Conditional Central Algorithms for Worst-Case Set Membership Identification and Filtering

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Abstract—This paper deals with conditional central estimators in a set membership setting. The role and importance of these algorithms in identification and filtering is illustrated by showing that problems like worst-case optimal identification and state filtering, in contexts where disturbances are described through norm bounds, are reducible to the computation of conditional central algorithms. The conditional Chebishev center problem is solved for the case when energy norm bounded disturbances are considered. A closed form solution is obtained in terms of finding the unique real root of a polynomial equation in a semi-infinite interval.

Keywords—Identification, filtering, set membership, bounded disturbances, conditional estimation.

I. INTRODUCTION

Identification for robust control has assumed a key role in the past few years. The basic objective of this research area is to design identification schemes providing variables tailored to the robust control design techniques developed in the last two decades. The essential feature of these methods is to construct a mixed model of uncertainty, consisting of a parametric model and an associated nonparametric perturbation. Loosely speaking, the parametric part accounts for the frequency behavior of the system within the control bandwidth, while the nonparametric part accounts for the unmodelled dynamics always present in any model of a real system. The objective is to derive an estimate of the parametric model and of the magnitude of the associated nonparametric perturbation, such that the real system is guaranteed to belong to the set of admissible plants associated to the estimated model.

First efforts along this line of research can be found in [1], [2], [3]. Both $\mathcal{H}_\infty$ and $\ell_1$ error criteria have been adopted (see [4], [5], [6] for more extensive lists of references). The main problem with these approaches is the overparameterization of the estimated models, generally of high order, with a number of parameters larger or equal to the number of data. Obviously, this fact makes synthesis control techniques like $\mathcal{H}_\infty$ or $\ell_1$ more difficult to be used in practical applications. Moreover, for a number of additional reasons, a “parsimonious” model of the system is more acceptable in real situations. Easy interpretation of the dominant system dynamics, limited complexity of the synthesized controller and tractability of the robustness problem for the overall control system are only some of the reasons why simple models are used in common practice.

These reasons motivate the use of restricted (or reduced) complexity models, which have been investigated in many contributions, either in a “hard bound” context or in a “soft bound” setting of the identification problem [7], [8]. For a given model parameterization, the problem consists in choosing an estimate in a subset of the parameter space with a prescribed structure, and an associated bound on the nonparametric perturbation. The simplest example of this situation is when the estimate of the unknown variable is constrained to belong to a linear manifold in the parameter space.

In this paper an approach founded on Information Based Complexity (IBC) [9], [10] with set membership description of uncertainty is taken. A linear framework is considered. The purpose is to characterize restricted complexity optimal estimators in a worst-case setting and to evaluate the associated nonparametric estimation error. Reduced complexity estimators are called conditional algorithms in the IBC literature [11]. Conditional algorithms deserve special attention also for the role they play in the important area of state estimation with norm bounded disturbances. In fact, it can be shown that the optimal filtering problem in a set membership setting can be reduced to the computation of a conditional central algorithm (see [12]). Unfortunately, these algorithms are difficult to compute, except for some special situations (see e.g. [11]). In several papers, suboptimal conditional estimators have been proposed for the problem of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ estimation ([8], [13], [14]). Although in many cases suboptimal estimators provide acceptable results, the problem of quantifying how much their worst-case error differs from the optimal one is at a large extent unsolved (see [14], [15] for preliminary results).

The main objective of this paper is to characterize the solution of the conditional optimal estimation problem and provide an efficient procedure for the computation of the conditional center of the feasible problem element set, i.e., the set of all problem elements compatible with the a priori assumptions and the measured data, for the case when disturbances are bounded according to the energy norm. For this case, it turns out that the conditional optimal estimator can be derived in closed form, as a function of the unique real solution of an algebraic equation in a semi-infinite interval.

The paper is organized as follows. Section 2 illustrates the problem formulation and shows how two important estimation problems, namely worst-case identification and set membership state filtering, lead naturally to conditional optimal estimators. Section 3 provides the main results on conditional central estimators and a procedure for comput-
ing the conditional center of an ellipsoidal set. In Section 4 numerical simulations on worst-case system identification are shown, while concluding remarks are reported in Section 5.

The notation in the paper is standard: ′ denotes transpose, $I_n$ is the $n \times n$ identity matrix, $|| \cdot ||_X$ is the norm adopted in the normed space $X$. The linear subspace spanned by the columns of $A$ is denoted by $\mathcal{R}(A)$. If $\mathcal{A}$ is a set, $\delta \mathcal{A}$ denotes its boundary.

II. Problem formulation

Consider the following linear set-membership estimation setting [10]. Let $X$ and $Y$ be linear normed spaces, and

$$ y = Fx + e, $$

where $y \in Y$ is the measurement vector, $x \in X$ is the unknown problem element that must be estimated, and $e$ is the error vector, which is supposed to be unknown but bounded in some norm

$$ ||e||_Y \leq \varepsilon. $$

Let $\mathcal{M} \subset X$ be a $p$-dimensional linear manifold. The aim is to find a restricted complexity algorithm (or estimator) $\phi : Y \rightarrow \mathcal{M}$, providing an estimate $z = \phi(y) \in \mathcal{M}$ of the problem element $x$. The class of restricted complexity (or conditional) estimators will be denoted by $\Phi^R$.

The quality of the estimate is measured according to the worst-case $Y$-local error

$$ E_y(\phi) = \sup_{x \in FPS_y} ||\phi(y) - x||_X $$

where

$$ FPS_y = \{x \in X : ||Fx - y||_Y \leq \varepsilon\} $$

is the feasible problem element set, i.e. the set of all the problem elements, compatible with the available measurement $y$ and the error bound.

Let us now introduce the notion of conditional Chebyshev center of a set. Given two sets $S, Z \subseteq X$, the center of $S$, “conditioned” to $Z$, is defined as

$$ c_Z(S) = \arg\inf_{z \in Z} \sup_{x \in S} ||x - z||_X. $$

Hence, the conditional central algorithm

$$ \phi_{cc}(y) = c_{\mathcal{M}}(FPS_y) $$

enjoys the following property [11].

Proposition 1: $\phi_{cc}$ is $Y$-locally optimal in the class of conditional estimators $\Phi^R$

$$ E_y(\phi_{cc}) \leq E_y(\phi) \quad \forall y, \forall \phi \in \Phi^R. $$

The optimal error $E_y(\phi_{cc})$ is called conditional radius of information.

