On Neutral Functional–Differential Equations with Proportional Delays

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In this paper we develop a comprehensive theory on the well-posedness of the initial-value problem for the neutral functional-differential equation

\[ y'(t) = ay(t) + \sum_{i=1}^{\infty} b_i y(q_i t) + \sum_{i=1}^{\infty} c_i y'(p_i t), \quad t > 0, \quad y(0) = y_0, \]

and the asymptotic behaviour of its solutions. We prove that the existence and uniqueness of solutions depend mainly on the coefficients \( c_i, i = 1, 2, \ldots, \) and on the smoothness of functions in the solution space. As far as the asymptotic behaviour of analytic solutions is concerned, the \( c_i \) have little effect. We prove that if \( \text{Re} \, a > 0 \) then the solution \( y(t) \) either grows exponentially or is polynomial. The most interesting result is that if \( \text{Re} \, a \leq 0 \) and \( a \neq 0 \) then the asymptotic behaviour of the solution depends mainly on the characteristic equation

\[ a + \sum_{i=1}^{\infty} b_i q_i^a = 0. \]

These results can be generalized to systems of equations. Finally, we present some examples to illustrate the change of asymptotic behaviour in response to the variation of some parameters. The main idea used in this paper is to express the solution in either Dirichlet or Dirichlet–Taylor series form.

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1. INTRODUCTION

Initial-value problems for functional-differential equations of constant time delays of the form

\[
\frac{d}{dt} \left( y(t) - \sum_{i=1}^{\infty} c_i y(t - p_i) \right) = ay(t) + \sum_{i=1}^{\infty} b_i y(t - q_i), \quad t > 0, \quad (1.1)
\]

\[
y(t) = y_0(t), \quad x \leq 0, \quad (1.2)
\]

where \(a, b_i, c_i\) are complex constants, \(p_i, q_i > 0, i = 1, 2, \ldots\), satisfying \(\sup_{i \geq 1} p_i, \sup_{i \geq 1} q_i < \infty\), and \(y_0(t)\) is a given function, have been studied extensively. It is well known that (1.1)–(1.2), under some minor restrictions, has a unique solution and the asymptotic behaviour of this solution depends mainly on the corresponding characteristic equation

\[
\lambda \left( 1 - \sum_{i=1}^{\infty} c_i \exp(-\lambda p_i) \right) = a + \sum_{i=1}^{\infty} b_i \exp(-\lambda q_i).
\]

Details for this and for more general results can be found in Hale [11]. In this paper we study the initial-value problem for neutral functional-differential equations of the form

\[
y'(t) = ay(t) + \sum_{i=1}^{\infty} b_i y(q_i t) + \sum_{i=1}^{\infty} c_i y'(p_i t), \quad t > 0, \quad (1.3)
\]

\[
y(0) = y_0, \quad (1.4)
\]

where \(a, b_i, c_i\) are complex constants, \(p_i, q_i \in (0, 1), i = 1, 2, \ldots\), and \(y_0\) is a given initial value. To guarantee convergence of the series in (1.3), we assume that \(\sum_{i=1}^{\infty} |b_i|, \sum_{i=1}^{\infty} |c_i| < \infty\). A list of applications for this kind of equation features in Iserles [12]. One remarkable difference between (1.1) and (1.3) is that the latter has unbounded variable time delays.

The objective of this paper is to develop a fairly comprehensive theory on the well-posedness of the initial-value problem (1.3)–(1.4) and, most importantly of all, the asymptotic behaviour of its solutions. We prove that the existence and uniqueness of solutions of (1.3)–(1.4) depend mainly on the coefficients \(c_i, i = 1, 2, \ldots\), and the smoothness of functions in the solution space. However, the \(c_i\) have little effect on the asymptotic behaviour of analytic solutions of (1.3)–(1.4). We prove that if \(\Re a > 0\) then the solution \(y(t)\) either grows exponentially, i.e., \(\lim_{t \to \infty} y(t)e^{-at} \neq 0\), or is polynomial. Our most interesting result is that if \(\Re a \leq 0\) and \(a \neq 0\) then the asymptotic behaviour of the solution depends mainly on the characteristic equation

\[
a + \sum_{i=1}^{\infty} b_i q_i^{-\lambda} = 0.
\]
Our results can be generalized to systems of functional-differential equations with proportional time delays. Finally, we specialize our discussion by paying closer attention to the generalized pantograph equation [12]

\[ y'(t) = ay(t) + by(qt) + cy'(qt), \quad t > 0, \quad y(0) = 1. \quad (1.5) \]

We present some examples that illustrate the change of asymptotic behaviour in response to the variation of some parameters.

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Analogous to the existence and uniqueness theorem of Nussbaum [22], we have the following result (see also Iserles and Liu [14] and [17]).

**Theorem 1.** If \( \sum_{i=1}^{\infty} p_i^n c_i < 1 \) for some \( m \in \mathbb{Z}^+ \) then in the function space \( C^{m+1}[0, \infty) \) the solution of (1.3)–(1.4) exists and is unique if and only if

\[ \sum_{i=1}^{\infty} p_i^n c_i \neq 1, \quad n = 0, 1, \ldots, m - 1. \quad (2.1) \]

If (2.1) holds then the unique solution is analytic and can be written as

\[ y(t) = y_0 + y_0 \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} \left( 1 - \sum_{i=1}^{\infty} p_i^k c_i \right)^{-1} \left( a + \sum_{i=1}^{\infty} q_i^k b_i \right) \frac{t^n}{n!}. \quad (2.2) \]

Kuang and Feldstein [20] studied the scalar initial-value problem

\[ y'(t) = ay(t) + \sum_{i=1}^{M} b_i y(q_i t) + \sum_{i=1}^{K} c_i y'(p_i t), \quad t > 0, \quad y(0) = y_0, \quad (2.3) \]

where \( a, b_i, i = 1, \ldots, M, \) and \( c_i, i = 1, \ldots, K, \) are real constants. By transforming (2.3) into an integral equation, they proved that it has a unique solution (in the function space \( C^1[0, \infty) \)) if \( \sum_{i=1}^{K} |c_i p_i^{-1}| < 1 \). They then posed the problem of establishing the same result in the case of \( \sum_{i=1}^{K} |c_i p_i^{-1}| \geq 1 \). Our result shows that the condition \( \sum_{i=1}^{K} |c_i p_i^{-1}| < 1 \) can be modified to \( \sum_{i=1}^{K} |c_i| < 1 \).

