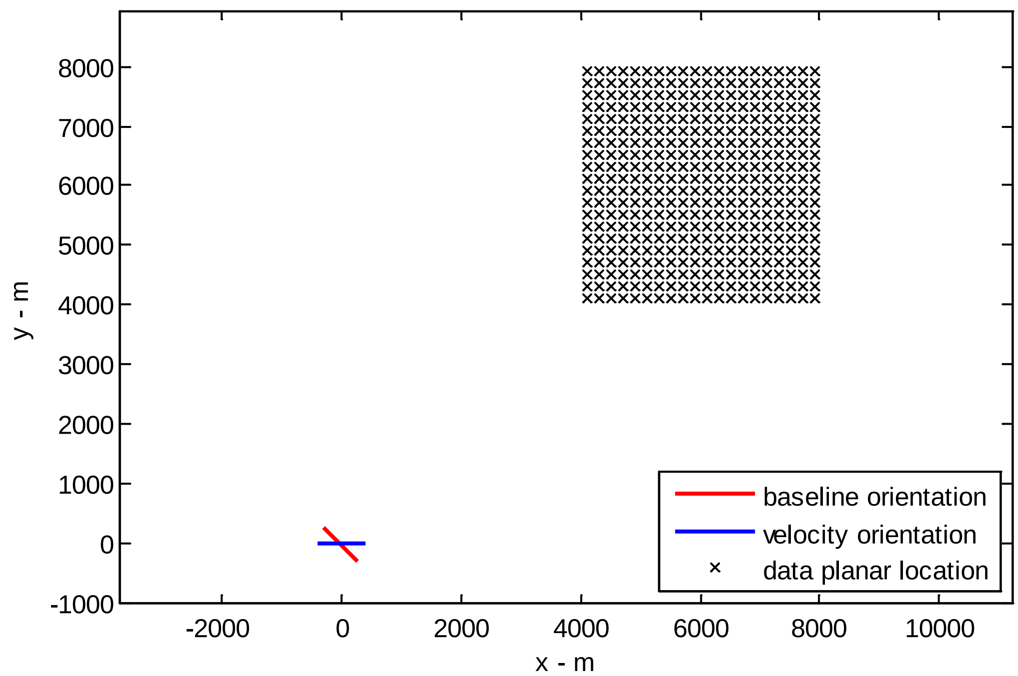

Figure 11. Plan view of sample locations and orientations of velocity vector and baseline vector.

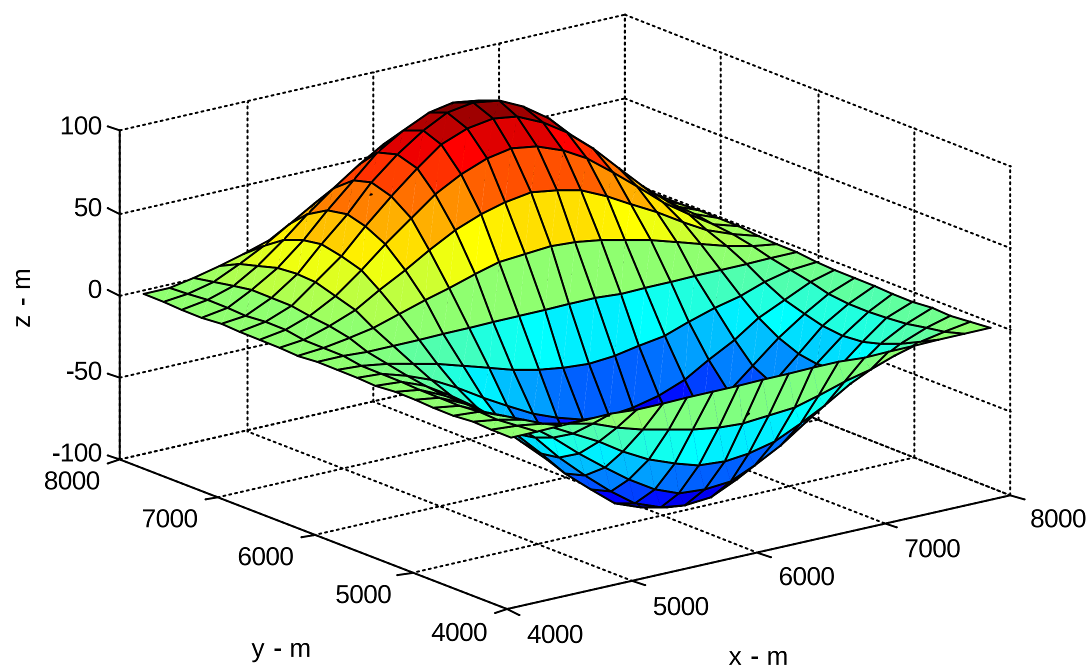

Figure 12. Simulated topography of sample locations.

