Advances and Applications in Statistics
© 2019 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/AS055020237
Volume 55, Number 2, 2019, Pages 237-251 **ISSN: 0972-3617**

PROCESS CAPABILITY INDICES FOR FATIGUE LIFE MODEL

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Abstract

In time to failure analysis the word 'fatigue' commonly depicts a situation that may arise due to shock accumulation or component failures. Factors that produce this adversity may be cyclical stress, fracture, faulty repair, mechanical overload, overheating, and many others. There are variety of statistical distributions which cover time to failure analysis as lognormal, Weibull, exponential, Poisson and few more but these distributions fails to ensure the failures due to fatigue. This failure strengthen the investigation about factors that cause fatigue measurements in statistical processes further intend to know the status of these processes in-control and later capable. This paper exemplifies a procedure to estimate process capability indices through a data set containing fatigue measurements.

Received: March 5, 2019; Revised: April 3, 2019; Accepted: April 5, 2019

2010 Mathematics Subject Classification: 65C60.

Keywords and phrases: fatigue measurements, statistical process control, process capability indices.

1. Introduction

Process capability indices PCIs are the unitless and dimensionless measures explaining the relationship between actual and allowable spread with the functional association of pre-determined specification limit(s) and process parameter(s). Estimating PCIs always have been playing vital role for quality practitioners.

Numerous authors proposed, developed and modified various procedures and superstructures in diversity of estimating PCIs for symmetric distributions; examples are Juran [1], Kane [2], Hsiang and Taguchi [3], Chan et al. [4], Pearn et al. [5] and many others. For skewed distributions, Pearn and Chan [6], Pearn and Kotz [7] and Perakis and Xekalaki [8] with others proposed procedures in the field of estimating PCIs. Ahmed and Safdar [9-11] worked on estimating PCIs under diverse distributional conditions.

In viewing the published literature in the field of estimating PCIs it is observed that proposed PCIs for diverse distributions can only specify failures due to fatigue henceforth a family of distribution is needed which specify monotonic failure rates that fatigue distributions do and Weibull does not.

Fatigue life model covers those processes that accumulate shocks or stress due to high fatigue measurements, Fleck et al. [12] analyzed crack growth due to loading, Bäumel et al. [13] considered material data for cyclic loading, Stephens et al. [14] worked on crack nucleation, and many others.

See for detail Vilca-Labra and Leiva [15]. These fatigue measurements are the quality characteristics in time to failure phenomenon and characterize fatigue distribution. It is observed that estimating PCIs for these measurements is overlooked in published literature although for making processes of these fatigue measurements the designers or quality practitioners are always been very inquisitive to keep measurements in statistical control. This gap initiated to work for those processes that exhibit fatigue measurements therefore in this paper a simple and straightforward procedure is presented to identify the status of a controlled process as capable or in-capable.

In Section 2 a brief description of fatigue model is summarized.

2. Fatigue Life Model

Birnbaum and Saunders [16] derived flexible distribution from a phenomenon of physical fatigue where failures were attributed by crack growth and named as Birnbaum-Saunders distribution. Later this distribution was discussed by Fleck et al. [12], Bäumel et al. [13] and Stephens et al. [14] and others. This distribution is very influential and persuasive in its proposition as Diaz Garcia and Leiva [17] discussed variety of generalized fatigue life models for example as Pearson VII, Cauchy, Kotz type and normal.

This Birnbaum-Saunders distribution is popularly known as fatigue life distribution with density function.

$$f(t;\, \alpha, \beta) = \frac{1}{2\alpha\beta}\left(\frac{t}{\beta}\right)^{-1/2}\left[1 + \left(\frac{t}{\beta}\right)^{-1}\right] \cdot \frac{1}{\sqrt{2\pi}} Exp\left[-\frac{1}{2\alpha^2}\left[\left(\frac{t}{\beta}\right)^{-1/2} - \left(\frac{t}{\beta}\right)^{1/2}\right]^2\right]; \quad t > 0. \tag{1}$$

Figure 1 displays density curves of fatigue distribution with three different shape parameters and fixed scale parameter.