Next we discuss the necessity of the assumption \( \sum_{i=1}^{\infty} p_i^n c_i < 1 \) made in Theorem 1. For simplicity, we only consider the case \( m = 0 \). First, we introduce the following lemma, which is almost the same as a result of Nussbaum [22].
Lemma 2. Let $p \in (0,1)$ be given and consider the homogeneous functional equation

$$y(t) = cy(pt), \quad t \geq 0,$$

in the function space $C[0, \infty)$.

1. If $|c| < 1$ but $c \neq 1$ then (2.4) has only the trivial solution $y(t) = 0$.
2. If $c = 1$ then all solutions of (2.4) are constant.
3. If $|c| > 1$ then there is a one-to-one correspondence between solutions of (2.4) and functions in the space $C^* = \{ f(t) \in C[0,1]; f(1) = cf(p) \}$.

Proof. The first two parts of Lemma 2 are easy to prove. Consider part 3. For every solution $y(t) \in C[0, \infty)$ of (2.4), there corresponds a function $f(t) = y(t)$, $t \in [0,1]$, which belongs to $C^*$. On the other hand, for every function $f(t) \in C^*$, it is easy to see that the function $y(t)$ given by the following recurrence relation:

$$y(0) = 0,$$
$$y(t) = c^k f(p^{-k}t), \quad p^{k+1} \leq t \leq p^k, \quad k = 0, \pm 1, \pm 2, \ldots,$$

is a continuous solution of (2.4).

Example 1. Consider solutions of the initial-value problem

$$y'(t) = cy'(pt), \quad t > 0, \quad y(0) = y_0,$$

in the function space $C^1[0, \infty)$. It follows from Lemma 2 that

1. The only solution of (2.5) is $y(t) = y_0$ if $|c| \leq 1$ but $c \neq 1$.
2. All the solutions of (2.5) can be written as $y(t) = y_0 + y_1 t$ if $c = 1$, where $y_1$ is an arbitrary constant.
3. There is a one-to-one correspondence between solutions of (2.5) and functions in the space $C^*$ if $|c| > 1$.

This example shows that the initial-value problem (2.5) loses uniqueness of solutions if either $c = 1$ or $|c| > 1$.

Example 2. Consider the homogeneous problem

$$y'(t) = \sum_{i=1}^{\infty} c_i y'(p_it), \quad t > 0, \quad y(0) = 0.$$

Suppose that $\sum_{i=1}^{\infty} c_i > 1$. By the intermediate-value theorem there exists a constant $\lambda > 1$ such that $\sum_{i=1}^{\infty} p_i^\lambda c_i = 1$. Hence, the homogeneous problem (2.6) has the nontrivial solution $y(t) = t^{\lambda+1}$. 
Example 3. Consider solutions of the initial-value problem
\[ y'(t) = c_1 y'(p_2 t) + c_2 y''(p_2 t) - c_1 c_2 y'(p_1 p_2 t), \quad t > 0, \]
\[ y(0) = y_0, \]
(2.7)
in the function space \( C^[[0, \infty]] \). Suppose that \( |c_1| \leq 1 \) and \( |c_2| \leq 1 \) but \( c_1, c_2 \neq 1 \). Let \( y(t) \in C^[[0, \infty]] \) be a solution of (2.7). Then \( x(t) = y'(t) - c_1 y'(p_1 t) \) satisfies the functional equation
\[ x(t) = c_2 x(p_2 t), \quad t \geq 0. \]
Lemma 2 implies that \( x(t) = 0 \). Therefore,
\[ y'(t) = c_1 y'(p_2 t), \quad t \geq 0, \quad y(0) = y_0. \]
It follows from Lemma 2 that \( y(t) = 0 \). Hence, the only \( C^[[0, \infty]] \) solution of (2.7) is \( y(t) = y_0 \). This example shows that the condition \( |c_1| + |c_2| + |c_1 c_2| < 1 \) is unnecessary for the uniqueness of solution of (2.7) in the function space \( C^[[0, \infty]] \).

In the rest of this paper, we assume that \( a \neq 0 \), that (2.1) holds, and that \( \lambda_+ = \sup_{t \geq 1} \max\{p_i, q_i\} < 1 \). We shall only consider the analytic solution of (1.3)-(1.4).

