1. Introduction

Simulations of some hydraulic and hydropower components are extremely difficult in multi-dimensional cases, while the system is connected to irregular boundary conditions, such as in the turbine spiral case, which is connected to the pressure shaft. It would be, therefore, quite reasonable to focus on an estimation of 2D and 3D transient flows in hydraulic systems to gain excellent achievements in this regard. Thus, this kind of complex domain is valuable, principally, during research work.

Studies on one dimensional transient flow in every type of piping system were performed by Wylie [1] and Chaudhry [2], and other previous developments on two and three dimensional transient flows are illustrated by Streeter and Wylie [3], Shin et al. [4,5] and Wylie [1] as well.

On the other hand, during the growth of CFD techniques, much valuable work was conducted in the past, for instance, by Thompson et al. [6,7], Jurgens et al. [8], Villamizar and Rojas [9] and Lee and Soni [10] consequently on visualization of the spatial domain.

After innovation by Lee [11] in non-orthogonal grids and its application to the solution of electromagnetic scattering problems, subsequently, the boundary-fitted Laplace transformation method was enhanced in this area by Grote and Kirsch [12], Villamizar and Acosta [13–16]. Thus, considering this background, this research has been kept up to date in a proper manner.

In this paper, a general method in polar boundary-fitted coordinate systems is implemented for analysis. The elliptic partial differential equation in the physical domain, \( D \), is used by means of Dirichlet boundary conditions on all boundaries. One coordinate is specified to be constant on each of the boundaries and a monotonic variation of the other coordinate around each boundary is imagined by polar slicing on \( D \).

The new statement of the initial method, which was presented by Thompson et al. [6,7], at this time, is redeveloped within the rank of higher performances. As a consequence, polarization around annular boundaries, by utilizing the Finite Difference Method (FDM), is fairly able to represent a new set of formulas that are functional for this target.

The following model is utilized to simulate, with the assistance of the continuous growth of computer rules, to improve the appearance of results in the matter of practical and real projects.

1.1. Assumptions

Throughout the following formulization, in the next chapter, some assumptions are considered, such as:
Nomenclature

\[ D, D' \] Physical and computational domains, respectively
\[ x, y \] Coordinate directions on \( D \)
\[ \eta \] \( y \)-co-ordination on \( D' \)
\[ \zeta \] \( x \)-co-ordination on \( D' \)
\[ \theta \] Angles on polar routing
\[ \Delta \theta \] Angular step
\[ \Delta x \] Spatial step in \( x \)-direction
\[ \Delta y \] Spatial step in \( y \)-direction
\[ r_a \] Radius of inner boundary condition
\[ \Phi, \psi \] Control functions
\[ K_1, K_2 \] The constants denoted of \( ax \)-grids
\[ J \] Total amount of assumed boundary layer on the particular thickness
\[ r(y) \] Length of boundary layers for arbitrary \( y \)-curvilinear coordination
\[ u \] Axial component of velocity
\[ U(r) \] Velocities through different boundary layers
\[ c \] Decay factor

Subscripts

\[ i \] Values through rows on grids
\[ j \] Values through columns on grids
\[ T \] Values at transfer grids
\[ BL \] Boundary Layer

Abbreviation

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>FDM</td>
<td>Finite Difference Method</td>
</tr>
<tr>
<td>CFD</td>
<td>Computational Fluid Dynamic</td>
</tr>
<tr>
<td>LTDT</td>
<td>Linearized Two-Dimensional Transient Flow</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\Delta x_T &= \Delta y_T, \\
\Delta \zeta_{ij} &= \Delta x_T, \\
\Delta y_T, 1 = \Delta \eta_i, j = \Delta Y_T, \quad i = 1, 2, \ldots, N_2, \quad j = 1, 2, \ldots, N_1.
\end{align*}
\]

A typical sketch of transformation in the computational process is shown in Figure 1.

1.2. Boundary fitted coordination transfer method

A universal method to obtain formal gridlines, by Thompson [6,7], is developed. Rectangular coordinates \((\xi, \eta)\) (1 \( \leq \xi \leq N_1 \) and \( 1 \leq \eta \leq N_2 \)) in which \( \Delta \xi = \Delta x_T \) and \( \Delta \eta = \Delta Y_T \) in the physical domain, \( D \), with curvilinear coordinates \((x(\xi, \eta), y(\xi, \eta))\) could be expected. A variety of numerical solutions were utilized in many working fields during the 1980s, also, the finite difference method has been obtained as being the most popular technique for engineers. Numerous works have already been undertaken in electromagnetic articles, but, this method is still sensible for use in hydraulic and hydropower systems.