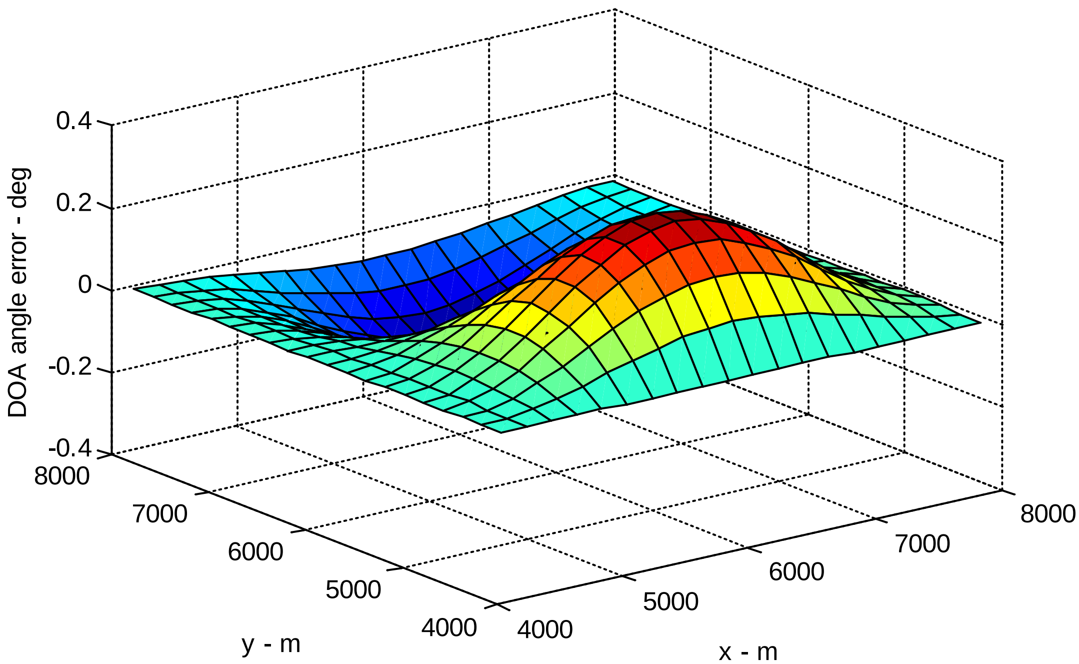

Figure 13. DOA angle error assuming nominal target height.

