Error Bounds for Conditional Algorithms in Restricted Complexity Set Membership Identification

A. Garulli * B. Z. Kacewicz † A. Vicino * G. Zappa ‡

Abstract

Restricted complexity estimation is a major topic in control-oriented identification. Conditional algorithms are used to identify linear finite dimensional models of complex systems, the aim being to minimize the worst-case identification error. High computational complexity of optimal solutions suggests to employ suboptimal estimation algorithms. This paper studies different classes of conditional estimators, and provides results that assess the reliability level of suboptimal algorithms.

Keywords. Control-oriented identification, restricted complexity, set membership, linear models, projection algorithms.

1 Introduction

A considerable amount of the recent literature is focused on control-oriented identification, either in a ‘soft bound’ or in a ‘hard bound’ setting of the problem (see e.g. [1, 2, 3, 4, 5]). One of the most important topics in this area of research is restricted complexity (or reduced order) model estimation. Although there exist several settings in which this

* Dipartimento di Ingegneria dell’Informazione, Università di Siena. Via Roma 56, 53100 Siena, Italy. E-mail: garulli@ing.unisi.it, vicino@unisi.it. Tel. (39) 577-263612; Fax: (39) 577-263602. Corresponding author is Andrea Garulli.
† Department of Mathematics, Informatics and Mechanics, Warsaw University. Banacha 2, 02-097 Warsaw, Poland. E-mail: blacew@mimuw.edu.pl. Tel./Fax: (48) 22-6583296.
‡ Dipartimento di Sistemi e Informatica, Università di Firenze. Via di S. Marta 3, 50139 Firenze, Italy. E-mail: zappa@dsi.ing.unifi.it. Tel. (39) 55-4796263; Fax: (39) 55-4796363.
problem can be studied, the basic assumption is that the true system lives in the world of real systems, including infinite dimensional and possibly nonlinear systems, while the set of models within which we look for an ‘optimal’ estimate lives in a finite, possibly low dimensional linear world, whose structure and topology is compatible with modern robust control synthesis techniques. The main problem consists in finding the element in the model set whose distance from the true system is minimal.

In this paper an information based complexity perspective is taken to deal with the restricted complexity estimation problem in a worst case setting. Several classes of algorithms have been proposed in the literature ([6, 7, 8]), such as conditional central estimators, central projection estimators, restricted projection estimators, interpolatory projection estimators. Conditional central estimators are known to be optimal [6]. Although for specific choices of norms these estimators are computable in an effective way, in general their computational complexity is formidable, which prevents their use in practical applications. For this reason, suboptimal estimators have often been adopted, exhibiting lower computational complexity at the price of larger estimation errors. In spite of this, surprisingly little attention has been devoted to derive bounds on the estimation errors of these algorithms ([7, 9]). The aim of this paper is to provide a number of results in this direction. The main result regards the central projection estimator when both the estimation error and the noise are measured according to the Euclidean norm. For this case, it is shown that there exists a tight upper bound on the ratio of the error of the central projection estimator and the optimal error, i.e. the conditional radius of information of the problem.

The paper is organized as follows. Restricted complexity estimation and conditional algorithms are introduced in Section 2, together with some preliminary results in a general setting. A rather complete characterization of conditional $\ell_2$ estimates for energy bounded noise is given in Section 3. Some concluding remarks are reported in Section 4.

2 Problem formulation and general results

Let $\mathcal{X}$ and $\mathcal{Y}$ be linear finite dimensional normed spaces, with $\dim(\mathcal{X}) \leq \dim(\mathcal{Y})$, and $x$ be an unknown element in $\mathcal{X}$. Then, suppose that a set of noisy measurements $y \in \mathcal{Y}$ are available and satisfy

\begin{equation}
    y = Fx + e
\end{equation}

(1)
where \( e \in \mathcal{Y} \) is an additive noise vector and \( F : \mathcal{X} \to \mathcal{Y} \) is a known linear mapping, usually referred to as the information operator. It is assumed that \( F \) as a matrix is full rank. The noise vector is supposed to be unknown but bounded in some norm:

\[
\| e \|_\mathcal{Y} \leq \varepsilon
\]

with given \( \varepsilon > 0 \). Now, consider the following estimation problem.