In this case, grid generation is controlled by an area of cells, and in addition, the gridline separation along the coordinates \( \xi \) and \( \eta \), respectively, when \( \alpha = X_\alpha x + Y_\alpha y, \beta = X_\beta x + Y_\beta y, \) and \( \gamma = X_\gamma x + Y_\gamma y \) represent scale metric factors for transformation \( T \),

\[
\begin{align*}
\alpha X_\xi - 2\beta X_\eta + \gamma X_\eta &= -\alpha \psi X_y - \gamma \phi X_y, \\
\alpha Y_\xi - 2\beta Y_\eta + \gamma Y_\eta &= -\alpha \psi Y_x - \gamma \phi Y_y,
\end{align*}
\]

2. Formulization

2.1. Polar transformation on grid generation method

The general model described in the previous chapter is implemented, utilizing boundary value problems. New derivations in Eqs. (11)-(14) are derived by the finite difference, time dependent, numerical method on the polar transformation technique.

Herein, \( r_a \) is the constant radius, and \( \theta \) is the angular spacing. A new formulation in terms of \( \psi \) and \( \phi \) is derived to find the hydraulic behavior at every point along their gridlines.

Suitable polar grid generation is estimated with essential assumptions on LTDT [1] to meet the logical and accurate CFD on 2D transient flow in the computational plane.

Figures 2 and 3 show the schematic of the waterway region on gridlines, and corresponding points on the physical and computational domains.

Tables 1 and 2 are derived based on specific assumptions in Figures 2 and 3 to perform the process of a new polar transformation method in circular shaped domains. Thus, further formulas are gained in accordance with Figure 4, respectively.

The concept of nearly uniform cell area and line spacing would be able to be employed in cases of numerical modeling and be a solution for the polar formulization method.

![Figure 1: Schematic of computational (a), and physical domain (b).](image-url)
Table 1: Formulation in the case of a polar curvilinear coordinate system in the physical domain, \( D \), along the y-line.

<table>
<thead>
<tr>
<th>Polarization along y-line</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_{1,j} = r_0 + (i-1)\Delta Y )</td>
<td></td>
</tr>
<tr>
<td>( y_{i,j} = r_0 + (i-1) \left( \frac{Y_i}{N_2} \right) )</td>
<td></td>
</tr>
<tr>
<td>( y_{i,j} - y_{i,j-1} = \frac{Y_i}{N_2} (j - j_{-1}) )</td>
<td></td>
</tr>
<tr>
<td>( y_{i,j} - y_{i-1,j} = \frac{Y_i}{N_2} j )</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Formulation in the case of a polar curvilinear coordinate system in the physical domain, \( D \), along the x-line.

<table>
<thead>
<tr>
<th>Polarization along x-line</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{1,j} = 0 )</td>
<td></td>
</tr>
<tr>
<td>( x_{i,j} = \left( r_0 + (i-1) \left( \frac{X_i}{N_2} \right) \right) \theta_j )</td>
<td></td>
</tr>
<tr>
<td>( x_{i,j} - x_{i,j-1} = \frac{X_i}{N_2} (\theta_{j-1} - \theta_{j_{-1}}) + r_0 \Delta \theta )</td>
<td></td>
</tr>
<tr>
<td>( x_{i,j} - x_{i-1,j} = \frac{X_i}{N_2} r_{\theta_{j-1}} \theta_j )</td>
<td></td>
</tr>
</tbody>
</table>

\[ \Delta \theta = \theta_{j+1} - \theta_j = \theta_j - \theta_{j-1} \cdot r_{\theta_{j-1}/2} \] is introduced by applying the same angular gridline distribution in the y-line to the physical geometry, \( D \).

To estimate the length of the arch between each y-line, Eq. (6) may be valid for further developments.

\[ r_a + r (\theta_{j}/2) \theta_j \approx y_j, \quad \text{Eq. (6)} \]

\[ R(i) = (N_2 - 1)/(2(i - 1)) r_a \Delta \theta. \quad \text{Eq. (7)} \]

In Eq. (7), \( R \) is the important parameter, which depends on the radius of the inner boundary condition, \( r_a \). \( N_2 \) is the number of gridlines and \( i \) is an arbitrary gridline in the y-direction, respectively. In accordance with the above discussion, Tables 1 and 2 are valuable for replacing the essential variables into Eqs. (2) and (3) with the assistance of Eq. (7).