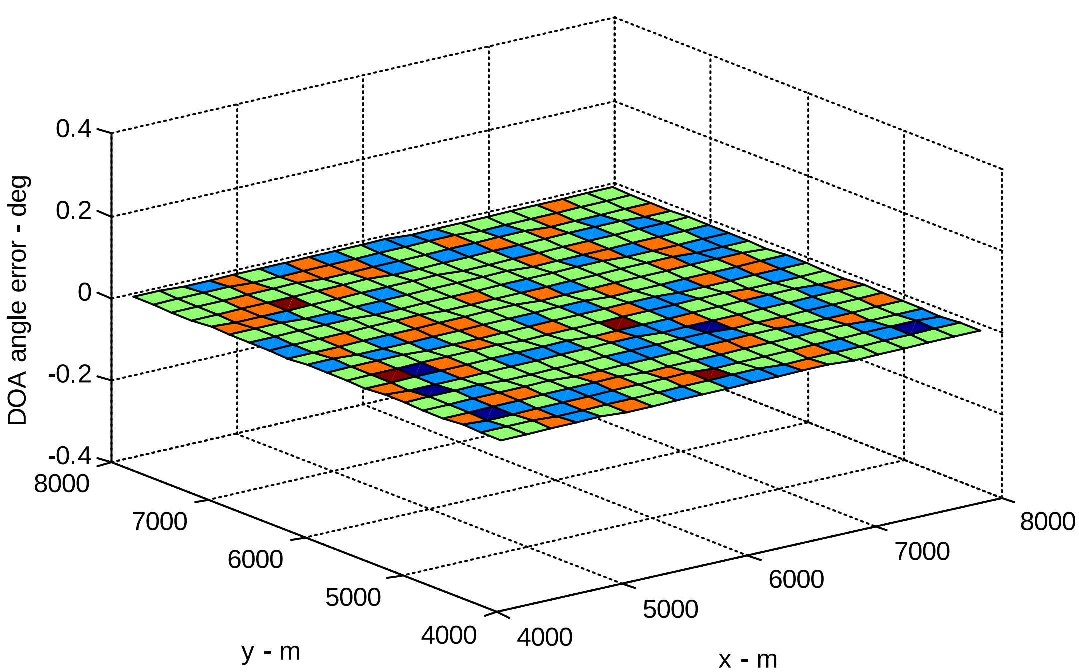

Figure 14. DOA angle error assuming accurate target location height.

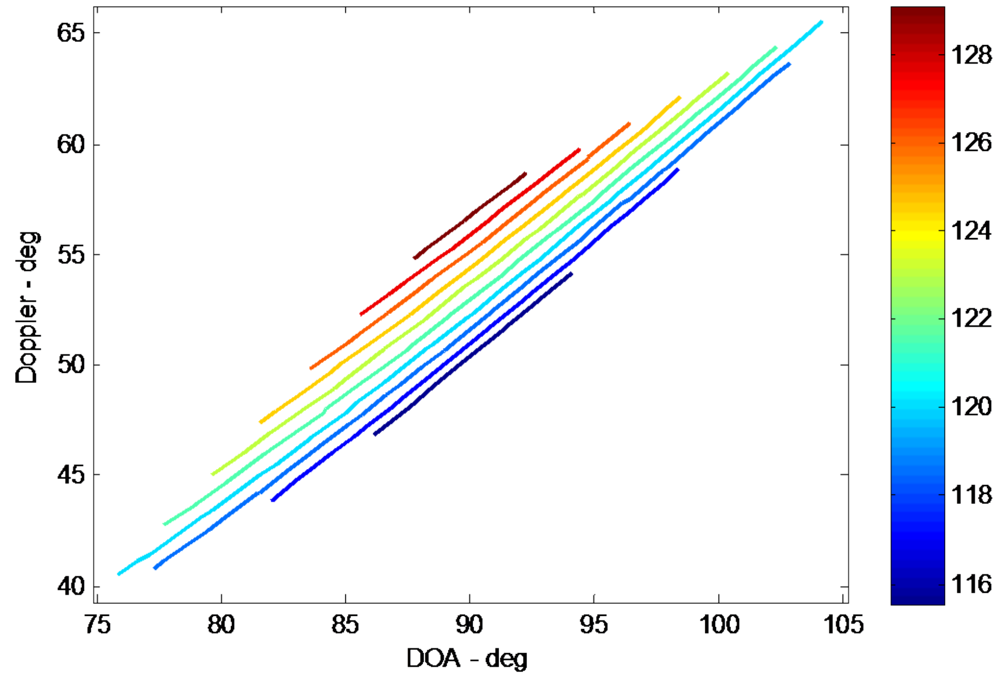

Figure 15. Relationship of Doppler angle to true DOA angle for various depression angles. All angle measures are in degrees.