When $X = \mathbb{R}^n$ and energy-bounded errors are of concern, the set $FPS_y$ is a $n$-dimensional ellipsoid. Therefore, the conditional central algorithm $\phi_{cc}$ requires the computation of the Chebyshev center of an ellipsoid, conditioned to belong to a linear manifold.

Before we attempt to characterize the solution of this problem, we show two examples of restricted complexity set-membership estimation, in which the algorithm $\phi_{cc}$ plays an important role.

A. Set-membership worst-case identification

Consider the linear, time-invariant, discrete-time, SISO system, described by the input/output relationship

$$ y_k = \sum_{i=0}^{\infty} h_i u_{k-i}. $$

It is assumed that the impulse response sequence $h = \{h_i\}_{i=0}^{\infty}$ belongs to a linear normed space $\mathcal{H}$, equipped with the norm $|| \cdot ||_\mathcal{H}$. An identification experiment consists in collecting $N + 1$ noisy input/output pairs $\{(u_k, y_k), k = 0, 1, \ldots, N\}$, related by

$$ y_k = \sum_{i=0}^{k} h_i u_{k-i} + v_k $$

where $v_k$ is the disturbance affecting the $k$-th measurement, and it has been assumed that $u_k = 0$, for $k < 0$. The relationship (8) can be rewritten as

$$ y^N = U_N h^N + v^N $$

where

$$ y^N = [y_0 \; y_1 \; \ldots \; y_N]' $$

$$ h^N = [h_0 \; h_1 \; \ldots \; h_N]' $$

$$ v^N = [v_0 \; v_1 \; \ldots \; v_N]' $$

$$ U_N = \begin{pmatrix} u_0 & 0 & \ldots & 0 \\ u_1 & u_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_N & u_{N-1} & \ldots & u_0 \end{pmatrix}. $$

The aim of the identification experiment is to estimate the unknown impulse response $h$ on the basis of measurements and suitable a priori information. A priori information is necessary in order to bound the set of feasible systems, because measurements provide information only on the first $N + 1$ samples of the system impulse response. More specifically, we assume that $v^N$ is bounded in some norm, $||v^N||_Y \leq \varepsilon$, and the unknown impulse response $h$ belongs to a given subset $K$ of $\mathcal{H}$. It is convenient to introduce the truncation operator $T_N$ in $\mathcal{H}$, as $T_N : h \rightarrow h^N$. Hence, $(I - T_N)h$ denotes the tail of $h$ and $T_N(K)$ is the set of truncated sequences obtained by applying the operator $T_N$ to every element of $K$. A mild assumption on $K$ adopted here, according to most of the literature, is that $(I - T_N)(K)$ is a bounded balanced set. Moreover, in order to get rid of unnecessary mathematical complications, it is assumed that a priori information concerns only the tail
of the impulse response sequence, i.e. $T_NK = T_NH$ (see e.g. [8]).

Now, define the Feasible System Set $FSS_y$ corresponding to the measurements \{w^N, y^N\}, i.e. the set of impulse response sequences compatible with the measurements and the a priori information, as

$$FSS_y = \{h \in K : \|y^N - U_nh^N\|_Y \leq \varepsilon\}. \quad (9)$$

Notice that $FSS_y$ is bounded if and only if $u_0 \neq 0$. We look for an estimate of $h$ in a restricted class of linearly parameterized models $\mathcal{M}$

$$\mathcal{M} = \{h : h = M_h\alpha, \alpha \in \mathbb{R}^p\} \quad (10)$$

where $M_h$ is a linear operator, $M_h : \mathbb{R}^p \rightarrow \mathcal{H}$. $\mathcal{M}$ is a linear manifold of $\mathcal{H}$ and $\alpha$ is the parameter vector to be identified. In system identification theory, the basis of the parameterized models is typically chosen as a collection of impulse responses of linear filters, such as Laguerre, Kautz, and other orthonormal functions (see e.g. [16], [17]).

In the above context, an estimator is a mapping $\phi$ from $y^N$ to the parameter space $\mathbb{R}^p$. For a given $\phi$, the local worst-case error is defined as

$$E_y(\mathcal{M}, \phi) = \sup_{h \in FSS_y} \|h - M_h\phi(y^N)\|_\mathcal{H} \quad (11)$$

and it accounts for the maximum “gap” between the selected model and the set of feasible systems.

The local error (11) depends on the choice of the model class $\mathcal{M}$. The following result says that, for many norms, it is convenient to select $\mathcal{M}$ such that $\mathcal{M} \subset T_NH$ (see [8], [18]).

**Proposition 2:** Assume that the set $(I - T_N)(K)$ is balanced and that the norm $\| \cdot \|_\mathcal{H}$ satisfies

$$\|h\|_\mathcal{H}^r = \|T_Nh\|_\mathcal{H}^r + \|(I - T_N)h\|_\mathcal{H}^r \quad (12)$$

for some positive integer $r$. Then

$$E_y(T_N\mathcal{M}, \phi) \leq E_y(\mathcal{M}, \phi) \quad \forall \phi, \forall y.$$ 

Notice that assumption (12) is satisfied for every $\ell_q$ norm, $1 \leq q < \infty$. On the contrary, for the $\mathcal{H}_\infty$ norm

$$\|h\|_{\mathcal{H}_\infty} = \sup_{\omega \in [0, 2\pi]} \sum_{k=0}^{\infty} h_ke^{-j\omega k},$$

the R.H.S. of (12) is only an upper bound (for $r = 1$).