**Problem (Restricted complexity estimation)** Let \( \mathcal{M} \) be a given set in \( \mathcal{X} \). Find optimal or suboptimal (in some sense) estimation algorithms \( \phi : \mathcal{Y} \to \mathcal{M} \), that provide an estimate \( z = \phi(y) \) of the unknown element \( x \in \mathcal{X} \).

In the following, these algorithms are called conditional or restricted complexity estimators; the set of all conditional estimators will be denoted by \( \Phi_{\mathcal{M}} \). In the set membership setting, the quality of the estimate provided by an algorithm \( \phi \) is usually evaluated according to the \( \mathcal{Y} \)-local worst-case estimation error, given by

\[
E(\phi) = \sup_{x \in F} \| x - \phi(y) \|_\mathcal{X}
\]

where

\[
F = F(y, \varepsilon) = \{ x \in \mathcal{X} : \| Fx - y \|_\mathcal{Y} \leq \varepsilon \}
\]

is the feasible parameter set, i.e. the set of all elements in \( \mathcal{X} \) that are compatible with the information provided by the measurements \( y \) and the noise bound \( \varepsilon \) (it will be assumed that \( F \neq \emptyset \)). The minimum error associated with the class of conditional estimators \( \Phi_{\mathcal{M}} \) is called conditional radius of information and is given by

\[
R_{\mathcal{M}} = \inf_{\phi \in \Phi_{\mathcal{M}}} E(\phi).
\]

If \( \mathcal{M} = \mathcal{X} \), we denote by \( R = R_{\mathcal{X}} \) the (unconditional) radius of information, i.e. the minimum error that can be achieved by any estimation algorithm. Obviously, \( R_{\mathcal{M}} \geq R \), for any set \( \mathcal{M} \).

A conditional estimator that achieves the minimum error \( R_{\mathcal{M}} \) is said to be conditionally optimal. Moreover, if an algorithm \( \phi' \) satisfies the inequality

\[
E(\phi') \leq k R_{\mathcal{M}} \quad \forall y,
\]

for some fixed \( k \in \mathbb{R}, k > 1 \), then \( \phi' \) is said to be \( k \)-almost optimal (or optimal within a factor \( k \)).
It is useful to recall that the Chebyshev center of a set \( S \subset \mathcal{X} \) is defined as
\[
c(S) = \arg \inf_{x \in \mathcal{X}} \sup_{s \in S} \| x - s \|_X.
\]
and \( r(S) = \sup_{s \in S} \| c(S) - s \|_X \) is called Chebyshev radius of the set \( S \). The diameter of \( S \) is given by \( d(S) = \max_{x, w \in S} \| x - w \|_X \), and \( d(S) \leq 2r(S) \). Analogously, given two sets \( S, Z \subset \mathcal{X} \), the conditional Chebyshev center of \( S \), with respect to \( Z \), is given by
\[
c_Z(S) = \arg \inf_{z \in Z} \sup_{s \in S} \| z - s \|_X.
\]
and \( r_Z(S) = \sup_{s \in S} \| c_Z(S) - s \|_X \) denotes the conditional Chebyshev radius of \( S \).

Now, let us introduce some algorithms, usually employed in restricted complexity estimation problems.

**Definition 1** A conditional central estimator is defined as \( \phi_c(y) = z_{cc} \) where
\[
            z_{cc} = \arg \inf_{z \in \mathcal{M}, x \in \mathcal{F}} \| z - x \|_X = c_M(\mathcal{F}).
\]

**Definition 2** An interpolatory projection estimator is defined as \( \phi_i(y) = z_{ip} \) where
\[
            z_{ip} = \arg \inf_{z \in \mathcal{M}} \| z - x_i \|_X
\]
and \( x_i \) is an element of the set \( \mathcal{F} \).