3. Dirichlet and Dirichlet–Taylor Series Solutions

The Taylor series solution (2.2) is of little use in the study of asymptotic behaviour. On the other hand, it seems that the solution cannot be found through Laplace transform as was done in the case of (1.1)-(1.2). Fortunately, it can be expressed in either Dirichlet or Dirichlet–Taylor series form. To begin with, we introduce some notation. For sequences \( (x_i)_{i=1}^{\infty} \) of complex numbers and \( (j_i)_{i=1}^{\infty} \) of integers, we define
\[ x = (x_1, x_2, \ldots) \in \mathbb{C}^\infty, \quad y = (y_1, y_2, \ldots) \in \mathbb{C}^\infty, \]
\[ j = (j_1, j_2, \ldots) \in \mathbb{Z}^\infty, \]
and the standard multi-index operations
\[ |x| = \sum_{i=1}^{\infty} |x_i|, \quad |j| = \sum_{i=1}^{\infty} |j_i|, \quad x^j = \prod_{i=1}^{\infty} x_i^{j_i}, \]
\[ (x, y) = \sum_{i=1}^{\infty} x_i y_i, \]
\[ x y = (x_1 y_1, x_2 y_2, \ldots), \quad x^k = (x_1^k, x_2^k, \ldots), \quad k \in \mathbb{Z}^+. \]
First, we seek Dirichlet series solution of the form
\[ y(t) = \alpha y_0 \sum_{n=0}^{\infty} \sum_{\|j+k\|=n} d_{j,k} \exp(atq^k), \tag{3.1} \]
where \( \alpha \) and \( d_{j,k}, j, k \in (\mathbb{Z}^+)^n \), are coefficients to be determined. It is understood that \( p_i = 1, i = 1 \) if \( c_i = 0 \) \((b_i = 0)\) for some \( i \in \mathbb{N} \). Let \( d_{a,0} = 1 \), where \( 0 = (0,0,\ldots) \), and \( d_{j,k} = 0 \) if \( j_i < 0 \) or \( k_i < 0 \) for some \( i \in \mathbb{N} \). It is easy to see that
\[ y'(t) = a \alpha \sum_{n=0}^{\infty} \sum_{\|j+k\|=n} q^j d_{j,k} \exp(atq^k), \]
and for \( i \in \mathbb{N} \) that
\[ y(q_i,t) = \alpha y_0 \sum_{n=0}^{\infty} \sum_{\|j+k\|=n} d_{j,-e_i,k} \exp(atq^k), \]
\[ y'(p_i,t) = a \alpha y_0 \sum_{n=0}^{\infty} \sum_{\|j+k\|=n} q^j d_{j,-e_i,k} \exp(atq^k), \]
where \( e_i = (\delta_{1,i}, \delta_{2,i}, \ldots) \) with \( \delta_{n,i} \) denoting the Kronecker symbol. By substituting (3.1) into (1.3) and comparing the coefficients of \( \exp(atq^k) \) on both sides, we obtain the recurrence relation
\[ (q^k - 1)d_{j,k} = \sum_{i=1}^{\infty} \left( \frac{b_i}{a}d_{j,-e_i,k} + q^{j}d_{j,-e_i,k}q^d_{j,-e_i,k} \right), \quad j, k \in (\mathbb{Z}^+)^n, \tag{3.2} \]
which leads to the following result.

**Theorem 3.** If \( |b| < |a| \) then the solution of (1.3)–(1.4) has a Dirichlet series expansion of the form (3.1), where \( \{d_{j,k}\} \) are determined by (3.2) and
\[ \alpha = \prod_{n=0}^{\infty} \frac{1 + (b,p^n)/a}{1 - (c,p^n)}. \tag{3.3} \]

**Proof.** It is not difficult to see from (3.2) that
\[ |d_{j,k}| \leq \rho \max_{\|j+m\| < \|j+k\| - 1} |d_{j,m}|, \quad j, k \in (\mathbb{Z}^+)^n, \]
for \( \rho = (|b|/|a| + |c|/(1 - \lambda_j)), \) which implies that
\[ |d_{j,k}| \leq \rho^{|j+k|}, \quad j, k \in (\mathbb{Z}^+)^n. \]
Therefore, the generating function
\[ g(x, y) = \sum_{n=0}^{\infty} \sum_{|j| + |k| = n} d_{j, k} x^j y^k \]
is analytic for all \( x, y \in \mathbb{C}^n \) satisfying \( |x| + |y| < \rho \). Multiplying (3.2) by \( x^j y^k \) and summing up for \( j, k \in (\mathbb{Z}^+)^n \) yields
\[ g(qx, py) - g(x, y) = \frac{(b, x)}{a} g(x, y) + (c, y) g(qx, py), \]
which implies that
\[ g(x, y) = \frac{1 - (c, y)}{1 + (b, x)/a} g(qx, py). \]

Due to the fact that \( \lim_{x, y \to 0} g(x, y) = g(0, 0) = 1 \), we deduce that
\[ g(x, y) = \prod_{n=0}^{\infty} \frac{1 - (c, p^n y)}{1 + (b, q^n x)/a}, \] (3.4)
where the convergence of the product follows from our assumption that \( \lambda_n^2 < 1 \). (Note that formula (3.4) is, in fact, a generalization of the \( q \)-binomial theorem [10]. A further generalization is (5.9), which features in Section 5.) Hence, \( g(x, y) \) can be extended analytically to \( x, y \in \mathbb{C}^n: |b| |x| < |a| \). Since \( |a| > |b| \), we deduce that the series \( \sum_{n=0}^{\infty} \sum_{|j| + |k| = n} d_{j, k} \) converges absolutely. Therefore, the Dirichlet series (3.1) converges absolutely for all \( t \in \mathbb{R} \). Our assumption that (2.1) holds implies that \( g(1, 1) \neq 0 \), where \( 1 = (1, 1, \ldots) \). It is easy to verify that the Dirichlet series (3.1) satisfies (1.3)–(1.4) if we let \( \alpha = 1/g(1, 1) \).