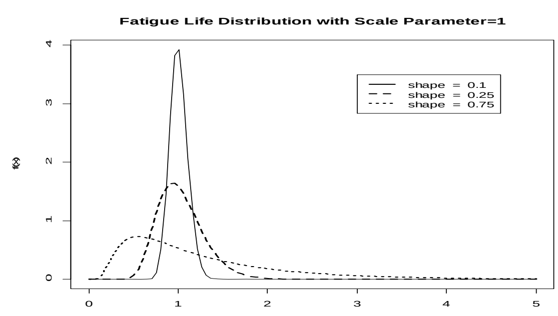

Figure 1. Density curve of fatigue life distribution.

Now the cumulative distribution of fatigue life model will be

$$\Phi\left[\frac{(t/\beta)^{\frac{1}{2}} - (t/\beta)^{-\frac{1}{2}}}{\alpha}\right]. \tag{2}$$

Here $\Phi(\bullet)$ is the standard normal cumulative distribution function of t.

Most convenient and popular generalized model with supposition of a standard normal variable y such that $y = \frac{(t/\beta)^{\frac{1}{2}} - (t/\beta)^{-\frac{1}{2}}}{\alpha}$, the fatigue density function follows standard normal density function. Here α and β are shape and scale parameter, respectively.

It can be shown that equation (1) could be written as

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}. \tag{3}$$

Now the cumulative distribution of fatigue life model is

$$\Phi(y) = \Phi\left[\frac{(t/\beta)^{\frac{1}{2}} - (t/\beta)^{-\frac{1}{2}}}{\alpha}\right]. \tag{4}$$

Here $\Phi(y)$ is the standard normal cumulative distribution function.

Many researchers worked on estimation of parameters of fatigue life distribution, Birnbaum and Saunders [16] using likelihood method and mean-mean-estimator. Chang and Tang [19] used the graphical method by least square. Ng et al. [20] worked on modified moment method to estimate the parameters of fatigue distributions.

In Section 3, developed and modified PCIs are summarized proposed earlier for normal processes.

3. Process Capability Indices

For normal processes Vannman [21] constructed a superstructure form as

$$C_P(u, v) = \frac{d - u|\mu - m|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, \quad u \geq 0; \quad v \geq 0 \tag{5}$$

such that,

$$C_P(0, 0) = C_p,\ C_P(1, 0) = C_{pk},\ C_P(0, 1) = C_{pm} \text{ and } C_P(1, 1) = C_{pmk}.$$

In this superstructure μ and σ are the process mean and process standard deviation respectively, T is target value, $d = (USL - LSL)/2$ is half length of specification interval and $m = (USL + LSL)/2$ is the midpoint between upper and lower specification limits.

The confidence interval for four basic indices are summarized as under:

The $100(1 - \alpha)\%$ confidence interval for C_p,

$$\left(\frac{\chi_{n-1, \alpha/2}}{(n-1)^{\frac{1}{2}}}\hat{C}_p,\ \frac{\chi_{n-1, 1-\alpha/2}}{(n-1)^{\frac{1}{2}}}\hat{C}_p\right). \tag{6}$$

Here $\chi^2_{n-1, \alpha/2}$ and $\chi^2_{n-1, 1-\alpha/2}$ are the upper $\alpha/2$ and $1 - \alpha/2$ quantile of a chi square distribution with $(n - 1)$ degrees of freedom, respectively. For details see Pearn et al. [5].

Confidence interval for C_{pk} is as follows:

$$\hat{C}_{pk} \pm z_{1-\alpha/2}\left\{\frac{n-1}{9n(n-3)} + \hat{C}^2_{pk}\frac{1}{2(n-3)}\left(1 + \frac{6}{n-1}\right)\right\}^{\frac{1}{2}}, \tag{7}$$

see Nagata and Nagahata [22, 23].

Confidence interval for C_{pm} will be

$$\left(\left(\frac{1}{n(1+\delta^2)}\right)\chi'_{n, \alpha/2}(n\delta^2)\hat{C}_{pm},\ \left(\frac{1}{n(1+\delta^2)}\right)\chi'_{n, 1-\alpha/2}(n\delta^2)\hat{C}_{pm}\right), \tag{8}$$

see Boyles [24], Subbaiah and Taam [25] and Patnaik [26].