**Definition 3** A central projection estimator is given by \( \phi_{cp}(y) = z_{cp} \) where
\[
            z_{cp} = \arg \inf_{z \in \mathcal{M}} \| z - c(\mathcal{F}) \|_X.
\]

**Definition 4** A restricted projection estimator is given by \( \phi_{rp}(y) = z_{rp} \) where
\[
            z_{rp} = \arg \inf_{z \in \mathcal{M}} \| Fz - y \|_Y.
\]

In the following, optimality and almost optimality of the estimation algorithms are studied in a general setting, i.e. without specifying the set \( \mathcal{F} \), the estimate set \( \mathcal{M} \) and the norms adopted in the spaces \( \mathcal{X} \) and \( \mathcal{Y} \). Since the norms are general, they will be omitted when clear from the context. First, let us recall the following property that follows from the definition of conditional central algorithm [6].
**Theorem 1** The conditional central algorithm \( \phi_{cc} \) is conditionally optimal in the class of conditional estimators \( \Phi_{\mathcal{M}} \)

\[
E(\phi_{cc}) \leq E(\phi) \quad \forall y, \forall \phi \in \Phi_{\mathcal{M}}.
\]

Hence, the conditional central algorithm provides the minimum estimation error (3), among all the conditional estimators. Notice that, from the definitions of conditional and unconditional radius of information, we get respectively \( R_{\mathcal{M}} = r_{\mathcal{M}}(\mathcal{F}) = E(\phi_{cc}) \) and \( R = r(\mathcal{F}) \). The aim of the following theorems is to characterize the quality of the estimates provided by the algorithms introduced by Definitions 2-4, in terms of the quantities \( R \) and \( R_{\mathcal{M}} \).

**Theorem 2** Any interpolatory projection estimator satisfies

\[
E(\phi_{ip}) \leq R_{\mathcal{M}} + 2R.
\]

*Proof.* Let \( x \in \mathcal{F} \) and \( z \in \mathcal{M} \). From Definition 2, one has \( \| z_{ip} - x \| \leq \| z_{ip} - x_i \| + \| x_i - x \| \leq \| z - x_i \| + d(\mathcal{F}) \leq \sup_{w \in \mathcal{F}} \| z - w \| + 2R \). The result is obtained by maximizing over \( x \in \mathcal{F} \) and minimizing over \( z \in \mathcal{M} \).

**Theorem 3** Any central projection estimator satisfies

\[
E(\phi_{cp}) \leq R_{\mathcal{M}} + 2R.
\]

*Proof.* Let \( x \in \mathcal{F} \) and \( z \in \mathcal{M} \). From Definition 3, one has \( \| z_{cp} - x \| \leq \| z_{cp} - c(\mathcal{F}) \| + \| c(\mathcal{F}) - x \| \leq \| z - c(\mathcal{F}) \| + \| c(\mathcal{F}) - x \| \leq \| z - x \| + 2\| c(\mathcal{F}) - x \|. \) The result is obtained by maximizing over \( x \in \mathcal{F} \) and minimizing over \( z \in \mathcal{M} \).

Notice that the previous result cannot be obtained from Theorem 2, since the Chebishev center of a set does not necessarily belong to the set itself (i.e., \( \phi_{cp} \) is not an interpolatory projection algorithm, in general). When this occurs, the bound on the worst-case performance of the central projection algorithm can be improved.

**Corollary 1** If \( c(\mathcal{F}) \in \mathcal{F} \), then

\[
E(\phi_{cp}) \leq R_{\mathcal{M}} + R.
\]

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Proof. The result immediately follows from the proof of Theorem 3, by noting that 
\[ \| z_{cp} - c(\mathcal{F}) \| \leq R_M. \]

Since \( R \leq R_M \), the above results can be restated in terms of “almost optimality” of the estimation algorithms. In particular, interpolatory projection and central projection algorithms are optimal within a factor 3. Moreover, if \( c(\mathcal{F}) \in \mathcal{F} \), the central projection algorithm is optimal within a factor 2. In general, it is not possible to give a bound on the estimation error provided by the restricted projection algorithm \( \phi_{rp} \). In Sect. 3, it will be shown that it can not be found a \( k \) such that this algorithm is \( k \)-almost optimal.