\[ r_a (\theta_{j+1} + \theta_{j-1} - 2\theta_j) = r_a [(\theta_j + \Delta \theta) + (\theta_j - \Delta \theta) - 2\theta_j] = 0. \quad \text{Eq. (8)} \]

Eq. (8) can easily be realized as a reasonable issue in the polar system as well. This equation also is matched with the model of nearly uniform cell area and line spacing.

\[ -\beta (r_{\theta_{j+1}/2} \theta_{j+1} - r_{\theta_{j-1}/2} \theta_{j-1}) + y' (r_{\theta_{j+1}/2} \theta_{j+1} + r_{\theta_{j-1}/2} \theta_{j-1} - 2r_{\theta_{j}/2} \theta_j) \]
\[ = -\Delta X(i-1) (r_{\theta_{j}/2} \theta_j) \alpha \psi_{i,j} \]
\[ = -\Delta X(i-1) r_a \psi_{i,j} - \Delta X(i-1) (\theta_j - \theta_{j-1}) y \phi_{i,j} \]
\[ i = 2, 3, \ldots, N_2, j = 2, 3, \ldots, N_1. \quad \text{Eq. (9)} \]

Finally, Eqs. (9) and (10) are illustrated to calculate accurate line spacing in the computational domain by using the defined variables in Table 3.
Table 3: Polar parameters to possess easier presentation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$r_{ij}/\theta_i - r_{ij}/\theta_j + R^2 + (r_j - r_i)^2$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$(r_{ij}/\theta_i - r_{ij}/\theta_j + R^2 + (r_j - r_i)^2 + r_j^2)(r_j - r_i)^2$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$r_{ij}/\theta_i^2 + r_j^2$</td>
</tr>
</tbody>
</table>

2.2. Controlling system by $\alpha \gamma$-grid

The $\alpha \gamma$-grid model, based on [14], is essentially used to control gridline accuracy and seems equivalent to this case of study. In addition, this progress is a suitable technique for controlling accepted values of variables $\Phi$ and $\psi$ during the grid generation procedure by Eqs. (2) and (3).

To deal with $\alpha \gamma$-grids, two additional Poisson type equations are illustrated by the gradient of metric factors $\alpha$ and $\gamma$ in Eqs. (4) and (5). The finite difference time dependent method is utilized in this derivation as:

$$\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - 4\Phi_{i,j} = k_1 C_{\Phi_{i,j}},$$

$$\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{i,j} = k_2 C_{\psi_{i,j}},$$

where $C_{\Phi_{i,j}}$ and $C_{\psi_{i,j}}$ are found in terms of $-k_1 \alpha_{i,j}$ and $-k_2 \gamma_{i,j}$ in Eqs. (4) and (5). $K_1$ and $K_2$ at interval (0.1) are expressed for each row and column on the curvilinear coordinate in physical domain, $D$, respectively. This formulization easily shows that the value of $k_1 C_{\Phi_{i,j}}$ is almost zero in the polar transformation method.

On the other hand, $C_{\Phi_{i,j}}$ is derived in Eq. (13) in the same manner as the following:

$$C_{\Phi}(i,j) = \frac{-4(i-1)^2}{\Delta X^2} C_{\Phi}(i,j),$$

$$C_{\psi}(i,j) = \frac{(r_{ij}/\theta_i - r_{ij}/\theta_j + R^2)(r_j - r_i)^2}{(r_{ij}/\theta_i - r_{ij}/\theta_j + R^2)(r_j - r_i)^2} C_{\psi}(i,j).$$

2.3. Polar transformation method on curvilinear coordinate line in a boundary layer

A new definition on the polar transformation method is applied for specified boundary layer thickness, when the boundary layer is required for a particular model (low Reynolds number).

The automatic gridline concentrated in the boundary layer is logical for development in the existing case. To create coordinate system generation and a specified distance for each gridline along the $y$-coordinate, concentric circular boundaries are yielded.

The universal concept of $r(y)$ was generated for the first time by Thompson et al. [6,7] as:

$$r(y) = C_2 e^{y/J} 1 \leq y \leq J,$$

where $C_1$ and $C_2$ are two constants, $J$ is the total number of assumed boundary layer on the particular thickness. The constants of Eq. (15) may be evaluated from boundary conditions, $r(i) = r_1$, $r(J) = r_2$, thus:

$$r(y) = r_1 \left[ \left( \frac{r_2}{r_1} \right)^{\left( \frac{y-y_1}{y_J-y_1} \right)} \right].$$

Therefore, a combined function was considered by $r(y)$. It is formed by joining a logarithmic function to a quartic polynomial, near to the edge of the boundary layer.