The Dirichlet series may fail to converge if \( |b| \geq |a| \). However, we can express a high-order derivative of the solution of (1.3)–(1.4) into a Dirichlet series, and find the solution through the Taylor expansion formula with integral remainder. Let \( m \in \mathbb{N} \) be such that \( \langle b, q^m \rangle \) \( < |a| \), and let \( y_m(t) \) be the \( m \)th derivative of the solution \( y(t) \) of (1.3)–(1.4). Therefore,
\[ y(t) = \sum_{n=0}^{m-1} \frac{y_n(0)}{n!} t^n + \frac{1}{(m-1)!} \int_0^t (t-\tau)^{m-1} y_m(\tau) d\tau, \] (3.5)
where, by Theorem 1,
\[ y_n(0) = y_0 \prod_{k=0}^{n-1} \left( 1 - \sum_{i=1}^{\infty} p_i^k c_i \right)^{-1} \left( a + \sum_{i=1}^{\infty} q_i^k b_i \right), \quad n = 1, 2, \ldots \] (3.6)
Differentiating both sides of (1.3) \( m \) times yields

\[ y'_m(t) = ay_m(t) + \sum_{i=1}^{\infty} b_i q^m_i y_m(q_i t) + \sum_{i=1}^{\infty} c_i p^m_i y'_m(p_i t), \quad t > 0. \quad (3.7) \]

According to Theorem 3, we have

\[ y_m(t) = \alpha_m y_m(0) \sum_{n=0}^{\infty} \sum_{|j| + |k| = n} d_{j,k,m} \exp(atq^j p^k), \]

where \( d_{a,0,m} = 1, d_{j,k,m} = 0 \) if \( j_i < 0 \) or \( k_i < 0 \) for some \( i \in \mathbb{N} \), \( \{d_{j,k,m}\} \) satisfies the recurrence relation

\[ (q^j p^k - 1) d_{j,k,m} = \sum_{i=1}^{\infty} \left( \frac{b_i q^m_i}{a} d_{j-i,e,k,m} + q^j p^k - e_i p_i d_{j,k-e,m} \right), \]

\[ j, k \in (\mathbb{Z}^+) \]

and \( \alpha_m \) satisfies \( \alpha_m y_m(0) = a^m \alpha y_0 \). Noting that \( d_{j,k,m} = (q^j p^k)^m d_{j,k} \) and

\[ \frac{1}{(m-1)!} \int_0^t (t - \tau)^{m-1} e^{a\tau} d\tau = a^{-m} \left( e^{at} - \sum_{l=0}^{m-1} \frac{a^l}{l!} \right), \]

we have the following result.

**Theorem 4.** If \( m = \min(n \in \mathbb{Z}^+: |b|, |q^j|) < |a| \geq 1 \), then the solution of (1.3)–(1.4) has the Dirichlet–Taylor series expansion

\[ y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n \]

\[ + \alpha y_0 \sum_{n=0}^{\infty} \sum_{|j| + |k| = n} d_{j,k} \left( \exp(atq^j p^k) - \sum_{l=0}^{m-1} \frac{(aq^j p^k)^l}{l!} \right), \]

where \( y^{(n)}(0) \), \( \alpha \) and \( \{d_{j,k}\} \) are given by (3.6), (3.3), and (3.2), respectively.

**4. ASYMPTOTIC BEHAVIOUR**

In this section we demonstrate how to use Dirichlet and Dirichlet–Taylor series solutions to analyse the asymptotic behaviour of the neutral equation (1.3).
THEOREM 5. Let $y(t)$ be the unique solution of \((1.3)-(1.4)\) and $m := \min\{n \in \mathbb{Z}^+: |b_n, q^n| < |a|\}$. Then

1. \(\lim_{t \to \pm\infty} y(t)e^{-\alpha t} = \alpha y_0\) if $\text{Re} \ a > 0$, where $\alpha$ is given by \((3.3)\);
2. $y(t) = \sigma(t^m)$ as $t \to \infty$ if $\text{Re} \ a = 0$. Moreover, $y^{(m)}(t)$ is almost periodic; and
3. $y(t) = o(t^m)$ as $t \to \infty$ if $\text{Re} \ a < 0$.

Proof. We only give proof of the case of $\text{Re} \ a > 0$, since the other two cases can be proved similarly. If $m = 0$ then we deduce from \((3.1)\) that

\[
|y(t)\exp(-\alpha t) - \alpha y_0| \leq \exp(-\text{Re} a(1 - \lambda_+)t) \sum_{n=1}^{\infty} \|d_{1,k}\|
\]

which implies that $\lim_{t \to \infty} y(t)e^{-\alpha t} = \alpha y_0$. If $m \geq 1$ then we deduce from \((3.7)\) that $\lim_{t \to \infty} y_m(t)e^{-\alpha t} = \alpha^m y_m(0) = a^m \alpha y_0$. Using the Taylor expansion \((3.5)\), we deduce that $\lim_{t \to \infty} y(t)e^{-\alpha t} = \alpha y_0$. 

Part 1 of the last theorem shows that the solution $y(t)$ either grows exponentially, i.e., $\lim_{t \to \infty} y(t)e^{-\alpha t} \neq 0$, or is polynomial.

THEOREM 6. Let $y(t)$ be the unique solution of \((1.3)-(1.4)\) and let

\[
\mu_0 := \left\{ \sup \left\{ \text{Re} \lambda: a + \sum_{i=1}^{\infty} b_i q_i^\lambda = 0 \right\}, \qquad b \neq 0, \right. \\
-\infty \quad \left. \left. b = 0. \right. \right\}
\]

1. If $\text{Re} \ a = 0$ then $y(t) = \sigma(t^\mu)$ as $t \to \infty$ for all $\mu \in (\mu_0, \infty) \cap [0, \infty)$.
2. If $\text{Re} \ a < 0$ then $y(t) = o(t^\mu)$ as $t \to \infty$ for all $\mu > \mu_0$.

To prove the theorem, we introduce three lemmas.

LEMMA 7. Suppose that $y(t) \in C[0, \infty)$ satisfies the functional equation

\[
ay(t) + \sum_{i=1}^{\infty} b_i y(q_it) = h(t), \quad t \geq 0,
\]

where $a$, $b_i$, and $q_i$, $i = 1, 2, \ldots$, are as before, and $h(t) \in C[0, \infty)$ satisfies $h(t) = \sigma(t^\mu)$ as $t \to \infty$ for some constant $\mu_\gamma$. Then $y(t) = \sigma(t^\mu)$ as $t \to \infty$ for any $\mu \in (\mu_0, \infty) \cap [\mu_\gamma, \infty)$, where $\mu_0$ is the same as in Theorem 6.