An asymptotically unbiased interval for C_{pmk} is

$$\hat{C}_{pmk} \mp z_{\alpha/2}\frac{\hat{\sigma}_{pmk}}{\sqrt{n}},$$

$$\hat{\sigma}^2_{pmk} = \left[\frac{1}{9(1+\delta^2)} + \frac{2\delta}{3(1+\delta^2)^{3/2}}\right]\hat{C}_{pmk} + \frac{72\delta^2 + D\left(\frac{m_4}{s_n^4} - 1\right)}{72(1+\delta^2)^2}\hat{C}^2_{pmk}. \quad (9)$$

Here $\hat{\sigma}^2_{pmk}$ is the asymptotic estimator of $Var(\hat{C}_{pmk})$, $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution,

$$m_4 = \sum_{i=1}^{n}(X_i - \overline{X})^4/n, \quad \delta = (\overline{X} - T)/S_n\delta = (\overline{X} - T)/S_n,$$

$$S_n^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2/n.$$

See for details Chen and Hsu [27].

Here in this article PCIs based on fatigue life model are estimated on two main lines: first translate fatigue life density function to normal (see equation (2)) and second estimate four basic PCIs with their respective confidence intervals earlier developed for normal processes.

Based on these two lines a program is listed and the required analysis is followed by R- with required packages *gbs* and VGAM in R-3.0.3 [28].

In Section 4, a straightforward procedure is written to estimate PCIs of fatigue measurements.

4. Steps for Estimate PCIs for Fatigue Life Model

The steps involved in estimating PCIs using this model are as follows:

(i) For data whose measurements come from two parameter fatigue distribution $f(t;\, \alpha,\, \beta)$ specify tolerance region of quality characteristics i.e. preset values; target T, mid-point m and specifications (LSL, USL).

(ii) Estimate shape and scale parameters $(\hat{\alpha}, \hat{\beta})$ of fatigue measurements by maximum likelihood estimation (MLE) method [18], draw a density curve and simulate fatigue samples "*t*" of size 100, 200, 500 and 1000.

(iii) Apply chi-square goodness of fit test to asses each simulated sample "*t*" exhibiting fatigue distribution and transform each sample and preset values to standard normal "*y*" as in equation (2).

(iv) Construct histogram and QQ-plot for graphical assessment and apply Shapiro-Wilk (SW) normality test for statistical assessment to check assumption of normality for both samples of "*t*" and "*y*".

(v) Make subgroups of each sample '*y*' of size 10 and construct $\overline{X} - R$ control chart to check the second assumption of estimating PCIs that measurements should be in statistical control. (The program is designed so that it exclude those observations which are beyond the control limits and reconstruct control charts.)

(vi) Estimate PCIs for each transformed fatigue sample '*y*' from equation (5) and from control charts using $\overline{\overline{Y}}$ as an estimator of process mean and $\hat{\sigma} = \overline{R}/d_2$ ($\overline{R}$ is the mean of ranges of each sample and d_2 is a constant for size of the sample 10) as an estimator of process standard deviation.

(vii) Construct 95% and 99% confidence intervals of each PCIs under the assumption that for each sample '*y*', the target value, T and midpoint of specification intervals '*m*' are set to be equal using equations (6) to (9).

In Section 5, an example for a fatigue data set is presented.

5. Illustration Example

A data set *psi*21 is earlier used by Birnbaum and Saunders [16] (BISA) as fatigue life (*T*) of 6061-T6. Aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second were exposed to pressure with maximum stress of 21,000 *psi* (pounds per square inches).

For that data set the preset specification limits are assumed as $LSL =$

415, $USL = 2417$ and estimated fatigue parameters using MLE are $\hat{\alpha} = 0.31$, $\hat{\beta} = 1336.377$.

Table 1. Fatigue time of aluminum coupons exposed stress (psi21)

370	706	716	746	785	797	844	855	858	886
886	930	960	988	999	1115	1120	1134	1140	1199
1115	1120	1134	1140	1199	1115	1120	1134	1140	1199
1200	1200	1203	1222	1235	1238	1252	1258	1262	1269
1270	1290	1293	1300	1310	1313	1315	1330	1355	1390
1416	1419	1420	1420	1450	1452	1475	1478	1481	1485
1502	1505	1513	1522	1522	1530	1540	1560	1567	1578
1594	1602	1604	1608	1630	1642	1674	1730	1750	1750
1763	1768	1781	1782	1792	1820	1868	1881	1890	1893
1895	1910	1923	1924	1945	2023	2100	2130	2215	2268
2440									

Figure 2 displays the fatigue density curve for measurements of time of aluminum coupouns exposed stress *psi*21.