In the following discussion, we will assume $\mathcal{M} \subset T_NH$, i.e.

$$\mathcal{M} = \{h^N \in \mathbb{R}^N : h^N = M\alpha, \alpha \in \mathbb{R}^p, p < N\}$$

where $M \in \mathbb{R}^{N \times p}$ defines the chosen $p$th-order model class $\mathcal{M}$. Then, if Proposition 2 applies, the worst-case error is given by

$$E_y(\mathcal{M}, \phi) = \left( \sup_{h \in T_N(FSS_y)} \|h^N - M\phi(y^N)\|_\mathcal{H}^r + \sup_{h \in K} \|(I - T_N)h\|_\mathcal{H}^r \right)^{1/r} \quad (13)$$

where $T_N(FSS_y) = \{h^N \in T_NH : \|y^N - U_nh^N\|_Y \leq \varepsilon\}$. Since the term depending on $K$ can be computed a priori, the domain of search for the optimal estimate can be restricted to the finite dimensional space $\mathbb{R}^N$. In particular, Proposition 1 guarantees that the conditional central algorithm

$$\phi_{cc}(y^N) = \arg \inf_{\alpha \in \mathbb{R}^p} \sup_{h \in T_N(FSS_y)} \|h^N - M\alpha\|_\mathcal{H}$$

provides the worst-case optimal impulse response estimate, among the reduced-order models $\mathcal{M}$. If Proposition 2 does not apply, even if $\mathcal{M} \subset T_NH$, the R.H.S. of (13) provides an upper bound of the worst-case error $E_y(\mathcal{M}, \phi)$, and $\phi_{cc}$ minimizes the value of this upper bound.

**B. Set-membership state filtering**

Consider the linear, time-varying system

$$x_{k+1} = A_kx_k + w_k \quad k = 0, \ldots, N \quad (15)$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $y_k \in \mathbb{R}^{n_y}$ the output, $w_k$ the process error and $v_k$ the measurement noise. Let $\hat{x}_0$ be a given estimate of the initial state $x_0$. According to the framework introduced above, equations (15) can be written as $y^N = F_Nx^N + e^N$, where

$$x^N = [x^N_0 \ x^N_1 \ \ldots \ x^N_N]'$$

$$y^N = [\hat{x}_0^N \ 0 \ \ldots \ 0 \ y^N_0 \ \ldots \ y^N_N]'$$

$$e^N = [(\hat{x}_0 - x^N_0)^T w^N_{1-N} \ \ldots \ v^N_0 \ \ldots \ v^N_N)'$$

$$F_N = \begin{pmatrix} I_{n_x} & 0 & \cdots \\ 0 & -I_{n_x} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ C_0 & 0 & \cdots & A_{N-1} - I_{n_x} \\ 0 & C_1 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & C_N \end{pmatrix}.$$ 

The vector $e^N$ contains all the uncertainties in the system. An upper bound on some norm of $e^N$ is assumed to be known, $\|e^N\|_Y \leq \varepsilon$.

Assume that one wants to estimate $x_k$ on the basis of $x_0, y_0, y_1, \ldots, y_k$, for each $k = 1, \ldots, N$ (filtering problem). Let $\hat{x}_{j|j}$ denote the estimate of $x_j$ on the basis of the information up to time $j$ (i.e. $y^j$), and let

$$FSS_{y^j} = \{x^j : \|F_jx^j - y^j\|_Y \leq \varepsilon\}$$

be the Feasible Filtering Set, containing the state sequences compatible with $y^j$. Then, at time $k$ the desired state sequence estimate has the form $\hat{x}^k = [\hat{x}^k_0 | \hat{x}^k_1 | \ldots | \hat{x}^k_k]'$ and can not be freely selected among all the elements of $FSS_{y^k}$. 
In fact, at time $k$ the filtered estimates $\hat{x}_{0|0} \ldots \hat{x}_{k-1|k-1}$ are fixed, since they have been already computed at previous time instants. Therefore, the new estimate $\hat{x}^k$ must belong to the set
\[
\mathcal{M}_k = \{ \{ x'_0 \ldots x'_k \} : x_i = \hat{x}_{i|i}, \ i = 0, \ldots, k-1 \}
\]
which is a $n_x$-dimensional linear manifold in $\mathbb{R}^{(k+1)n_x}$.

The set-membership state filtering problem can be summarized as follows: at each time $k$, find an estimate $\hat{x}^k \in \mathcal{M}_k$ of $x^k \in \mathcal{F} \mathcal{S} \mathcal{S}_{y^k}$. According to Proposition 1, the conditional central algorithm
\[
\hat{x}_{k|k} = \phi_{cc}(y^k) = c_{\mathcal{M}_k}(\mathcal{F} \mathcal{S} \mathcal{S}_{y^k})
\]
is the optimal set-membership state filter, according to the worst-case error defined in (3).

### III. Conditional Central Algorithm for Energy Bounded Disturbances

In this section, an efficient procedure for the computation of the conditional central algorithm is presented for the case when energy bounded disturbances are considered and errors are measured in the energy norm. This means that the $\ell_2$ norm is used in (2) and (3). In the worst-case identification context, this corresponds to the $\mathcal{H}_2$ identification problem which has been widely addressed in the recent literature (see e.g. [17], [19], [20]). For ease of notation, the $\ell_2$ norm will be denoted by $\| \cdot \|$ in the following.

Without loss of generality, the problem can be reformulated as follows, through a suitable change of coordinates.

**Problem [CCC (Conditional Chebishev Center)]:** For a given axis-aligned ellipsoid
\[
\mathcal{E} = \{ x \in \mathbb{R}^n : x^T Q x \leq 1 \} \quad (16)
\]
and a $p$-dimensional linear manifold ($p < n$)
\[
\mathcal{M} = \{ z \in \mathbb{R}^n : z = z^0 + M \alpha, \ \alpha \in \mathbb{R}^p \} \quad (18)
\]
and a $n_x$-dimensional linear manifold ($p < n$)
\[
\mathcal{M}' M = I_p \quad (19)
\]
find $z_{cc} \in \mathcal{M}$ such that
\[
z_{cc} = \arg \min_{z \in \mathcal{M}} \max_{x \in \mathcal{E}} \| x - z \|^2. \quad (21)
\]

If the distance of a point $z$ from a set $\mathcal{E}$ is defined as
\[
d_\mathcal{E}(z) = \max_{x \in \mathcal{E}} \| x - z \|^2, \quad (22)
\]
we can say that $z_{cc}$ is the element of the linear manifold $\mathcal{M}$, whose distance from the ellipsoid $\mathcal{E}$ is minimum. Notice that, while in general (22) is nonconvex, the problem
\[
\min_{z \in \mathcal{M}} d_\mathcal{E}(z)
\]
is a convex optimization problem, because $d_\mathcal{E}(z)$ is convex in $z$. The aim of this section is to exploit the special structure of this problem, in order to characterize the solution of the conditional center problem.

