Fourier Transform For Signals On Dynamic Graphs

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Abstract—Signal processing on graphs offers a new way of analyzing multivariate signals. The different relationships among the sources generating the multivariate signals can be captured by weighted graphs where the nodes are the signal sources and the edges correspond to the relationships between these signals. Classical signal processing concepts need to be adapted to signals on graphs. In this paper, we propose a graph Fourier transform for signals on dynamic graphs, where the relationships vary over time. The proposed transform is evaluated on both simulated and real dynamic social networks with signal defined on its nodes.

I. INTRODUCTION

Classical multivariate signal processing methods analyze signal properties over time. However, examining signal characteristics using the information about their sources is of interest in many applications. For example, it is often desirable to analyze the spectrum of the air temperature values over a set of sensors rather than evaluate the spectrum of a sensor’s values over time. Moreover, multivariate signal processing methods consider the dependency among the signals without making any assumption about their sources. These methods assume that the sources are in a regular domain, and their relationships are uniform. However, in most applications, the relationships among the sources are not the same and needs to be taken into account while processing the signals generated by these sources. For instance, in social networks, friendships among individuals influence their behavior or opinion. Therefore, it is important to consider their friendships while analyzing the individuals’ behaviors and opinions. One way to represent the relationship among the sources of multivariate signals is to use graphs. Recently, this has led to a new research area, called signal processing on graphs [1].

Signal processing on graphs assume that the signal sources are the vertices of a graph and each node has a value, and edges show the relationship among the sources [1]. Figure 1 shows a typical signal on a graph. Signal processing on static graphs have looked at the signal samples over sources assuming that the relationship among sources does not change. The irregular topology of signals on graphs makes use of well-known signal processing methods such as filtering, denoising, and compressing difficult [2], which is why, signal processing concepts like graph filtering [3], downsampling [2], inpainting [4], sparse signal representation [5], and Fourier Transform have been defined based on the graph Laplacian matrix [1], [6], [7].

However, the relationships among the sources may change with time, and can be represented by dynamic graphs. For example, wireless sensor networks are redeployable sources whose proximities may change over time. Social networks are another example of dynamic graphs, where members’ friendships change with time. Generally, the extension of signal processing concepts for graphs relies on the graph Laplacian matrix. The graph Laplacian matrix of dynamic graphs change with their adjacency matrices, and cannot be directly used for signal processing on dynamic graphs. Therefore, in this paper, we introduce an approach to find a common subspace representation for multiple graphs across time [8], [9] and define a common Laplacian matrix for them. The eigenvectors of this modified Laplacian matrix are employed to derive Fourier transform for signals on dynamic graphs.

![An example of signal on a graph](image)

II. GRAPH THEORY

Let G = {V, E, A} be an undirected, weighted graph with a set of vertices V, |V| = N, a set of edges E, and the adjacency matrix A. A signal on the vertices of the graph \( f: V \rightarrow \mathbb{R} \) is defined as a vector \( f \in \mathbb{R}^{N \times 1} \), where its \( i^{th} \) component is the signal value on the \( i^{th} \) vertex.

The normalized graph Laplacian of the graph G is defined as \( L = D^{-\frac{1}{2}}(D - A)D^{-\frac{1}{2}} \), where D is the degree matrix. The complete set of orthonormal eigenvectors of the Laplacian matrix \( \{u_k\}_{k=0,1,...,N-1} \) and their corresponding eigenvalues \( \lambda_0 \leq \lambda_1 \leq ... \leq \lambda_{N-1} \) have been used for numerous purposes such as graph spectral clustering [10]. The spectral clustering algorithms project the graph Laplacian matrix onto a new subspace such that the clusters within the graph become distinguishable. The basis of this new subspace is obtained through solving the optimization problem \( \min_{U \in \mathbb{R}^{N \times K}} \text{tr} (U'LU) \)
subject to \( U^T U = I \), where \( I \) is a unitary matrix, and \( K \) is the number of clusters.

III. DYNAMIC GRAPH FOURIER TRANSFORM

Dynamic graphs is a time series of static graphs

\[
\mathbf{G}^{(t)} = \{ \mathbf{V}, \mathbf{E}^{(t)}, \mathbf{A}^{(t)} \} \quad \forall t \in \{1, 2, ..., T\},
\]

where it is assumed that the number of vertices remains constant while the edges and the adjacency matrices vary with time.

The graph Fourier transform theory relies on the spectral content of graphs. However, since the spectral content of the time series of graphs change with time in dynamic graphs, a common subspace has to be found in order to obtain a unique spectral representation for all time steps.