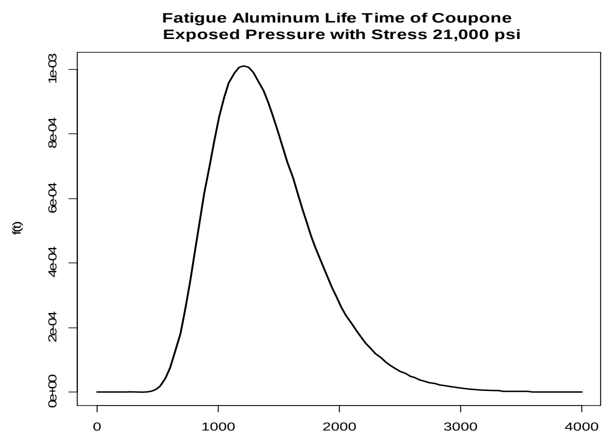

Figure 2. Density curve of fatigue time of aluminum coupons (psi21).

Before estimating PCIs, histograms and QQ-plots are constructed to graphically assess the normality assumption and for statistical assessment Shapiro-Wilk normality test is performed for both samples "*t*" and "*y*" of each size. Form Figure 3, it is shown that fatigue samples violate the assumption of normality where transformed fatigue sample do not.

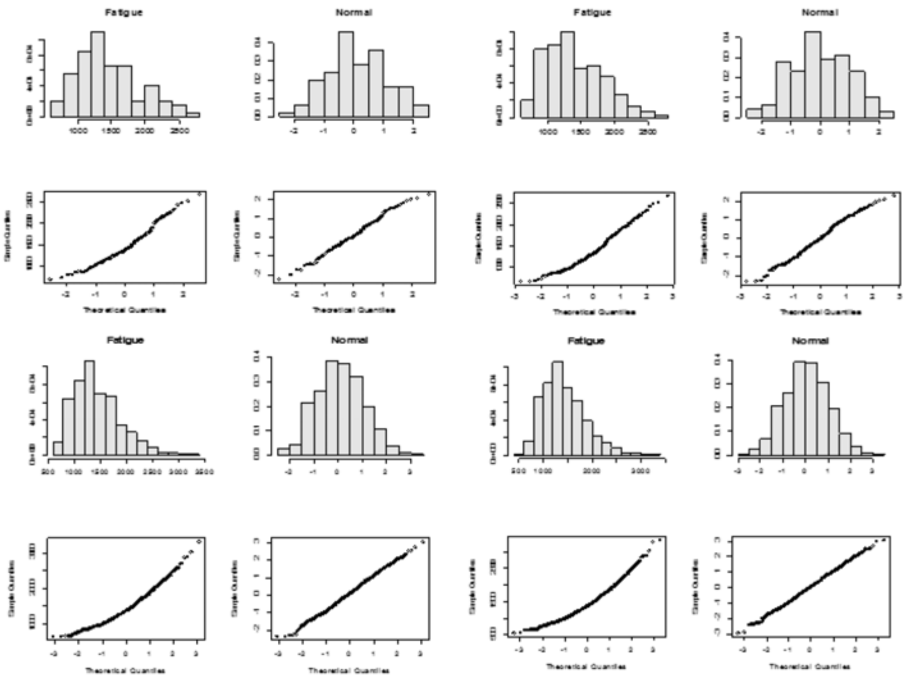

Figure 3. Histogram and QQ-plot of t and y (psi21) of size $n = 100,\ 200,$ 500, and 1000 (from left to right).

Furthermore in Table 2 SW-test corroborated the graphical assessment that transformed fatigue measurements are normally distributed.

Table 2. Shapiro-Wilk (SW) normality test (psi21)

n	100		200		500		1000	
Distribution	t	y	t	y	t	y	t	y
SW-statistics	0.960	0.991	0.966	0.989	0.959	0.997	0.961	0.999
P-value	0.004	0.741	0.000	0.146	0.000	0.538	0.000	0.697

Figure 4 shows $\overline{X} - R$ control charts for standard normal samples of each size to assess the foremost assumption of estimating PCIs that the measurements should in statistical control.