#### A. Characterization of the conditional Chebishev center

To start with, let us recall a necessary condition for the solution of a general min-max problem (see [21]). Let
\[
\mathcal{S}_\mathcal{E}(z) = \{ z \in \mathcal{E} : \| z - z \|^2 = d_\mathcal{E}(z) \}
\]
be the set of points of $\mathcal{E}$ where the maximum in (22) is achieved.

**Theorem 1:** A necessary and sufficient condition for the function $d_\mathcal{E}(z)$ to achieve its minimum on $\mathcal{M}$, at a point $z_m \in \mathcal{M}$, is that
\[
\min_{y \in \mathcal{E}(M)} \max_{\| g \| = 1} g'(z - z_m) \geq 0. \quad (24)
\]

According to the previous result, we need to characterize the set $\mathcal{S}_\mathcal{E}(z)$ for a generic point $z$. In order to simplify the discussion, let us assume that $q_1 < q_2 \leq q_3 \leq \ldots$, i.e. there is a unique largest semiaxis of the ellipsoid $\mathcal{E}$ (the extension to the general case being straightforward). Moreover, let us define $\mathcal{E} = [z_2 \ldots z_n]'$ and $\mathcal{Q} = \text{diag}(q_1, \ldots, q_{n-2})$.

The next preliminary lemmas will be used in the following derivations.

**Lemma 1:** Let $z$ be given. Define the functions
\[
c(\lambda) = z'(I_n - \lambda Q)^{-2} Q z - 1 \quad (25)
\]
and
\[
d_1(\lambda) = \| (I_n - \lambda Q)^{-1} \lambda Q z \|^2. \quad (26)
\]
If $\lambda_1, \lambda_2$ are two real solutions of $c(\lambda) = 0$, with $\lambda_1 > \lambda_2$, then $d_1(\lambda_1) > d_1(\lambda_2)$.

**Proof:** See Appendix A.

**Lemma 2:** Let $c(\lambda)$ and $d_1(\lambda)$ be defined as in Lemma 1 and set
\[
d_2 = \frac{1}{q_1} \left[ 1 - z'(I_{n-1} - \frac{1}{q_1} \mathcal{Q})^{-1} \mathcal{Q} z \right]. \quad (27)
\]
Then, if $z_1 = 0$, for any $\lambda \in \mathbb{R}$ satisfying $c(\lambda) = 0$
\[
d_2 \geq d_1(\lambda)
\]
and the inequality is strict if $\lambda \neq \frac{1}{q_1}$.

**Proof:** See Appendix B.

Now, we are ready to characterize the function $d_\mathcal{E}(z)$.

**Theorem 2:** Let $z \in \mathbb{R}^n$ be given and define the $(n-1)$-dimensional ellipsoid
\[
\mathcal{E} = \{ z : z_1 = 0 \text{ and } z'(I_{n-1} - \frac{1}{q_1} \mathcal{Q})^{-2} \mathcal{Q} z \leq 1 \}. \quad (28)
\]
Then:
- If $z \in \mathcal{E}$, then $\mathcal{S}_\mathcal{E}(z)$ is a pair and
\[
d_\mathcal{E}(z) = d_2, \quad (29)
\]
with $d_2$ given by (27).
- If $z \notin \mathcal{E}$, then $\mathcal{S}_\mathcal{E}(z)$ is a singleton and
\[
d_\mathcal{E}(z) = d_1(\hat{\lambda}), \quad (30)
\]
where $d_1(\lambda)$ is given by (26) and $\tilde{\lambda}$ is the largest real solution of the equation $c(\lambda) = 0$, with $c(\lambda)$ given by (25).

Proof: Let $z$ be fixed. From the theory of constrained optimization [22], the first-order necessary condition for $\hat{x}$ to be a local maximum of $\|x-z\|^2$, subject to the constraint $x'Qx \leq 1$, is that there exists $\lambda \in \mathbb{R}$ such that

\[
(I_n - \lambda Q)\hat{x} = z \quad (31)
\]
\[
\hat{x}'Q\hat{x} = 1. \quad (32)
\]

Let us discuss the possible solutions of (31)-(32), for different values of $\lambda$.

(i) Let $\lambda \neq \frac{1}{q_i}$, $i \in \{1, 2, \ldots, n\}$. Then, from (31)

\[
\hat{x} = (I_n - \lambda Q)^{-1} z \quad (33)
\]

and $\|\hat{x} - z\|^2 = d_1(\lambda)$, defined by (26). Substituting (33) into (32), one gets $c(\lambda) = 0$. Then, due to Lemma 1, the real solution of $c(\lambda) = 0$ which maximizes the cost $d_1(\lambda)$ is the largest one, namely $\tilde{\lambda}$. Hence, for $\lambda \neq \frac{1}{q_i}$, the unique candidate maximum is

\[
\hat{x} = (I_n - \lambda Q)^{-1} z. \quad (34)
\]

(ii) Let $\lambda = \frac{1}{q_i}$. From (31) it must be $z_1 = 0$. Then, from (31)-(32) one gets two real solutions

\[
\hat{x}_1 = \pm \frac{1}{q_i} \left(1 - \frac{1}{q_i} \right)^{-2} \frac{Qz_i}{(I_n - \frac{1}{q_i} Q)^{-1} z} \quad (35)
\]

\[
\begin{bmatrix}
\hat{x}_2 \\
\vdots \\
\hat{x}_n
\end{bmatrix} = \left( I_n - \frac{1}{q_i} Q \right)^{-1} z
\]

provided that $\frac{1}{q_i} \left(1 - \frac{1}{q_i} \right)^{-2} Qz_i \leq 1$. It is easy to verify that $\|\hat{x} - z\|^2 = d_2$, given by (27). Then, from Lemma 2 one has that $d_2 \geq d_1(\lambda)$ for any real $\lambda$ satisfying $c(\lambda) = 0$. Hence, when $z \notin \mathcal{E}$, the solutions (35) exist and the corresponding cost $d_2$ is larger than the cost $d_1(\lambda)$, relative to the solution (34) in (i). Conversely, when $z \notin \mathcal{E}$ and $z_1 = 0$, one has $c(\lambda) = \frac{1}{q_i} \left(1 - \frac{1}{q_i} \right)^{-2} Qz_i - 1$, with $c(\frac{1}{q_i}) > 0$ and $\lim_{\lambda \to \frac{1}{q_i}} c(\lambda) = -1$. Since $c(\lambda)$ is a continuous decreasing function in $[\frac{1}{q_i}, +\infty)$, the unique candidate maximum is given by (34), with $\lambda = \frac{1}{q_i}$. In this case, $\hat{x}_1 = 0$.