Table 4: Formulation in the case of a polar curvilinear coordinate system for the boundary layer in the physical domain, $D$, along the $y$-line.

<table>
<thead>
<tr>
<th>Polarization along $y$-line for boundary layer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_{ij} = r_i + r_{BL,ij}$</td>
</tr>
<tr>
<td>$y_{ij} - y_{ij-1} = r_{BL,ij} - r_{BL,ij-1}$</td>
</tr>
<tr>
<td>$y_{ij} - y_{ij-1} = r_{BL,ij} - r_{BL,ij-1}$</td>
</tr>
</tbody>
</table>

Assume that the velocity profile in the boundary layer is approximated by the exponential equation as:

$$U(r) = 1 - e^{-\frac{r}{\delta}}, \quad U = 0.99 \text{ at } r = \delta.$$  

Then, the decay factor, $C$, is given by:

$$C = -\frac{1}{\delta} \ln(0.01).$$

To achieve the same velocity changes from each gridline, Eq. (17) is solved easily while $u = 0.99 (y - 1/\gamma_b - 1)$. Here, $\gamma_b$ is the line at the edge of the boundary layer:

$$u(r) = -\frac{1}{C} \ln(1 - u).$$

Substitute $u$ into Eq. (19) and write, in a general form, for each row and column $(i, j)$, in $D$. The result yields:

$$r_{BL,ij} = -\frac{1}{C} \ln \left[ 1 - 0.99 \left( \frac{y - 1}{y_{\gamma_b,ij} - 1} \right) \right], \quad 1 \leq y \leq y.$$  

Boundary layer gridline generation by Eq. (20) likely shows that the space between the boundary layer is moderately increased from the boundary conditions to the inner region.

Thus, Eq. (20) can be visualized to possess an essential formula in the case of the boundary layer in the recent polar-transformation system.

Compatible variables are defined for the boundary layers region in Tables 4 and 5 for employment in further formulization.

Subsequently, new forms of Eqs. (9) and (10) are derived easily by replacing $\alpha_{BL,ij} \beta_{BL,ij}$ into Eqs. (2) and (3) as:

$$\alpha_{BL}[r_{BL,(i+1,j)} + r_{BL,(i-1,j)} - 2r_{BL,(i,j)}] \beta_{BL} - \frac{1}{2} \beta_{BL}[r_{BL,(i+1,j+1)}]$$

$$- r_{BL,(i-1,j-1)} \theta_{ij+1} + (r_{BL,(i+1,j)} - r_{BL,(i,j-1)}) \theta_{ij+1} + \gamma_{BL}[r_{\psi_{i,j+1}} + \psi_{ij} - 2\theta_{ij} + r_{BL,(i+1,j)} \theta_{ij+1} + r_{BL,(i,j-1)} \theta_{ij+1} + r_{BL,(i,j)} \theta_{ij}) \theta_{ij}$$

$$= -\Delta X \gamma_{BL} \psi_{ij}[r_{BL,(i,j)} - r_{BL,(i,j-1)}] \theta_{ij}$$

$$- \Delta X \gamma_{BL} \psi_{ij}[r_{BL,(i,j)} - r_{BL,(i-1,j)}] \theta_{ij}$$

Table 5: Formulation in the case of a polar curvilinear coordinate system for the boundary layer in the physical domain, $D$, along the $x$-line.

<table>
<thead>
<tr>
<th>Polarization along $x$-line for boundary layer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{ij} = 0$</td>
</tr>
<tr>
<td>$x_{ij} = \left[ r_i + r_{BL,ij} \right] \theta_{ij}$</td>
</tr>
<tr>
<td>$x_{ij} - x_{ij-1} = r_{BL,ij} \theta_{ij} - r_{BL,ij-1} \theta_{ij-1}$</td>
</tr>
<tr>
<td>$x_{ij} - x_{ij-1} = r_{BL,ij} - r_{BL,ij-1}$</td>
</tr>
</tbody>
</table>

$$1 \leq x \leq N, \quad i = 1, 2, 3, \ldots, N_x, \quad j = 1, 2, 3, \ldots, N_t.$$
Finally, Eqs. (21) and (22) are illustrated to calculate accurate line spacing in the computational domain in the case of the boundary layer by using the defined variables in Table 6.

3. Application

This is applied to gridline transformation that is analyzed in the spiral case, which is called a physical domain in the Francis turbine type of hydropower plant. This physical domain, $D$, is picked up from a real project in China, with two Francis turbine units, the essential data of which are listed in Table 7.