A. Common Subspace Across Time-Varying Graphs

In [9], the vertex connectivity of a graph \( \mathbf{G} = \{ \mathbf{V}, \mathbf{E}, \mathbf{A} \} \) with \( |\mathbf{V}| = N \) is represented in a \( N \)-dimensional subspace defined by \( N \) eigenvectors of the graph Laplacian matrix. The information carried by the graph connectivity is summarized in this \( N \)-dimensional subspace. In this paper, we propose an approach based on the Grassmann manifold to obtain a unique subspace representing a time series of graphs [9]. A Grassmann manifold is a set of \( N \)-dimensional subspaces in \( \mathbb{R}^n \). Each graph \( \mathbf{G}^{(t)} = \{ \mathbf{V}, \mathbf{E}^{(t)}, \mathbf{A}^{(t)} \} \) is a unique point on the Grassmann manifold.

Let \( \mathbf{U}^{(t)} \in \mathbb{R}^{N \times N} \) be the matrix of the eigenvectors of the Laplacian matrix \( \mathbf{L}^{(t)} \) corresponding to the \( t \)th graph \( \mathbf{G}^{(t)} \). Let \( \hat{\mathbf{U}} \) be the basis vectors that span a common subspace across all time, where the modified graph \( \hat{\mathbf{G}} \) with the modified Laplacian matrix \( \hat{\mathbf{L}} \) is located. The common subspace \( \text{span}(\hat{\mathbf{U}}) \) represents a point on the Grassmann manifold. The goal is to find this subspace \( \text{span}(\hat{\mathbf{U}}) \) such that it is close to all subspaces \( \text{span}(\mathbf{U}^{(t)}) \) \( \forall t \in \{1, 2, ..., T\} \) by minimizing the projection distance defined as follows [9]:

\[
d^2_{\text{proj}}(\hat{\mathbf{U}}, \mathbf{U}^{(t)}) = \sum_{t=1}^{T} N - tr(\hat{\mathbf{U}}^T \mathbf{U}^{(t)} \mathbf{U}^{(t)T}) = NT - \sum_{t=1}^{T} tr(\hat{\mathbf{U}}^T \mathbf{U}^{(t)} \mathbf{U}^{(t)T}). \tag{1}
\]

However, in order to preserve the information about the vertex connectivity of each individual graph, the Laplacian quadratic form \( \sum_{t=1}^{T} tr(\hat{\mathbf{U}}^T \mathbf{L}^{(t)} \hat{\mathbf{U}}) \) must be minimized as well [9]. Therefore, the problem of finding the common subspace can be addressed through solving the following minimization problem:

\[
\min_{\mathcal{O} \in \mathbb{R}^{N \times K}} \sum_{t=1}^{T} tr(\hat{\mathbf{U}}^T \mathbf{L}^{(t)} \hat{\mathbf{U}}) + \alpha \left[ KT - \sum_{t=1}^{T} tr(\hat{\mathbf{U}}^T \mathbf{U}^{(t)} \mathbf{U}^{(t)T}) \right]
\]

subject to \( \hat{\mathbf{U}}^T \hat{\mathbf{U}} = I \).

Rearranging the terms in Eq. 2 results in the following equation:

\[
\min_{\mathcal{O} \in \mathbb{R}^{N \times K}} \sum_{t=1}^{T} tr(\hat{\mathbf{U}}^T \mathbf{L}^{(t)} \hat{\mathbf{U}}) + \alpha \left[ KT - \sum_{t=1}^{T} tr(\hat{\mathbf{U}}^T \mathbf{U}^{(t)} \mathbf{U}^{(t)T}) \right]
\]

subject to \( \hat{\mathbf{U}}^T \hat{\mathbf{U}} = I \).

B. Dynamic Graph Fourier Transform

The classical Fourier transform is the inner product of the signal with the eigenfunction of the 1D Laplace operator \( e^{-j2\pi xt} \). Similarly, the graph Fourier transform is defined as the inner product of the signal on the graph \( f \) and the eigenvectors of the corresponding Laplacian matrix [1]. However, dynamic graphs consist of a time series of graphs with multiple Laplacian matrix. In order to obtain a unique frequency representation for the signals on the set of graphs, we propose to use the Laplacian matrix \( \hat{\mathbf{L}} \).

Let \( f^{(t)} \in \mathbb{R}^{N \times 1} \) be a signal on the vertices of \( \mathbf{G}^{(t)} \) at time step \( t \), where \( f^{(t)}(n) \) is the signal value on the \( n \)th vertex of the \( t \)th graph. The dynamic graph Fourier transform of the signal \( f^{(t)} \) is defined as:

\[
\hat{f}^{(t)}(\lambda_k) = \mathbf{\hat{F}}(f) = \sum_{n=0}^{N-1} f^{(t)}(n) \hat{u}_k(n), \quad k = 0, 1, ..., N-1, t = 1, 2, ..., T. \tag{4}
\]

where \( \hat{u}_k \) is the \( k \)th eigenvector of \( \hat{\mathbf{L}} \). \( \hat{f}^{(t)}(\lambda_k) \) \( \forall t \in \{1, 2, ..., T\} \) shows the variation of the \( k \)th spectral content of the signal on the set of graphs. Similar to classical Fourier transform, smaller eigenvalues correspond to low frequencies. The smallest eigenvalue \( \lambda_0 \), which is zero, represents the DC component of the signal.