(iii) Let $\lambda = \frac{1}{q_i}$, $i \in \{2, \ldots, n\}$. In order to complete the proof, it remains to show that in this case the solutions of (31)-(32) do not provide a maximum point for $d_2(z)$. From constrained optimization theory [22], recall that the solution $\hat{x}$, $\lambda$ of (31)-(32) must satisfy also the second-order necessary condition, i.e. the matrix $I_n - \lambda Q$ must be negative semidefinite on the tangent plane $\{\xi : \xi'Q\xi = 0\}$.

It is easy to see that this condition is violated by $\lambda = \frac{1}{q_i}$, $i \in \{3, \ldots, n\}$, for any $\hat{x}$, and by $\lambda = \frac{1}{q_2}$ for all $\hat{x}$ such that $\hat{x}_2 \neq 0$. Then, the only remaining case to be examined is $\lambda = \frac{1}{q_2}$, $\hat{x}_2 = 0$ (which implies $z_2 = 0$). Now, the solution of (31)-(32) is given by

\[
\begin{align*}
\hat{x}_1 &= \frac{z_i}{1 - q_i/q_2} \quad i \neq 2 \\
\hat{x}_2 &= 0
\end{align*}
\]

and its cost is $d_3 = \sum_{i \neq 2} \frac{z_i^2 q_i^2}{(q_i/q_2)^2}$. It can be shown that it is always $d_2(z) > d_3$. In fact, define the functions

\[
\tau(\lambda) = \sum_{i = 1}^{n} \frac{z_i^2 q_i^2}{(1 - \lambda q_i)^2} - 1, \quad \overline{d}_2(\lambda) = \sum_{i = 1}^{n} \frac{\lambda^2 z_i^2 q_i^2}{(1 - \lambda q_i)^2}.
\]

Notice that $c(\lambda) = \tau(\lambda)$, $d_1(\lambda) = \overline{d}_1(\lambda)$, for all $\lambda \neq \frac{1}{q_2}$. By substituting (36) into (32) one gets $c(\frac{1}{q_2}) = 0$. Then, if $z_2 \neq 0$, it is easy to see that the largest $\lambda \in \mathbb{R}$ such that $\tau(\lambda) = 0$ satisfies $\lambda > \frac{1}{q_2} > \frac{1}{q_i}$ and hence, due to Lemma 1, $d_1(\lambda) = \overline{d}_1(\frac{1}{q_2}) = d_3$. Otherwise, if $z_1 = 0$, let

\[
\overline{d}_2 = \frac{1}{q_1} \left[1 - \sum_{i = 1}^{n} \frac{z_i^2 q_i}{1 - q_i/q_1}\right].
\]

Then, following Lemma 2, one gets $\overline{d}_2 > \overline{d}_1(\lambda)$, for all $\lambda$ satisfying $\tau(\lambda) = 0$, $\lambda \neq \frac{1}{q_1}$, and hence also $d_2 = \overline{d}_2 > \overline{d}_1(\frac{1}{q_2}) = d_3$. Therefore, the maximum of $\|x - z\|^2$ cannot be achieved at $\hat{x}$ given by (36). This completes the proof.

According to Theorem 2, the general condition given by Theorem 1 becomes much simpler when applied to the minmax problem (21). In particular, if $\mathcal{S}_c(z_m)$ is a singleton, $\mathcal{S}_c(z_m) = \{\xi_m\}$, condition (24) boils down to

\[
\min_{\|g\|=1} g'(\xi_m - z_m) \geq 0
\]

which implies that the vector $\xi_m - z_m$ must be orthogonal to every element in the span of $M$, i.e.

\[
M'(\xi_m - z_m) = 0. \quad (37)
\]

We shall exploit condition (37) in order to characterize the solution of the CCC problem when $\mathcal{S}_c(z_m)$ is a singleton. For this purpose, let us introduce some geometrical quantities.

Let $M^+ = \{z \in \mathbb{R}^n : M'z = 0\}$ be the $(n-p)$-dimensional linear subspace orthogonal to $M$, and let $\Pi : \mathbb{R}^n \to M^+$ be the orthogonal projection operator on $M^+$. Notice that, by (20), $z^0 \in M^+$. Moreover, let $\mathcal{E}_c = \Pi(\mathcal{E})$ be the $(n-p)$-dimensional ellipsoid obtained by projecting $\mathcal{E}$ on $M^+$. Then, let

\[
\xi^* = \arg \max_{\xi \in \mathcal{E}_c} \|\xi - z^0\|^2 \quad (38)
\]

be the point of $\mathcal{E}_c$ having maximum distance from $z^0$ (if $\mathcal{S}_c(z^0)$ is not a singleton, let $\xi^*$ be any element of the solution set). Necessarily, $\xi^* \in \partial \mathcal{E}_c$ and therefore there exists a unique point $x^* = \delta \mathcal{E}_c$ such that $\Pi(x^*) = \xi^*$. Finally, let
$z^*$ be the orthogonal projection of $x^*$ onto $\mathcal{M}$. A sketch of the above geometric construction is reported in Fig. 1 for the case $n = 2$, $p = 1$. The expressions of $z^*$ and $x^*$ will be derived in Sect. III-B.

Now we can prove the following lemma.

**Lemma 3:** i) If $S_E(z_{cc})$ is a singleton, then $z_{cc} = z^*$.

ii) If $S_E(z^*)$ is a singleton, then $z_{cc} = z^*$ if and only if $S_E(z^*) = \{x^*\}$.