Proof. For any $\mu \in (\mu_0, \infty) \cap [\mu_\gamma, \infty)$ we define $x(t) = e^{-\mu t} y(e^t)$, $-\infty < t < \infty$. The function $x(t)$ satisfies the difference equation

\[
ax(t) + \sum_{i=1}^{\infty} b_i q_i^\mu x(t + \log q_i) = H(t), \quad 0 \leq t < \infty,
\]
where \( H(t) := e^{-\mu t}h(e^t) = \mathcal{O}(1) \) as \( t \to \infty \). Since \( \sup(\Re \lambda; a + \sum_{i=1}^{\infty} b_i q^{i+1} \lambda) = 0 \), it follows from Theorem 4.1 of Hale [11, p. 287] that \( x(t) = \mathcal{O}(1) \) as \( t \to \infty \). Therefore, \( y(t) = \mathcal{O}(t^\mu) \) as \( t \to \infty \) for any \( \mu \in (\mu_0, \infty) \cap [\mu_1, \infty) \).

**Lemma 8.** If \( \beta, \lambda \in (0, 1) \) then \( \sum_{n=0}^{\infty} \beta^n \exp(-t\lambda^n) = \mathcal{O}(t^\mu) \) but not \( o(t^\mu) \) as \( t \to \infty \) for \( \mu = -\log \beta/\log \lambda < 0 \).

**Proof.** If \( \mu \leq -1 \) then
\[
 t^{-\mu} \sum_{n=0}^{\infty} \beta^n \exp(-t\lambda^n) = \sum_{n=0}^{\infty} (t\lambda^{n+1})^{-1} \exp(-t\lambda^n)(t\lambda^n - t\lambda^{n+1}) \\
 \leq \sum_{n=0}^{\infty} \int_{t\lambda^n}^{t\lambda^{n+1}} x^{-1} \exp(-x) \, dx \\
 < \int_{0}^{\infty} x^{-1} \exp(-x) \, dx < \infty,
\]
and, for \( t \geq \lambda \),
\[
 t^{-\mu} \sum_{n=0}^{\infty} \beta^n \exp(-t\lambda^n) = \sum_{n=0}^{\infty} (t\lambda^{n-1})^{-1} \exp(-t\lambda^n)(t\lambda^{n-1} - t\lambda^n) \\
 \geq \sum_{n=0}^{\infty} \int_{t\lambda^n}^{t\lambda^{n-1}} x^{-1} \exp(-x) \, dx \\
 \geq \int_{0}^{\infty} x^{-1} \exp(-x) \, dx > 0.
\]
Hence, \( \sum_{n=0}^{\infty} \beta^n \exp(-t\lambda^n) = \mathcal{O}(t^\mu) \) but not \( o(t^\mu) \) as \( t \to \infty \). The case of \( \mu \in (-1, 0) \) can be proved similarly.

Alternatively, we can prove the preceding lemma in the following way. The function
\[
 x(t) := \sum_{n=0}^{\infty} \frac{1}{(\lambda; \lambda)_n} \beta^n \exp(-t\lambda^n)
\]
satisfies the functional-differential equation
\[
 x'(t) = \beta x(\lambda t) - x(t), \quad t > 0.
\]
(In what follows we use standard notation from the theory of basic hypergeometric functions [10], e.g.,
\[
 (a; q)_n = \begin{cases} 
 1, & n = 0, \\
 (1-a)(1-aq) \cdots (1-aq^{n-1}), & n = 1, 2, \ldots,
\end{cases}
\]
called the shifted factorial or \( q \)-factorial or Gauss–Heine symbol, and \( (a; q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \).
According to Theorem 3(i) and (ii) of Kato and McLeod [19], \(x(t) = \mathcal{O}(t^\mu)\) but not \(o(t^\mu)\) as \(t \to \infty\). Hence, Lemma 8 holds since
\[
(\lambda; \lambda) \cdot x(t) < \sum_{n=0}^{\infty} \beta^n \exp(-t\lambda^n) < x(t).
\]

In the remainder of this section we let \(\lambda_- = \inf_{t \geq 1} \max(p, q_t) > 0\).

Lemma 9. If \(\Re a < 0\) and \(|b| < |a|\) then \(y(t) = \mathcal{O}(t^\mu)\) as \(t \to \infty\) for \(\mu = (\log|a| - \log|b|)/\log \lambda_-\).

Proof. It is easy to see from (3.2) that for any \(\varepsilon \in (0, 1 - |b|/|a|)\) there exists a constant \(M > 0\) such that
\[
\sum_{|j| + |k| = n} |d_{j,k}| \leq M \left(\frac{|b| + \varepsilon}{|a|}\right)^n, \quad n \geq 0.
\]

According to Theorem 3, we have
\[
|y(t)| \leq |ay_0|M \sum_{n=0}^{\infty} \left(\frac{|b| + \varepsilon}{|a|}\right)^n \exp(\Re a t^n).
\]

It follows from Lemma 8 that \(y(t) = \mathcal{O}(t^\mu)\) as \(t \to \infty\) for \(\mu = (\log|a| - \log|b| + \varepsilon)/\log \lambda_-\). The arbitrariness of \(\varepsilon\) implies the desired result.