3 Conditional \( \ell_2 \) estimators with energy bounded noise

In several identification problems, linear parameterization of the estimated model is required, i.e. the set \( \mathcal{M} \) is supposed to be a linear subset of \( \mathcal{X} \). Let \( \mathcal{X} = \mathbb{R}^n \), \( \mathcal{Y} = \mathbb{R}^m \), with \( m \geq n \), and make the following assumptions.

(A1). The set \( \mathcal{M} \) is a \( p \)-dimensional linear manifold in \( \mathbb{R}^n \), with \( p < n \), i.e. \( \mathcal{M} = \{ z \in \mathbb{R}^n : z = z^o + M\alpha \} \), where \( M \in \mathbb{R}^{n \times p} \) is full rank.

(A2). \( \| \cdot \|_x = \| \cdot \|_y = \ell_2 \).

Assumption (A2) means that energy-bounded noise is considered in (2) and the estimation error (3) is measured in the energy norm. For ease of notation, the \( l_2 \) norm will be denoted by \( \| \cdot \| \) in the following.

According to the assumptions (A1) and (A2), the feasible parameter set (4) is given by the ellipsoid

\[ \mathcal{F} = \{ x \in \mathbb{R}^n : x'F'Fx - 2y'Fx + y'y \leq \varepsilon^2 \}. \]  

and the estimates \( z_{ip}, z_{cp}, z_{rp} \) in (8), (9) and (10) can be easily computed as

\[ z_{ip} = z^o + M(M'M)^{-1}M'(x_i - z^o) \]  
\[ z_{cp} = z^o + M(M'M)^{-1}M'(F'F)^{-1}F'(y - Fz^o) \]  
\[ z_{rp} = z^o + M(M'M)^{-1}M'(F'F)^{-1}F'(y - Fz^o). \]

On the other hand, a difficult min-max optimization problem must be solved in order to compute the conditional central estimate \( z_{cc} \) in (7). A complete characterization of the conditional center of \( \mathcal{F} \) and an efficient procedure for computing it has been recently proposed in [10].
Through a suitable change of coordinates in the space $\mathcal{X}$ (see the Appendix), the restricted estimation problem can be reformulated w.l.o.g. as follows. Let $\mathcal{F}$ be the ellipsoid

$$\mathcal{F} = \mathcal{E} = \{x \in \mathbb{R}^n : x'Qx \leq 1\}$$

and $\mathcal{M}$ be the linear manifold

$$\mathcal{M} = \{z \in \mathbb{R}^n : z = z^o + M\alpha, \; \alpha \in \mathbb{R}^p, \; p < n\}$$

Then $c(\mathcal{E}) = 0$, and the estimates (12)-(14) provided by the interpolatory projection, central projection and restricted projection algorithms are given by

$$z_{ip} = z^o + MM'x_i$$

$$z_{cp} = z^o$$

$$z_{rp} = [I_n - M(M'QM)^{-1}M'Q]z^o.$$ (21)

In the following, the evaluation of the reliability level of the above estimates is addressed. This is done by computing tight upper bounds on the worst-case estimation error of each conditional projection algorithm.

**Theorem 4** Let assumptions (A1)-(A2) hold. Then, the central projection algorithm $\phi_{cp}$ satisfies

$$E(\phi_{cp}) \leq \sqrt{\frac{4}{3}R_M}.$$ (22)

Moreover, there exist estimation problems in which the upper bound is achieved.

**Proof.** First, let us show that one can assume w.l.o.g. $p = \dim(\mathcal{M}) = n - 1$. In fact, suppose $p < n - 1$ and denote by $\mathcal{M}_n$ the $(n - 1)$-dimensional linear manifold containing $z_{cp} = z^o$ and orthogonal to $z^o$. By definition, the estimate provided by $\phi_{cp}$ and the error $E(\phi_{cp})$ do not change if we consider $\mathcal{M}_n$ instead of $\mathcal{M}$. Moreover, $\mathcal{M} \subset \mathcal{M}_n$ and hence $R_{\mathcal{M}_n} \leq R_{\mathcal{M}}$. Therefore, the maximum ratio between $E(\phi_{cp})$ and the conditional radius of information is always achieved for an $(n - 1)$-dimensional manifold.