**Proof:** i) By contradiction. Assume that $z_{cc} = z \in \mathcal{M}$ such that $z \neq z^*$ and $S_E(z) = \{x\}$. Then, according to Theorem 1, it must be $M'(z - x) = 0$. From the construction of $z^*$ and $x^*$, one has that for each $x \in \mathcal{E}$ and $z \in \mathcal{M}$, such that $(z - x) \in M^t$, it holds $\|z - x\| \leq \|z^* - x^*\|$. Therefore,

$$\|z - x\| \leq \|z^* - x^*\|. \tag{39}$$

But since $z^*$ is the projection of $x^*$ onto $\mathcal{M}$, we have also

$$\|z^* - x^*\| \leq \|z - x^*\|. \tag{40}$$

Therefore, form (39) and (40) we get

$$\|z - x\| \leq \|z - x^*\|. \tag{39}$$

But this contradicts $S_E(z) = \{x\}$.

ii) Since by construction $(x^* - z^*) \in M^t$, if $S_E(z^*) = \{x^*\}$ Theorem 1 guarantees that $z_{cc} = z^*$. Conversely, if $S_E(z^*) = \{x\}, \mathcal{E} \neq x^*$ and $z_{cc} = z^*$, one has $M'(z^* - x) = 0$ and hence, by construction $\|z^* - x\| \leq \|z^* - x^*\|$. But this contradicts $S_E(z^*) = \{x\}$. 

From Lemma 3, it is clear that $z^*$ is the only candidate solution of the CCC problem in the set $\{z \in \mathcal{M} : S_E(z) \text{ is a singleton}\}$. Moreover, it is sufficient to check the condition $S_E(z^*) = \{x^*\}$ to guarantee that $z^*$ solves (21). Otherwise, due to Theorem 2, the solution must belong to the set $\mathcal{M} \cap \mathcal{E}$, where $\mathcal{E}$ is given by (28). First notice that, if this intersection is empty, then $S_E(z)$ is a singleton for any $z \in \mathcal{M}$, and hence $z_{cc} = z^*$. Then, recall that, when $S_E(z)$ is a pair, $d_E(z) = d_2$ given by (27), and define

$$z^{**} = \arg \min_{z \in \mathcal{M}} d_2 \tag{41}$$

where

$$\mathcal{M} = \{z \in \mathcal{M} : z_1 = 0\}. \tag{42}$$

Notice that (41) is a quadratic optimization problem, whose solution can be easily computed in closed form (see e.g. [22]). Details of the complete derivation of $z^{**}$ are reported in Sect. III-B. The following lemma characterizes the role of $z^{**}$.

**Lemma 4:** If $S_E(z^*) \neq \{x^*\}$, then $z^{**} \in \mathcal{E}$.

**Proof:** Assume by contradiction that $z^{**}$ does not belong to $\mathcal{E}$. Then, according to Theorem 2, $S_E(z^{**})$ is a singleton and

$$d_E(z^{**}) = d_1(\lambda^{**})$$

for some $\lambda^{**} > \frac{1}{q_1}$ such that $c(\lambda^{**}) = 0$. If we consider $d_2$ in (27) as a function of $z$, we get from (41)

$$d_2(z) \geq d_2(z^{**}) \tag{42},$$

$$\forall z \in \mathcal{M} \cap \mathcal{E}.$$ Moreover, as $z_1^{**} = 0$ and $\lambda^{**} > \frac{1}{q_1}$, from Lemma 2 one has

$$d_2(z^{**}) > d_1(\lambda^{**}) \tag{42}.$$}

Since, according to Lemma 3, $S_E(z^*) \neq \{x^*\}$ guarantees that $z_{cc} \in \mathcal{E}$, we get

$$d_E(z_{cc}) = d_2(z_{cc}) > d_E(z^{**})$$

which is clearly a contradiction.

Finally, we are ready to state the main result of the paper.

**Theorem 3:** The solution of the CCC problem (16)-(21) is given by

$$z_{cc} = \begin{cases} z^*, & \text{if } S_E(z^*) = \{x^*\} \\ z^{**}, & \text{otherwise.} \end{cases} \tag{43}$$

**Proof:** If $S_E(z^*) = \{x^*\}$, Lemma 3 guarantees that $z_{cc} = z^*$. Otherwise, $z_{cc} = \arg \min_{z \in \mathcal{M} \cap \mathcal{E}} d_2$.

Since $(\mathcal{M} \cap \mathcal{E}) \subset \mathcal{M}$ and, by Lemma 4, $z^{**} \in \mathcal{E}$, we get $z_{cc} = z^{**}$ which completes the proof.

**Remark 1:** The theory developed in this section can be easily extended to the case when $q_1 = q_2 = \cdots = q_r < q_{r+1} \leq \cdots \leq q_n$, the only significative change concerning Theorem 2. It can be checked that $S_E(z)$ is not a singleton only if $z_1 = z_2 = \cdots = z_r = 0$ (i.e. $z$ lays on the $(n-r)$-dimensional hyperplane orthogonal to the $r$ largest semiaxes of $\mathcal{E}$). Moreover, in this case $S_E(z)$ is either a singleton or the boundary of an $r$-dimensional hypersphere. Nevertheless, all the reasoning following Theorem 2 still holds. Notice that $\mathcal{M}$ in (42) is now defined as $\{z \in \mathcal{M} : z_1 = \cdots = z_r = 0\}$ and, in general, it is empty if $r > p$.
B. A procedure for the computation of the conditional Chebyshev center

According to Theorem 3, the computation of \( z_{cc} \) boils down to that of \( z^* \), \( x^* \) and \( z^{**} \). The following propositions provide explicit expressions for these quantities. First, let us consider \( z^* \) and \( x^* \), defined through the geometric construction preceding Lemma 3.

**Proposition 3:** Define \( V \in \mathbb{R}^{n \times (n-p)} \) such that \( V'M = 0 \) and \( V'V = I_{n-p} \). Then

\[
\begin{align*}
    z^* &= z^o - M(M'QM)^{-1}M'QV\hat{\xi}_1, \\
    x^* &= [I_n - M(M'QM)^{-1}M'Q]V\hat{\xi}_1.
\end{align*}
\]

where

\[
\hat{\xi}_1 = \arg \max_{\xi_1 \in \mathcal{E}_1} \|\xi_1 - V'z^o\|^2
\]

(46)

and

\[
\mathcal{E}_1 = \{\xi_1 \in \mathbb{R}^{n-p} : \xi_1'V'[Q - QM(M'QM)^{-1}M'Q]V\xi_1 \leq 1\}.
\]

(47)