Consequently, the eigenvector, \( \mathbf{u}_0 \), associated with the smallest eigenvalue is constant and equal to \( 1/N \) at each vertex. Generally, the smaller the eigenvalue is, the lower the frequency it represents and the eigenvector associated with it varies slowly over time.

Similarly, the inverse Fourier transform is defined as:

\[
f^{(t)}(n) = \mathbf{\hat{F}}^{-1}(\hat{f}^{(t)}(\lambda_k)) = \sum_{k=0}^{N-1} \hat{f}^{(t)}(\lambda_k) \hat{u}_k(n), \quad n = 0, 1, ..., N-1, \quad t = 1, 2, ..., T. \tag{5}
\]

IV. EXPERIMENTAL RESULTS AND DISCUSSION

In order to evaluate the performance of the proposed dynamic graph Fourier transform, a simulated dynamic graph and the corresponding signals on its vertices as well as a real dataset is used in this section. Some discussion regarding the interpretation of the outcomes and the application of this new transform are provided as well.

A. Simulated Dataset

A path graph is generated with \( N = 128 \) vertices and \( T = 128 \) time steps. To change the graph structure over time, two edges are selected at each time step randomly from an uniform distribution and set to one while other edges remain unchanged. The signal on vertices is also defined as \( f^{(t)} = \mathbf{u}_5^{(1)} + \mathbf{u}_5^{(10)} ; t = 1, 2, ..., 32 \), where \( \mathbf{u}_5^{(1)} \) is the fifth eigenvector of the first graph, and \( \mathbf{u}_5^{(10)} \) is the 15th eigenvector of the Laplacian matrix of the graph at \( t = 10 \).
For the time interval \( t = 33, 34, \ldots, 64 \), the signal is \( f(t) = u^{(1)}_5 + u^{(10)}_{15} + u^{(40)}_{40} \), where \( u^{(40)}_{40} \) is the 40th eigenvector of the 40th graph. Similarly, the signal for other time steps are defined as: \( f(t) = u^{(1)}_{100} + u^{(40)}_{40} \); \( t = 65, 66, \ldots, 96 \), and \( f(t) = u^{(30)}_{100} + u^{(100)}_{45} \); \( t = 97, 98, \ldots, 128 \). Figure 2 shows the spectral content of the signal on the dynamic graphs for the whole time interval, \( t = 1, 2, \ldots, 128 \). The signal \( f(t) \) is also corrupted with white Gaussian noise with \(-20dB\), and its graph Fourier transform is shown in Fig. 3.

**B. Real Dataset: Mobility traces of taxi cabs in San Francisco**

The real dataset used in this experiment contains mobility traces of taxi cabs in San Francisco, USA in May 2008 [11], [12]. 500 taxi cabs are equipped with GPS devices, and their location and their occupancies are collected approximately every 30 seconds. Here, we use GPS coordinates of 64 cabs to calculate their proximity and build dynamic graphs with 128 time steps. Each time step is the average of information over 30 minutes. The cabs' occupancies are used as the signal on the graph at each time step. The dynamic graph Fourier transform is computed for analyzing the time-varying the occupancies of the taxi cabs.

Figure 4(a) shows the dynamic graph Fourier transform of the occupancies of the cabs, where the first spectral component has the highest value. In order to better demonstrate other spectral components of the cabs occupancies, the DC component is suppressed and the modified spectrum is displayed in Fig. 4(b). This figure shows the variation of the different spectral components over time. For instance, the 3rd and 5th spectral components are important within the time interval (16, 25), 14th and 47th are significant within the time interval (1, 10). Eigenvectors carry information about the structure of graphs, which is why the 14th eigenvector of the common Laplacian matrix is shown in Fig. 5. Figure 5 indicates that nodes 7, 14, 43 or equivalently the taxi cabs have contributed mostly to the significant value of the 14th spectral content within the time interval (1, 10). The signals on 7th, 14th, and 43rd are displayed over time in Fig. 6. This analysis shows that the correspondence between the spectral components and the individual nodes in the graph can be used to determine the time changing energy distribution of different nodes in the network.

**V. Conclusions**

Graphs are flexible tools to model the signals and their sources which are in topologically complicated domains. Signal processing on graphs has provided a framework to analyze such signals by modifying the classical signal processing concepts. However, many applications like wireless sensor networks are represented by dynamic graphs, where the topology
of sources changes with time. In this paper, we introduced a new approach to obtain a common subspace representation for a time series of graphs. Eigenvectors of the common subspace obtained from a modified Laplacian matrix is used as the bases to calculate the graph Fourier transform of the signals on the dynamic graph. Future work will focus on the graph Fourier transform for dynamic graphs with large number of time steps. Once the number of graphs increases, the common subspace across graphs has to be close to a larger set of subspaces, which reduced the accuracy of finding the common subspace. Thus, finding the modified graph should be limited to the graphs within a moving window.

REFERENCES