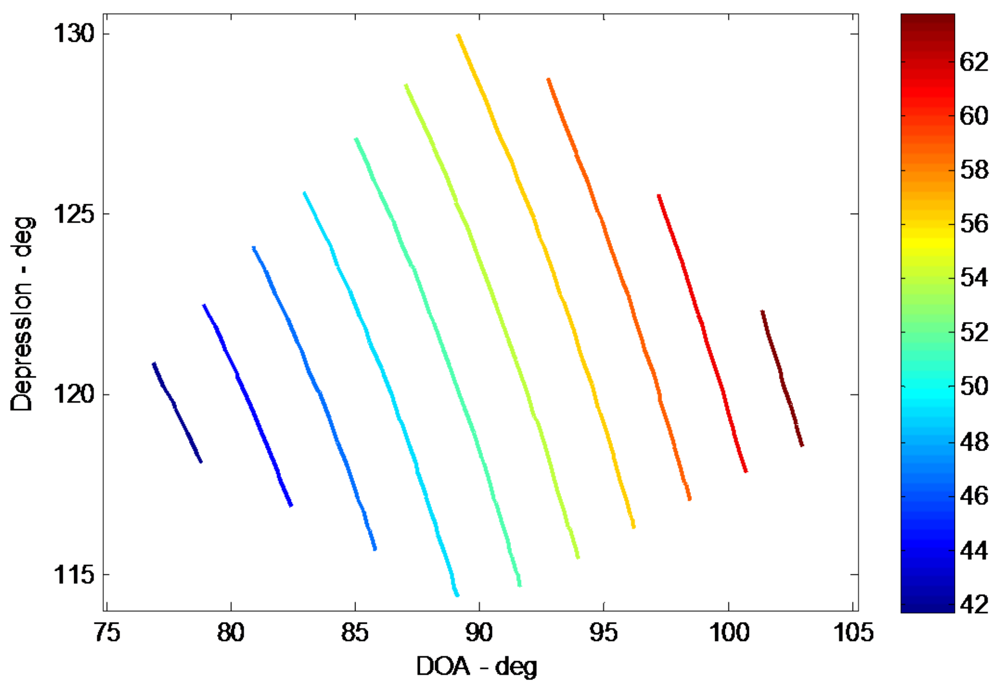

Figure 16. Relationship of depression angle to true DOA angle for various Doppler angles. All angle measures are in degrees.

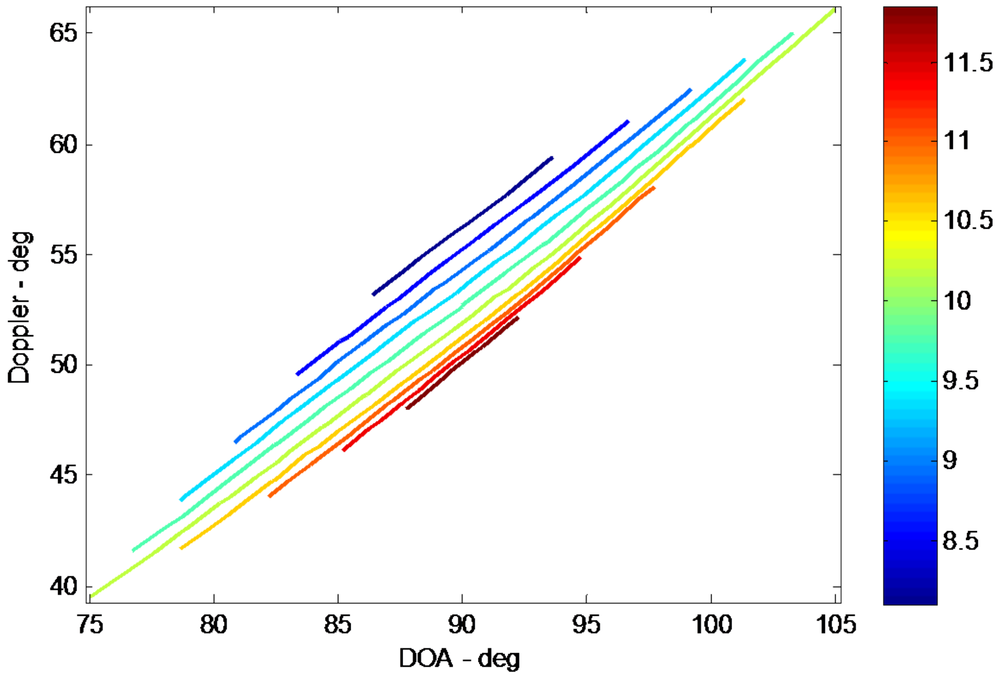

Figure 17. Clutter ridges at various ranges given in km.

"It is the set of the sails, not the direction of the wind that determines which way we will go."
-- Jim Rohn

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