**Proof:** First, notice that the column vectors of the matrix \( W = [V \ M] \) provide an orthonormal basis for \( \mathbb{R}^n \), and let us introduce the change of coordinates \( \tilde{x} = W'x \). In these new coordinates, the sets \( \mathcal{M} \) and \( \mathcal{E} \) are given respectively by

\[
\mathcal{M} = \{\tilde{z} : \tilde{z} = \begin{bmatrix} V'z^o \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_p \end{bmatrix} \alpha, \alpha \in \mathbb{R}^p\}
\]

\[
\mathcal{E} = \{\tilde{z} : \tilde{z}'W'QW\tilde{z} \leq 1\}.
\]

Now, it is not difficult to check that, in the new coordinates, the ellipsoid \( \mathcal{E}_\pi = \Pi(\mathcal{E}) \) is given by

\[
\tilde{\mathcal{E}} = \{\tilde{z} : \tilde{z} = \begin{bmatrix} \tilde{\xi}_1 \\ 0 \end{bmatrix}, \tilde{\xi}_1 \in \mathcal{E}_1\}.
\]

Define \( \hat{\xi}_1 \) as in (46) and consider the linear manifold

\[
\tilde{\mathcal{M}} = \{\tilde{z} : \tilde{z} = \begin{bmatrix} \hat{\xi}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_p \end{bmatrix} \alpha, \alpha \in \mathbb{R}^p\},
\]

passing through \( [\hat{\xi}_1' \ 0]' \) and parallel to \( \tilde{\mathcal{M}} \). By definition of \( \tilde{\mathcal{E}}_\pi \) and \( \hat{\xi}_1 \), we get that \( \tilde{\mathcal{M}}_1 \) is tangent to \( \tilde{\mathcal{E}} \), and the contact point is given by

\[
\tilde{\mathcal{M}}_1 \cap \tilde{\mathcal{E}} = \begin{bmatrix} \hat{\xi}_1 \\ -(M'QM)^{-1}M'QV\hat{\xi}_1 \end{bmatrix} = \tilde{x}^*.
\]

Moreover, the projection of \( \tilde{x}^* \) onto \( \tilde{\mathcal{M}} \) is

\[
\begin{bmatrix} V'z^o \\ -(M'QM)^{-1}M'QV\hat{\xi}_1 \end{bmatrix} = \tilde{z}^*.
\]

Finally, going back to the original coordinates, it is easy to verify that \( W'[\hat{\xi}_1]' = \xi^* \) in (38), and \( W\tilde{x}^* = x^* \), \( W\tilde{z}^* = z^* \), with \( z^* \) and \( x^* \) given by (44) and (45), respectively.

In order to derive an explicit expression for \( z^{**} \), one must solve the quadratic optimization problem (41). Standard calculations allow one to rewrite \( \tilde{\mathcal{M}} \) as

\[
\tilde{\mathcal{M}} = \{z = \begin{bmatrix} 0 \\ \xi \end{bmatrix}, \xi \in \mathbb{R}^{n-1} : \xi = \xi^o + \tilde{\mathcal{M}} \pi, \pi \in \mathbb{R}^{n-1}\}
\]

(48)

where \( \xi^o \) and \( \tilde{\mathcal{M}} \) can be determined from \( \xi^o \) and \( M \), by imposing the constraint \( z^o = 0 \). Then, the following proposition holds.

**Proposition 4:** Let \( D = (\tilde{Q} - q_1I_n)^{-1}\tilde{Q} \). Then

\[
z^{**} = \begin{bmatrix} I_{n-1} - M(M'DM)^{-1}M'D \end{bmatrix} \xi^o.
\]

(49)

**Proof:** By using (27) and the definition of \( D \), problem (41) can be rewritten as

\[
z^{**} = \arg \min_{z \in \tilde{\mathcal{M}}} \left[ \frac{1}{q_1} + \xi'D\xi \right]
\]

from which (49) easily follows.

Finally, the algorithm for the solution of the CCC problem can be summarized as follows:

1. Compute \( z^* \) and \( x^* \), according to Proposition 3.
2. Check if \( \mathcal{S}_\pi(z^*) = \{x^*\} \) by using Theorem 2. If the condition is satisfied, set \( z_{cc} = z^* \); otherwise go to step 3.
3. Compute \( z^{**} \) according to Proposition 4 and set \( z_{cc} = z^{**} \).

Notice that the main tasks that must be performed are related to the computation of the distance of a point from an ellipsoid (respectively of \( V'z^o \) from \( \mathcal{E}_1 \) in Proposition 3, and of \( z^* \) from \( \mathcal{E} \) for checking the condition in step 2). This can be easily accomplished according to Theorem 2. In particular, it is easy to verify that the equation \( c(\lambda) = 0 \) has a unique real solution for \( \lambda > \frac{1}{L} \), where \( f = \min\{j : \ z_j \neq 0\} \), and that \( \frac{dc(\lambda)}{d\lambda} < 0, \forall \lambda \in \left(\frac{1}{L}, +\infty\right) \).

Moreover, from the proof of Theorem 2, it can be argued that when \( z_1 = 0 \), \( \mathcal{S}_\pi(z) \) is a singleton if and only if \( \lambda > \frac{1}{q_1} \). Therefore, \( \lambda \) in (30) can be easily computed within the desired precision, using any search technique in the semi-infinite interval \( \left[\frac{1}{q_1}, +\infty\right) \). Finally, the set \( \mathcal{S}_\pi(z) \) can be obtained by using equations (34) or (35), depending on \( \mathcal{S}_\pi(z) \) being a singleton or a pair.

IV. Numerical examples

The conditional central algorithm derived in the previous section has been applied to the set membership worst-case identification problem of Section II-A. Input/output data have been generated according to the model (8), with the following assumptions:

- the impulse response \( h \) belongs to the a priori set

\[
K = \{h : \sum_{k=N+1}^{\infty} h_k^2 \rho^{-2k} < L^2\}
\]

with \( 0 < \rho < 1 \).
- the noise sequence is energy bounded

\[
\frac{1}{N} \sum_{k=0}^{N} v_k^2 \leq \varepsilon^2.
\] (50)

The worst-case error (11) has been measured in the $H_2$ norm. Recall that in this case Proposition 2 guarantees that the conditional central algorithm provides the optimal worst-case estimate.