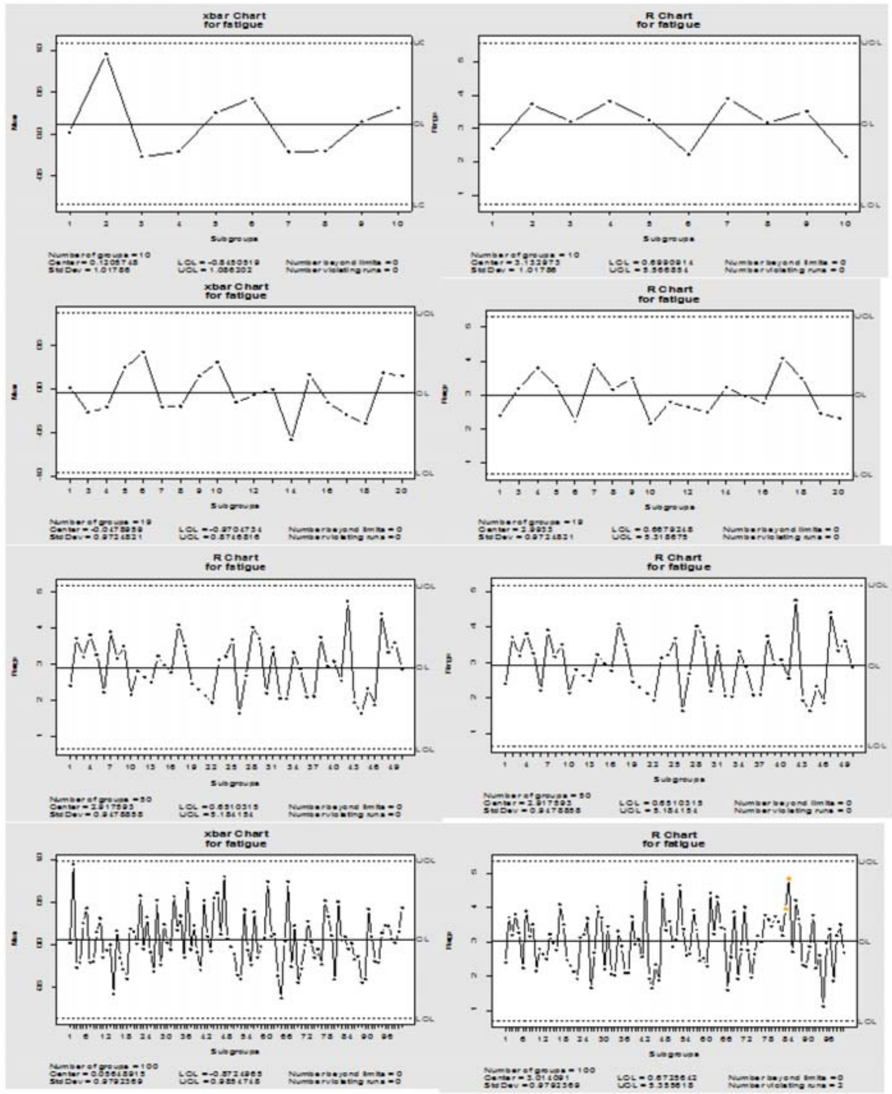

Figure 4. Control charts of transformed fatigue samples (psi21) for size of the sample $n = 100$, 200, 500 and 1000 (top to bottom).

The program in R is listed so that it excludes those samples which make the process not in statistical control and reconstruct control charts.

From the control chart for $n = 200$ i.e. 20 samples of size 10 upper control limits for $\overline{X} - R$ chart is found to be $UCL_{\overline{X}} = 0.9357$, where the mean subgroup 2 is found exceeding upper control limit and make the process out of control. To make the process in-control we exclude those 10 measurements for which the sample mean is found to be greater than UCL and re-construct the control charts for remaining 190 observations which make 19 subgroups each of size 10. We again obtain the control limits for 19 subgroups which are found to be

$$LCL_{\overline{X}} = -0.9704734, \quad CL_{\overline{X}} = -0.0478959, \quad UCL_{\overline{X}} = 0.87468160.$$

We observed that after excluding 10 measurements for new control limits the process lies in statistical control.

Now four basic PCIs based on Vannman [21] superstructure in equation (5) along with their respective confidence interval using equations (6) to (9) are obtained.

Table 3 comprises the results of PCIs for each simulated transformed fatigue sample with the assumption of $T = m$.