Consider the setting (15)-(18) and define

$$x^o = \arg\max_{x \in \mathcal{E}} \|z^o - x\|.$$ (22)
Denote by $z^+$ and $z^-$ the orthogonal projection of respectively $x^\alpha$ and $-x^\alpha$ onto $\mathcal{M}$. Set $\alpha = \arccos(|z^+ - z^-|/\|x^\alpha - z^-\|)$ and

$$z_m = z^- + \frac{1}{2 \cos^2 \alpha} (z^+ - z^-),$$

for $0 \leq \alpha < \frac{\pi}{2}$ (the case $\alpha = \frac{\pi}{2}$ corresponds to $z^+ = z^- = z^\alpha$ and is trivial). It is easy to verify that $z_m$ satisfies $\|z_m - x^\alpha\| = \|z_m - z^-\|$. Then, define $h = \|x^\alpha - z^+\|$, $l = \|x^\alpha - z_m\|$, $d = \|x^\alpha - z^\alpha\|$ (a sketch of the above situation is depicted in Fig. 1, for the case $n = 3, p = 2$). Now, let us consider the set $\tilde{\mathcal{E}} = \{z^-, x^\alpha\}$, containing only two points. By construction, the projection of its Chebyshev center onto $\mathcal{M}$ is $z^\alpha$ and hence (by definition of $x^\alpha$) the error $\tilde{E}(\phi_{cp})$ is the same for both $\mathcal{E}$ and $\tilde{\mathcal{E}}$, and is given by $d$. Moreover, the conditional Chebyshev center of $\tilde{\mathcal{E}}$ is given by

$$\tilde{z}_{cc} = \begin{cases} z_m & \text{if } 0 \leq \alpha \leq \frac{\pi}{4} \\ z^+ & \text{if } \frac{\pi}{4} < \alpha \leq \frac{\pi}{2} \end{cases}$$

and the corresponding conditional radius is

$$\tilde{R}_\mathcal{M} = \begin{cases} l & \text{if } 0 \leq \alpha \leq \frac{\pi}{4} \\ h & \text{if } \frac{\pi}{4} < \alpha \leq \frac{\pi}{2} \end{cases}.$$

Then, by noting that $-x^\alpha \in \mathcal{E}$, one has

$$R_\mathcal{M} \geq \max\{\|z_{cc} - x^\alpha\|, \|z_{cc} + x^\alpha\|\} \\ \geq \max\{\|z_{cc} - x^\alpha\|, \|z_{cc} - z^-\|\} \\ \geq \min_{z \in \mathcal{M}} \max\{\|z - x^\alpha\|, \|z - z^-\|\} = \tilde{R}_\mathcal{M}.$$
Hence,
\[
\frac{E(\phi_{cp})}{R_M} \leq \frac{E(\phi_{ip})}{R_M} = g(\alpha) = \begin{cases} \frac{d}{h} = \cos \alpha \sqrt{4 - 3 \cos^2 \alpha} & \text{if } 0 \leq \alpha \leq \frac{\pi}{4} \\ \frac{d}{h} = \frac{\sqrt{4 \tan^2 \alpha + 1}}{2 \tan \alpha} & \text{if } \frac{\pi}{4} < \alpha \leq \frac{\pi}{2} \end{cases}.
\]

Finally, standard calculations lead to
\[
\max_{0 \leq \alpha \leq \frac{\pi}{2}} g(\alpha) = \sqrt{\frac{4}{3}}
\]
and the maximum is achieved for \(\alpha = \arccos \sqrt{\frac{2}{3}}\). This means that the bound is tight. \(\square\)

**Remark 1** The above result can be extended to a more general class of sets. It can be observed that Theorem 4 still holds for a compact set \(\mathcal{E}\) such that \((-x^0 + \gamma z^0) \in \mathcal{E}\), for some \(\gamma \in \mathbb{R}\). In fact, the projection of \((-x^0 + \gamma z^0)\) onto \(\mathcal{M}\) is again \(z^-\) and the same reasoning follows. In particular, this means that Theorem 4 holds for all compact balanced sets.