Proof of Theorem 6. Denote \(y_n(t) = y^{(n)}(t), n \in \mathbb{N}\). An important observation is that \(y_n(t)\) satisfies the functional-differential equation
\[
y'_n(t) = ay_n(t) + \sum_{i=1}^{\infty} b_i q^n y_n(q_i t) + \sum_{i=1}^{\infty} c_i p^n y_n(p_i t), \quad t > 0, \quad (4.1)
\]
and the functional equation
\[
ay_n(t) + \sum_{i=1}^{\infty} b_i q^n y_n(q_i t) = h_n(t), \quad t \geq 0, \quad (4.2)
\]
where
\[
h_n(t) = y_{n+1}(t) - \sum_{i=1}^{\infty} c_i p^n y_{n+1}(p_i t).
\]

Another useful observation is that \(y_{n+1}(t) = \mathcal{O}(t^\mu)\) as \(t \to \infty\) implies \(h_n(t) = \mathcal{O}(t^\mu)\) as \(t \to \infty\) for any fixed \(\mu\) and \(n\). First, let us consider the case \(\Re a = 0\). Let \(m \in \mathbb{N}\) be such that \(|b, q^n| < |a|\). Applying Theorem 5 to (4.1) with \(n = m\) yields \(y_m(t) = \mathcal{O}(1)\) as \(t \to \infty\). Applying Lemma 7
repeatedly to (4.2) from $n = m - 1$ to $n = 0$ yields $y(t) = \theta(t^\mu)$ as $t \to \infty$ for any $\mu \in (\mu_0, \infty) \cap [0, \infty)$. Next, we consider the case $\text{Re} \, a < 0$. Let $m \in \mathbb{N}$ be such that

$$\frac{\log |a| - \log(b, q^m)}{\log \lambda_-} < \mu_0.$$  

Applying Lemma 9 to (4.1) with $n = m$ yields $y_m(t) = \theta(t^{\mu'})$ as $t \to \infty$ for any

$$\mu' \in \left(\frac{\log |a| - \log(b, q^m)}{\log \lambda_-}, \mu_0\right).$$

Applying Lemma 7 repeatedly to (4.2) from $n = m - 1$ to $n = 0$ yields $y(t) = \theta(t^{\mu'} m')$ as $t \to \infty$ for any $\mu > \mu_0$. Hence, $y(t) = o(t^{\mu'})$ as $t \to \infty$ for any $\mu > \mu_0$. This completes the proof of Theorem 6.

5. ON A SYSTEM OF EQUATIONS

In this section we generalize our results on the scalar initial-value problem (1.3)–(1.4) to the vector problem

$$y'(t) = Ay(t) + \sum_{i=1}^\infty B_i y(q_i t) + \sum_{i=1}^\infty C_i y'(p_i t), \quad t > 0, \quad (5.1)$$

$$y(0) = y_0, \quad (5.2)$$

where $A, B_i, C_i$ are $d \times d$ complex matrices, $p_i, q_i \in (0, 1), i = 1, 2, \ldots$, and $y_i$ is a column vector in $\mathbb{C}^d$. To guarantee convergence of the series in (5.1), we assume that $\sum_{i=1}^\infty \|B_i\| < \infty$ and $\sum_{i=1}^\infty \|C_i\| < \infty$, where $\| \cdot \|$ denotes the matrix norm induced by a vector norm, likewise denoted by $\| \cdot \|$, which is arbitrary. Note that this assumption is independent of the norm, since all norms in a finite-dimensional space are equivalent. In the sequel $\sigma(\cdot)$ denotes the spectrum, $\rho(\cdot)$ the spectral radius, $\alpha(\cdot)$ the maximal real part of the eigenvalues of the matrix (the spectral abscissa), and $\kappa(A)$ the geometric multiplicity of the eigenvalue with maximal real part. If there is more than one eigenvalue that attains the maximal real part, then $\kappa(A)$ denotes the maximal geometric multiplicity of all such eigenvalues.
Analogously to Theorem 1 we have the following result.

**Theorem 10.** If $\sum_{i=1}^{\infty} p_i^{m} ||C_i|| < 1$ for some $m \in \mathbb{Z}^+$, then in the function space $\mathcal{C}^{m+1}[0, \infty)$ the solution of (5.1)–(5.2) exists and is unique if and only if

$$I - \sum_{i=1}^{\infty} p_i^{n} C_i, \ n = 0, 1, \ldots, m - 1, \text{ are nonsingular.} \quad (5.3)$$

If (5.3) holds then the unique solution is analytic and can be written as

$$y(t) = \left( I + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} \left( I - \sum_{i=1}^{\infty} p_i^{k} C_i \right)^{-1} \left( A + \sum_{i=1}^{\infty} q_i^{k} B_i \right) \right) \frac{t^n}{n!} y_0, \quad t \geq 0, \quad (5.4)$$

where the multiplication of matrices in the product is carried out from left to right.

In the rest of this section, we assume that $A$ is nonsingular, (5.3) holds, $\lambda_+ = \sup_{i \geq 1} \max(p_i, q_i) < 1$, and

$$\sigma(A) \cap \sigma(q^1 p^k A) = \emptyset \quad (5.5)$$

holds for all $(j, k) \in (\mathbb{Z}^+)^n \times (\mathbb{Z}^+)^n \setminus \{(0, 0)\}$. Again we seek a Dirichlet or Dirichlet–Taylor series solution.

Analogously to Theorem 3 we have the following result.

**Theorem 11.** If $\sum_{i=1}^{\infty} ||A^{-1}B_i|| < 1$ then (5.1)–(5.2) has the Dirichlet series solution

$$y(t) = \sum_{n=0}^{\infty} \sum_{\|j+k\|=n} D_{j,k} \exp(At q^1 p^k) V y_0, \quad (5.6)$$

where $D_{0,0} = I$, $D_{j,k} = 0$ if $j_i < 0$ or $k_i < 0$ for some $i \in \mathbb{N}$, $(D_{j,k}), (j, k) \in (\mathbb{Z}^+)^n \times (\mathbb{Z}^+)^n \setminus \{(0, 0)\}$ are determined by the recurrence relation

$$q^1 p^k D_{j,k} A - A D_{j,k} = \sum_{i=1}^{\infty} (B_i D_{j-e_i,k} + q^1 p^k C_i D_{j,k-e_i} A), \quad j, k \in (\mathbb{Z}^+)^n, \quad (5.7)$$
and

\[
V = \lim_{m \to \infty} A^{-m} \prod_{n=0}^{m-1} \left( I - \sum_{i=1}^{\infty} C_i p_i \right)^{-1} \left( A + \sum_{i=1}^{\infty} B_i q_i \right). \tag{5.8}
\]