Figure 5 and the data through Table 8 show that the diameter of this existing domain is being gradually decreased in the present non-prismatic section by angular routing around the turbine runner.

The mathematical formula is easily obtained using MATLAB as the third degree polynomial fit equation, $D_{\text{mm}} = a + b \cdot \theta + c \cdot \theta^2 + d \cdot \theta^3$, with $a = 1220.7$, $b = -1.95$, $c = -4.33E - 04$, $d = -7.98E - 06$, Standard Error, $S = 4.65$, and correlation coefficient, $r = 0.999$. The corresponding graph is shown in Figure 6.

Control functions, $\Phi$, $\psi$, are estimated by FORTRAN codes for the current gridline transformation method.

Professional computer programming could find the different polar angles, $\theta$.

As there is a high Reynolds number in this structure, and water is being passed, with high velocity, through this non-prismatic section, it is quite correct to neglect the influence of the boundary layer and its estimation for this real condition.

As stated in [14], a new definition by $\alpha\gamma$-grids is expressed. The value of $k_1$ must be satisfied for the grid generation under $\alpha\gamma$-grids in the interval (0, 1). Thompson et al. [6,7] show the corresponding values of $k_1$ for each gridline at different angles, $\theta$. Routing is being performed through direction from the inner boundary (at nearest y-lines from the turbine runner) to the outer y-line coordinates.

The gridline transformation for a spiral case in this model could be broken down into two categories.

Firstly: the value of $K_1$ is simply illustrated in Figure 7 for the range of small values of $\Delta\theta$, [0°, 4°]. This graph shows that the quality of cells on the physical plane is suddenly enhanced from the y-line number 1 to $N_2 = 12$. However, for the slicing area, $\theta$ is from 30° to 130°, the value of $K_1$ looks to be almost negative, and could not be satisfied by the essential assumption of the nearly uniform cell area and line spacing or $\alpha\gamma$-grids.

Secondly: other existing situations are at higher values of $\Delta\theta$, [4°, 20°], which are also imaginable in this region. Graphs that are represented, corresponding to $K_1$ values, in Figure 8 are three pre-selected $\Delta\theta$ equal to 10, 14 and 20 on the physical domain. By considering these outcomes, Figure 8 potentially
Table 8: Designed dimensions in Figure 5, in the spiral case, with increment in the angle, $\theta$ (degrees), related to decreasing radius, $r$ (mm).

<table>
<thead>
<tr>
<th>$\theta$ (degrees)</th>
<th>$R_1$ (mm)</th>
<th>$R_2$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2300</td>
<td>2929.7</td>
</tr>
<tr>
<td>30</td>
<td>2277</td>
<td>2878.6</td>
</tr>
<tr>
<td>60</td>
<td>2250</td>
<td>2823.3</td>
</tr>
<tr>
<td>90</td>
<td>2224</td>
<td>2765.6</td>
</tr>
<tr>
<td>120</td>
<td>2195</td>
<td>2703.7</td>
</tr>
<tr>
<td>150</td>
<td>2161</td>
<td>2635.7</td>
</tr>
<tr>
<td>180</td>
<td>2126</td>
<td>2562.2</td>
</tr>
<tr>
<td>210</td>
<td>2084</td>
<td>2479.8</td>
</tr>
<tr>
<td>240</td>
<td>2034</td>
<td>2385.1</td>
</tr>
<tr>
<td>270</td>
<td>1973</td>
<td>2273</td>
</tr>
<tr>
<td>300</td>
<td>1930</td>
<td>2197.6</td>
</tr>
<tr>
<td>330</td>
<td>1914</td>
<td>2172</td>
</tr>
</tbody>
</table>

Figure 8: Routing along $y$-coordinate in the spiral case, while $\Delta \theta = 10^\circ$, $14^\circ$ and $20^\circ$.

shows that there is only a minor change in $K_1$ for different $\Delta \theta$, at this range of $\Delta \theta$. Thus, it is reasonable to state that using $\Delta \theta = 4$ is the accurate conclusion from the results of computation by the existing method, compared to other values for $\Delta \theta$ on physical domain, $D$.

4. Conclusion

It is of significant merit to utilize these outcomes, in order to present presentable accuracies of calculation in cases of a real circular boundary, by challenging the system involved with transient flow, such as in the turbine spiral case, and the worst load rejection, in addition, whilst hydraulic systems are being influenced by the effect of other kinds of hydraulic parameter.

Acknowledgments

This paper is potentially supported by the departments of Water Conservancy and Hydropower Engineering, Hohai University (Nanjing, China). The authors are grateful to every involved faculty member who supported us.

References


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