The nominal system generating the data is taken from [8]:

\[
H(z) = 0.486z^6 + 0.493z^4 + 0.779z^3 + 0.904z^2 + 0.074z + 0.383
\]

\[
-1.54z^6 + 2.128z^4 - 2.409z^3 + 1.453z^2 - 0.942z + 0.384.
\]

It can be checked that $\|H(z)\|_{H_2} \cong 7.26$ and $H(z)$ satisfies the a priori assumptions for $L = 2$ and $\rho = 0.995$. In the simulation experiments, $N = 50$ measurements have been collected and the disturbance $v$ has been chosen as an i.u.d. signal satisfying (50), with $\varepsilon = 0.2$. Two different input signals have been used: a step and a square wave of period 10, both satisfying $\frac{1}{\sqrt{N}} \|u^N\|_2 = 1$. Moreover, two linearly parameterized model classes, based on FIR and Laguerre series expansion have been considered. The Laguerre models have been formed according to [17], with Laguerre pole equal to 0.75 (the same value chosen in [8]). Simulation results relative to different input and model class choices are reported in Figures 2-5. The conditional radius of information (i.e. the worst-case error of the $\phi_{cc}$ algorithm) is compared with the estimated worst-case error of the central projection suboptimal algorithm provided in [8], for different parametric model orders. The reported plots show the average results over 10 different noise realizations.
As it can be checked by Figures 2-5, the conditional central algorithm provides a significant improvement of the worst-case estimation error. The error reduction, over model orders ranging from 1 to 10, varies from approximately 15% for the case of step input, to 25% for the case of square wave input. The improvement of estimate reliability is due to two distinct factors. First, the central projection estimator is known to be only suboptimal. Second, the available error associated to this suboptimal estimator is only an upper bound of the true error. Actually, it can be observed that the exact evaluation of the central projection estimator error shows the same level of complexity as the computation of the optimal solution.

In the above simulations, it was assumed that the bound \( \varepsilon \) is known exactly. In order to evaluate the effect of overestimation of the noise bound, the previous results are compared to those obtained in the case \( \varepsilon = 0.3 \) and \( \varepsilon = 0.4 \) (respectively, 50% and 100% overestimated noise bound) in Figure 6. It can be observed that while the increase of the estimation error is moderate, the rate of decrease of the error as the model order grows remains approximately unchanged.

V. CONCLUSIONS

In this paper, the role and optimality of conditional central algorithms in worst-case system identification and state estimation with set membership uncertainty have been investigated. The computation of optimal estimators is a very difficult task in general. If disturbances are bounded according to the energy norm, a closed form for the conditional central estimator can be derived in terms of the unique real root of a polynomial equation in a semi-infinite interval. A comparison of the error of the conditional central estimator with the estimated error of a suboptimal algorithm available in the literature is performed on simulated data, showing the improvement provided by the optimal solution.

Several interesting problems deserve attention for future work. The quality of the estimate provided by suboptimal algorithms, like for example the central projection estimator or the reduced least-squares estimator, needs a deeper investigation. Moreover, very little is known on conditional central algorithms when disturbances are bounded in different norms. For example, the case of \( L_\infty \) norm in the measurement space, which has been extensively investigated in the recent literature, appears of much interest in practical applications.

APPENDIX

A. Proof of Lemma 1

Consider \( \lambda_1, \lambda_2 \in \mathbb{R} \) such that \( \lambda_1 > \lambda_2 \) and \( c(\lambda_1) = c(\lambda_2) = 0 \). Then, from (25) one gets

\[
0 = c(\lambda_1) - c(\lambda_2) = \sum_{i=1}^{n} \gamma_i \left[ \frac{\lambda_1^2}{(1 - \lambda_1 q_i)^2} - \frac{\lambda_2^2}{(1 - \lambda_2 q_i)^2} \right]
\]

where

\[
\gamma_i = \frac{z_i^2 q_i^2 (\lambda_1 - \lambda_2)}{(1 - \lambda_1 q_i)^2 (1 - \lambda_2 q_i)^2} > 0.
\]

Then

\[
d_1(\lambda_1) - d_1(\lambda_2) = \sum_{i=1}^{n} \gamma_i (\lambda_1 + \lambda_2 - 2\lambda_1 \lambda_2 q_i).
\]

From (51) we get

\[
\sum_i \gamma_i = \frac{\lambda_1 + \lambda_2}{2} \sum_i \gamma_i q_i
\]

and hence, substituting into (52) we obtain

\[
d_1(\lambda_1) - d_1(\lambda_2) = \left[ \frac{(\lambda_1 + \lambda_2)^2}{2} - 2\lambda_1 \lambda_2 \right] \sum_{i=1}^{n} \gamma_i q_i > 0.
\]

as it was claimed. \qed

B. Proof of Lemma 2

Let \( z_1 = 0 \). From (25), \( c(\lambda) = 0 \) can be rewritten as

\[
\sum_{i=2}^{n} \frac{z_i^2 q_i}{(1 - \lambda q_i)^2} w ' s = 1
\]

where \( s = [z_2^\prime \ldots z_n^\prime]^\prime \) and \( w = [\frac{q_2}{(1 - \lambda q_2)^2} \ldots \frac{q_n}{(1 - \lambda q_n)^2}]^\prime \). For fixed \( \lambda \) satisfying (53), we want to show that \( \tilde{d}_2 - d_1(\lambda) \) is nonnegative in the set

\[
\mathcal{L} = \{ s \in \mathbb{R}^{n-1} : w ' s = 1, s_j \geq 0 \forall j \}.
\]

Since \( \tilde{d}_2 - d_1(\lambda) \) is a linear functional of \( s \), it suffices to prove that the claim is true in the vertices of \( \mathcal{L} \), i.e. for...
\[ s = s^i \equiv [0 \ldots 0 \frac{(1-\lambda_i)^2}{q_i} 0 \ldots 0], \quad 2 \leq i \leq n. \]

Substituting
\[ z_2^2 \ldots z_n^2 = s^i \] into (26) and (27) one gets
\[
d_1 = \frac{q_i \lambda^2}{q_1}, \\
d_2 = \frac{1}{q_1} + \frac{(1-\lambda_i)^2}{q_i-q_1}
\]

from which it is easily obtained that \( d_2 - d_1 \geq 0 \) for all \( \lambda \) satisfying \( c(\lambda) = 0 \), and \( d_2 - d_1 > 0 \) if \( \lambda \neq \frac{1}{q_1} \).

References