**Proof.** By substituting (5.6) into (5.1) and comparing the coefficients of \( \exp(A q^k p^k) \) on both sides, we obtain the recurrence relation (5.7). Clearly, (5.7) is obeyed in the case \( j, k = 0 \). It follows from a classical theorem \([9]\) that (5.7) is solvable for \( D_{j,k} \) if (5.5) holds. Introducing the generating function

\[
G(x, y) = \sum_{n=0}^{\infty} \sum_{|j|+|k|=n} D_{j,k} x^j y^k,
\]

we obtain from (5.7) the relation

\[
\left( A + \sum_{i=1}^{\infty} B_i x_i \right) G(x, y) = \left( I - \sum_{i=1}^{\infty} C_i y_i \right) G(qx, py) A,
\]

which implies that

\[
\det(G(x, y)) = \prod_{n=0}^{\infty} \frac{\det(I - \sum_{i=1}^{\infty} C_i p_i^n y_i)}{\det(I + \sum_{i=1}^{\infty} TA^{-1} B_i q_i^n x_i)} \neq 0. \tag{5.9}
\]

Hence, \( G(1, 1) \) is nonsingular. By letting \( V = G(1, 1)^{-1} \), we see that the Dirichlet series (5.6) satisfies (5.1)–(5.2). \[\blacksquare\]

Analogously to Theorem 4 we have the following result.

**Theorem 12.** If \( m := \min\{n \in \mathbb{Z}^+ : \sum_{i=1}^{n} q_i^n \|A^{-1} B_i\| < 1\} \geq 1 \), then (5.1)–(5.2) has the Dirichlet–Taylor series solution

\[
y(t) = \sum_{n=0}^{m-1} \frac{y^{(n)}(0)}{n!} t^n
\]

\[
+ \sum_{n=0}^{\infty} \sum_{|j|+|k|=n} D_{j,k} \left( \exp(A q^k p^k) - \sum_{l=0}^{m-1} \frac{(A q^k p^k)^l}{l!} t^l \right) V y_0,
\]

where \( y^{(n)}(0) \) can be obtained from (5.4), \( D_{j,k} \) are determined by (5.7), and \( V \) is given by (5.8).
Noting that
\[ \| e^{At} \| \leq M(t + 1)^{\kappa(A) - 1} e^{\alpha(A)t}, \quad t \geq 0, \]
for some constant \( M > 0 \), we have analogously to Theorems 5 and 6 the following results.

**Theorem 13.** Let \( \mathbf{y}(t) \) be the unique solution of (5.1)–(5.2). Then
1. \( \| \mathbf{y}(t) \| = O(t^{\kappa(A) - 1} e^{\alpha(A)t}) \) as \( t \to \infty \) if \( \alpha(A) > 0 \);
2. \( \| \mathbf{y}(t) \| = O(t^{m - \kappa(A) - 1}) \) as \( t \to \infty \) if \( \alpha(A) = 0 \); and
3. \( \| \mathbf{y}(t) \| = o(t^m) \) as \( t \to \infty \) if \( \alpha(A) < 0 \),
where \( m := \min(n \in \mathbb{Z}^+: \sum_{i=1}^{\infty} q_i^n \| A^{-1} B \| < 1) \).

**Theorem 14.** Suppose that \( \lambda := \inf_{i \geq 1} \max(p_i, q_i) > 0 \). Let \( \mu_0 = -\infty \) if
\[ \det \left( A + \sum_{i=1}^{\infty} B_i q_i^\lambda \right) = 0 \]
has no solution; otherwise, let
\[ \mu_0 := \sup \left\{ \Re \lambda: \det \left( A + \sum_{i=1}^{\infty} B_i q_i^\lambda \right) = 0 \right\}. \]

Then the solution \( \mathbf{y}(t) \) of (5.1)–(5.2) satisfies
1. \( \| \mathbf{y}(t) \| = O(t^{m - \kappa(A) - 1}) \) as \( t \to \infty \) for any \( \mu \in (\mu_0, \infty) \cap [0, \infty) \) if \( \alpha(A) = 0 \); and
2. \( \| \mathbf{y}(t) \| = o(t^\mu) \) as \( t \to \infty \) for any \( \mu > \mu_0 \) if \( \alpha(A) < 0 \).

Under further restrictions, we have the following result.

**Theorem 15.** If all eigenvalues of \( A \) have a positive real part, and \( A, B_i, \) and \( C_i \) commute with each other for all \( i \in \mathbb{N} \), then the solution \( \mathbf{y}(t) \) of (5.1)–(5.2) satisfies
\[ \lim_{t \to \infty} e^{-At} \mathbf{y}(t) = V \mathbf{y}_0, \]
where \( V \) is given by (5.8).

**Proof.** It is easy to prove by induction from (5.7) that, in the present situation, \( A \) and \( \{D_{j,k}\} \), \( (j, k) \in (\mathbb{Z}^+)^n \times (\mathbb{Z}^+)^n \), commute with each other.
We can then prove this theorem directly from Theorem 11 or 12, depending on whether the condition \( \sum_{i=1}^n \| A^{-1} B_i \| < 1 \) holds or not.