**Theorem 5** Let assumptions (A1)-(A2) hold. Then, the interpolatory projection algorithm \(\phi_{ip}\) satisfies
\[
E(\phi_{ip}) \leq R_M + R
\]
and there exist estimation problems in which the upper bound is achieved.

**Proof.** Let us consider again the setting (15)-(18). Denote by \(e^+\) and \(e^-\) the extremum points of \(d(\mathcal{E})\) (the diameter of \(\mathcal{E}\)), and for any \(z \in \mathcal{M}\), let \(\alpha\) be the angle between the vectors \(z\) and \(e^+\). One has
\[
\max_{x \in \mathcal{E}} \|z - x\| \geq \max \{\|z - e^+\|, \|z - e^-\|\}
= \sqrt{\|z\|^2 + r^2(\mathcal{E}) + 2 \|z\| r(\mathcal{E}) \cos \alpha}
\geq \sqrt{\|z^0\|^2 + r^2(\mathcal{E})},
\]
where \(r(\mathcal{E})\) is the length of the maximal semiaxis of \(\mathcal{E}\). Minimizing over \(z \in \mathcal{M}\) one obtains
\[
R_M \geq \sqrt{h^2 + R^2} \quad (23)
\]
where \(h = \|z^0\|\) and \(R = r(\mathcal{E})\). On the other hand, from (19) one gets for any \(x \in \mathcal{E}\)
\[
\|\tilde{z}_{ip} - x\| \leq \|z^0 + MMM^* x_i\| + \|x\|
= \sqrt{\|z^0\|^2 + \|MMM^* x_i\|^2 + \|x\|}
\leq \sqrt{\|z^0\|^2 + \|x_i\|^2 + \|x\|},
\]
and then, recalling that \( x_i \in \mathcal{E} \) by definition, and maximizing over \( x \in \mathcal{E} \)
\[
E(\phi_{ip}) \leq \sqrt{h^2 + R^2} + R \leq R_M + R. \tag{24}
\]
Moreover, it can be easily seen that the upper bound is achieved when \( \mathcal{M} \) contains the
diameter of the ellipsoid \( \mathcal{E} \), and \( x_i \) is one of the extremum points of the diameter. \( \square \)

**Remark 2** Theorem 5 can be extended to a more general class of sets. In particular, the
above proof still holds for any balanced set \( \mathcal{E} \).

Now, the restricted projection algorithm introduced in Definition 4 is considered. The
following straightforward proposition gives a simple geometric interpretation of the estimate \( z_{rp} \) and provides an upper bound on the estimation error of the restricted projection
algorithm, in the case when there are conditional estimates that belong to the feasible set \( \mathcal{E} \).

**Proposition 1** Let (A1) and (A2) hold. If \( \mathcal{E} \cap \mathcal{M} \neq \emptyset \), then

(i) \( z_{rp} \) in (21) is the symmetry center of the set \( \mathcal{E} \cap \mathcal{M} \);

(ii) \( E(\phi_{rp}) \leq 2R. \)

However, the following theorem shows that it is not possible to determine a bound on
the estimation error of \( \phi_{rp} \) in the general case.

**Theorem 6** Let (A1) and (A2) hold. For any \( k > 0 \), there exists a restricted complexity
estimation problem such that
\[
E(\phi_{rp}) > k \ R_M.
\]

**Proof.** Point (i) is just a corollary of Proposition 1. In order to prove point (ii), let us set
set \( n = 2, \ p = 1 \). Then consider the ellipsoid \( \mathcal{E} \) in (15) with matrix \( Q = Q' \) (generally
nondiagonal), and the manifold \( \mathcal{M} \) in (17) with \( z' = [-1 \ 0]' \) and \( M = [0 \ 1]' \). Let \( \lambda_1 = 1 \) and \( \lambda_2 > 1 \) be the eigenvalues of \( Q \), and \( \alpha \in (0, \frac{\pi}{2}) \) be the angle between the maximal
semiaxis of \( \mathcal{E} \) (of length 1) and \( M \) (see Fig. 2). Moreover, let \( \lambda_2 = \lambda_2(\alpha) \) be parameterized
with \( \alpha \). Rotating \( \mathcal{E} \) around its center so that \( \alpha \to 0 \), for \( \lambda_2(\alpha) \to +\infty \) sufficiently fast
one gets
\[
\begin{align*}
    z_{cc} &\to z' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
    R_M &\to \sqrt{\frac{1}{\lambda_1^2} + \|z'\|^2} = \sqrt{2}
\end{align*}
\]
Since by (21) \( z_{rp} \) is the projection of 0 onto \( \mathcal{M} \) in the weighted norm \( \| x \|^2_Q = x'Qx \), one also has \( \| z_{rp} \| \to +\infty \) and \( E(\phi_{rp}) \to +\infty \). Hence \( \lim_{\alpha \to 0} \frac{E(\phi_{cp})}{R_M} = +\infty \).