Table 3. PCIs of transformed fatigue samples

n	C_p	C_{pk}	C_{pm}	C_{pmk}
100	0.971	0.595	0.645	0.395
190	**1.004**	**0.656**	**0.694**	**0.454**
500	1.042	0.647	0.672	0.417
1000	1.009	0.641	0.677	0.430

Table 4 comprises the results of PCIs where process mean and process standard deviation are estimated from control charts for each simulated transformed fatigue sample with the assumption of $T = m$.

Table 4. PCIs of transformed fatigue samples based on control chart

n	C_p	C_{pk}	C_{pm}	C_{pmk}
100	0.971	0.595	0.645	0.395
190	**1.004**	**0.656**	**0.694**	**0.454**
500	1.042	0.647	0.672	0.417
1000	1.009	0.641	0.677	0.430

Tables 5 and 6 summarize 99% and 95% confidence interval for four basic PCIs for each simulated transformed fatigue sample with the assumption of $T = m$.

Table 5. 99% CI of PCIs of transformed fatigue samples (psi21)

Indices	C_p		C_{pk}		C_{pm}		C_{pmk}	
n	*LL*	*UL*	*LL*	*UL*	*LL*	*UL*	*LL*	*UL*
100	0.809	1.169	0.465	0.745	0.533	0.769	0.339	0.458
190	**0.899**	**1.173**	**0.583**	**0.805**	**0.629**	**0.819**	**0.436**	**0.533**
500	0.934	1.1	0.567	0.696	0.611	0.72	0.385	0.441
1000	0.95	1.066	0.594	0.685	0.638	0.716	0.409	0.45

Table 6. 95% CI of PCIs of transformed fatigue samples (psi21)

Indices	C_p		C_{pk}		C_{pm}		C_{pmk}	
n	*LL*	*UL*	*LL*	*UL*	*LL*	*UL*	*LL*	*UL*
100	0.849	1.124	0.498	0.712	0.559	0.739	0.353	0.444
190	**0.931**	**1.139**	**0.609**	**0.778**	**0.65**	**0.796**	**0.448**	**0.521**
500	0.954	1.08	0.583	0.68	0.624	0.706	0.392	0.434
1000	0.963	1.051	0.605	0.675	0.647	0.706	0.414	0.445

Tables 7 and 8 summarize 99% and 95 % confidence interval based on control charts for each simulated transformed fatigue sample with the assumption of $T = m$.

Table 7. 99% CI of PCIs based on $\overline{X} - R$ charts (psi21)

Indices	C_p		C_{pk}		C_{pm}		C_{pmk}	
n	*LL*	*UL*	*LL*	*UL*	*LL*	*UL*	*LL*	*UL*
100	0.796	1.15	0.457	0.734	0.547	0.742	0.337	0.454
190	**0.875**	**1.134**	**0.552**	**0.76**	**0.617**	**0.772**	**0.408**	**0.499**
500	0.958	1.128	0.582	0.713	0.628	0.717	0.389	0.446
1000	0.951	1.067	0.595	0.687	0.645	0.71	0.41	0.45

Table 8. 95% CI of PCIs based on $\overline{X} - R$ charts (psi21)

Indices	C_p		C_{pk}		C_{pm}		C_{pmk}	
n	*LL*	*UL*	*LL*	*UL*	*LL*	*UL*	*LL*	*UL*
100	0.836	1.106	0.49	0.701	0.57	0.718	0.351	0.44
190	**0.905**	**1.102**	**0.577**	**0.735**	**0.635**	**0.753**	**0.419**	**0.488**
500	0.978	1.107	0.598	0.697	0.638	0.706	0.396	0.439
1000	0.965	1.053	0.606	0.676	0.652	0.702	0.415	0.445

6. Concluding Remarks

Life time distributions which have been used in reliability, estimation of survival time or obtaining capability indices are encountered with failures which may be attribute due to assignable causes. Exponential, Weibull, lognormal and Poisson distribution cover mostly problems of failure analysis but for monotonic failure rates which may produce due to high fluctuation fatigue distribution has significance. This reason enquired to knowing the status of a fatigue process as capable or incapable in a very logical, straightforward and simple way.

Variety of fatigue generalizations are already proposed, and henceforth we use the simple transformation which translate this complicated distribution in standard normal distribution and allow using the conventional procedures developed for normal populations.

Acknowledgement

The authors thank the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

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