6. ON THE GENERALIZED PANTOGRAPH EQUATION

In this section we discuss the generalized pantograph equation (1.5). To guarantee the existence and uniqueness of the solution, we assume that \( c \neq q^{-k} \), \( k = 0, 1, \ldots \). Suppose that \( a \neq 0 \) and \( |b| < |a| \). Then (1.5) has the Dirichlet series solution

\[
y(t) = \frac{(-b/a; q)_\infty}{(c; q)_\infty} \sum_{n=0}^\infty \frac{(-ac/b; q)_n}{(q; q)_n} \left( -\frac{b}{a} \right)^n \exp( at^n q^n )
\]

if \( b \neq 0 \), or

\[
y(t) = \frac{1}{(c; q)_\infty} \sum_{n=0}^\infty \frac{(-1)^n c^n q^{n(n-1)/2}}{(q; q)_n} \exp( at^n q^n )
\]

if \( b = 0 \). Suppose that \( |b| \geq |a| > 0 \). Then (1.5) has the Dirichlet–Taylor series solution

\[
y(t) = \sum_{n=0}^{m-1} \frac{y^{(n)}(0)}{n!} t^n + \frac{(-b/a; q)_\infty}{(c; q)_\infty} \sum_{n=0}^\infty \frac{(-ac/b; q)_n}{(q; q)_n} \left( -\frac{b}{a} \right)^n \left( \exp( at^n q^n ) - \sum_{l=0}^{m-1} \frac{(aq^n)^l}{l!} t^l \right),
\]

where \( m := \lfloor \log(|a|/|b|)/\log q \rfloor + 1 \).

In the case where \( \Re a < 0 \) and \( b \neq 0 \), the result in Section 4 can be improved.

**Theorem 16.** Suppose that \( \Re a < 0 \) and \( b \neq 0 \). Let \( y(t) \) be the solution of (1.5), \( \mu = \log(|a|/|b|)/\log q \), \( \theta = 1/(2\pi) \arg(-q^a b^{-1}) \), and \( z(t) = t^{-\mu} y(t) \). Then

1. \( y(t) = \theta(t^\mu) \) as \( t \to \infty \).

2. If \( \theta \) is rational, \( \theta = l/k \), say, then the \( \omega \)-limit set of \( z(t) \) coincides with a closed continuous curve that can be represented parametrically by \((\Re z^*(t), \Im z^*(t))\) for \( t \in [1, q^{-k}] \) in the complex plane, where \( z^*(t) \in \)
If \( \theta \) is irrational then the \( \omega \)-limit set of \( z(t) \) is the annulus \( \Omega = \{ z : \underline{z} \leq |z| \leq \bar{z} \} \), where \( \underline{z} = \liminf_{t \to \infty} |z(t)| \), \( \bar{z} = \limsup_{t \to \infty} |z(t)| \).

This theorem is proved in Liu [16] for a slightly more general case. The Dirichlet and Dirichlet–Taylor series solutions make it possible to calculate the \( \omega \)-limit set of \( z(t) \) (the numerical methods presented in Liu [17] can be used in more general cases). In Figs. 1–6 we display the solutions for \( t \gg 1 \) in the “phase plane” \((\text{Re} \, y, \text{Im} \, y)\).

Finally, we discuss the change in asymptotic behaviour in response to the variation of some parameters. Denote by \( y(t; q) \) the solution of (1.5). It
is easy to see that

$$y(t; 1) = \exp\left(\frac{a + b}{1 - c} t\right)$$

and

$$y(t; 0) = -\frac{c + b/a}{1 - c} + \frac{1 + b/a}{1 - c} e^{at}.$$

Consider the case of $\text{Re} \ a > 0$. Obviously, the asymptotic behaviour of $y(t; q)$ is different from $y(t; 1)$ no matter how close $q$ is to 1 except when
Fig. 3. \( y'(t) = \exp(\frac{k}{T} \pi i) y(t) + \exp(\frac{i \pi}{T} \pi i) y(t/20), \ y(0) = 1. \)

\( b = -ac. \) Noting that

\[
\lim_{q \to 1-} (1 - q) \log(a; q) = \int_0^1 \log(1 - ax) \frac{dx}{x},
\]

we obtain from Theorem 5 that

\[
\lim_{q \to 1-} \left( \lim_{t \to \infty} y(t; q) \exp(-at) \right)^{1-q} = \exp \left( \int_0^1 \left( \log \left( 1 + \frac{bx}{a} \right) - \log(1 - cx) \right) \frac{dx}{x} \right).
\]
However, as $q \to 0^+$, the asymptotic behaviour of $y(t; q)$ is quite similar to $y(t; 0)$. In fact, we have

$$\lim_{q \to 0^+} \lim_{t \to \infty} y(t; q) e^{-at} = \lim_{t \to \infty} y(t; 0) e^{-at} = \frac{1 + b/a}{1 - c}.$$ 

Consider the case of $\Re a < 0$. If we treat $|b/a|$ as a parameter, it is clear from Theorem 16 that there exists a Hopf-like bifurcation as $|b/a|$ passes through 1. Another interesting phenomenon happens when $|b| = |a|$ and $a \to 0^+$. It seems that the $\omega$-limit set of $y(t; q)$ tends to a set (see Figures 3 and 4) which is obviously distinct from $-(c + b/a)/(1 - c)$, the limiting value of $y(t; 0)$. 

\[ y(t) = \exp\left(\frac{\pi}{2} i\right)y(t) + \exp\left(\frac{\pi}{10} i\right)y(10^{-7}t), y(0) = 1. \]
Remark. The case $a = 0$ has been studied by Morris, Feldstein, and Bowen [21], Kuang and Feldstein [20], Iserles [12], and Liu [16]. In the first two papers the authors proved the unboundedness of the nonpolynomial solution via the Phragmén–Lindelöf principle, and by the Ahlfors theorem in the third paper. In the fourth paper, the author proved by using the same technique as that used in Derfel [5] that for every nonpolynomial solution $y(t)$ of (1.3) there exists a sequence $(t_n)_{n=1}^\infty$, which tends to $\infty$ as $n \to \infty$, such that $|y(t)| \geq M(\exp(\beta \log t^2))$ at $t = t_n$, $n = 1, 2, \ldots$, for some positive constants $M$ and $\beta$.

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REFERENCES