Since in Theorem 4 it has been proven that the central projection algorithm is \( \sqrt{4/3} \)-almost optimal, one could conjecture that the estimate provided by the algorithm \( \phi_{cp} \) is always “better” than that of the algorithm \( \phi_{rp} \). This is not true, as it is shown in the following example.

**Example 1** Consider the ellipsoid \( E \) defined by (15) with \( Q = \text{diag}\{0.05, 0.25, 2.5\} \), and the linear set \( \mathcal{M} \) in (17), where \( z^o = [-0.47, -0.1, 0.94]' \), \( M = v /\|v\|_2 \) with \( v = [-0.17, 1.269, 0.05]' \). By computing the conditional central estimate \( z_{cc} \), following the procedure described in [10], and the estimates \( z_{cp} \) and \( z_{rp} \) according to (20)-(21), one obtains the following estimation errors

\[
\begin{align*}
E(\phi_{cc}) &= 25.0691, \\
E(\phi_{cp}) &= 25.3368, \\
E(\phi_{rp}) &= 25.1121.
\end{align*}
\]

which disproves the conjecture \( E(\phi_{cp}) \leq E(\phi_{rp}). \)

\[ \square \]

**4 Conclusions**

A rather complete characterization of the reliability level of conditional estimation algorithms in set membership identification has been provided. Computing tight upper
bounds on the worst-case estimation error is important for evaluating the degree of sub-
optimality of conditional estimators and allows to assess model quality. For the case when
energy-bounded noise is considered, it is shown that central projection and interpolatory
projection algorithms are almost optimal within a factor $\sqrt{4/3}$ and 2 respectively, while
the estimation error of restricted projection algorithms may be unbounded. Pointwise
bounded noise and different error norms will be the subject of future research.

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Appendix

For clarity of exposition, denote the new coordinates in (15)-(18) by $\tilde{x}$, $\tilde{z}$. Let us define

$$P = [\varepsilon^2 - y' F(F^t F)^{-1} F' y]^{-1} (F^t F)$$

and let $P = U Q U'$, where $Q$ is given by (16) and $U' U = I_n$. Moreover, let $x_s = (F^t F)^{-1} F' y$. Then, through the change of coordinates

$$\tilde{x} = U' (x - x_s) \quad , \quad x = U \tilde{x} + x_s$$

equation (11) becomes $\mathcal{F} = \{ \tilde{z} \in \mathbb{R}^n : \tilde{z}' Q \tilde{z} \leq 1 \}$, which is (15).

Now assume that $\mathcal{M} = \{ z \in \mathbb{R}^n : z = z^o + M \alpha , \quad \alpha \in \mathbb{R}^p , \quad p < n \}$, with $M' M = I_p$ and $M' z^o = 0$. In the new coordinates, one has

$$\tilde{z} = U' (z^o - x_s) + U' M \alpha = \tilde{z}^o + \tilde{M} \tilde{\alpha} , \quad \tilde{\alpha} \in \mathbb{R}^p$$

where $\tilde{z}^o = U' (z^o - x_s + M M' x_s)$, $\tilde{M} = U' M$ and $\tilde{\alpha} = \alpha - M' x_s$. It is easy to verify that $\tilde{M}' \tilde{M} = I_p$ and $\tilde{M}' \tilde{z}^o = 0$, which are the conditions (18).