Upper and lower bounds for the energy of bipartite graphs

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Abstract

Using Lagrange’s multiplier rule, we find upper and lower bounds of the energy of a bipartite graph \( G \), in terms of the number of vertices, edges and the spectral moment of fourth order. Moreover, the upper bound is attained in a graph \( G \) if and only if \( G \) is the graph of a symmetric balanced incomplete block design (BIBD). Also, we determine the graphs for which the lower bound is sharp.

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1. Introduction

The energy of a graph \( G \), denoted by \( E(G) \), is defined by

\[
E(G) = \sum_{i=1}^{n} |\lambda_i|,
\]

where \( n \) is the number of vertices of \( G \), and \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of the adjacency matrix of \( G \). This concept was introduced by Gutman [4], in connection to the so-called total \( \pi \)-electron energy. For details on the general theory of the total \( \pi \)-electron energy, as well as its chemical applications, we refer to [6].

The search of the upper and lower bounds for the energy of a graph and the search of graphs with minimal or maximal energy, is a wide field of spectral graph theory ([2]...
and [8]). Generally, these bounds for $E(G)$ are given in terms of the number of vertices and the number of edges of $G$, and sometimes, it also involves the fourth order spectral moment of $G$. For instance, see [5,7,9–12].

In this paper, we present an application of the Lagrange’s multiplier rule, to obtain bounds for the energy of bipartite graphs, which improve several known bounds that have appeared recently in the literature [9,11]. These bounds are given in terms of the number of vertices, edges and the fourth spectral moment. Moreover, we determine the graphs for which these bounds are sharp: the upper bound is attained in graphs of symmetric balanced incomplete block designs (BIBD); the lower bound is attained in graphs which are direct sums of $K_2$ or direct sums of complete bipartite graphs $K_{r,s}$ (for details on these classes of graphs, see [1]).

In what follows, $G$ denotes a bipartite graph with $n = 2N$ vertices ($N \geq 2$) and $m$ edges. The case $n = 2N + 1$ will be considered at the end of Section 3. It is well known that the spectrum of $G$ is symmetric with respect to the origin of $R$, and so, the eigenvalues of $G$ can be enumerated as

$$\pm \mu_1, \ldots, \pm \mu_N,$$

where

$$\mu_1 \geq \cdots \geq \mu_N \geq 0. \quad (1)$$

In particular, the energy of $G$ is given by

$$E(G) = 2(\mu_1 + \cdots + \mu_N).$$

For an even integer $k \geq 2$, the $k$-spectral moment of $G$ is defined as

$$M_k = 2 \sum_{i=1}^{N} \mu_i^k.$$

Of particular interest in this paper are the spectral moments $M_2$ (which is well known to satisfy the relation $M_2/2 = m$) and $M_4$. So, by setting $q = M_4/2$ we have the following relations:

$$m = \sum_{i=1}^{N} \mu_i^2 \quad \text{and} \quad q = \sum_{i=1}^{N} \mu_i^4. \quad (2)$$

We assume $m > 0$ to avoid the trivial case. Note also that $m^2 \geq q$. Moreover, using the Cauchy–Schwarz inequality, we have $m^2 \leq Nq$. The aim of this paper are the following results.

**Theorem 1.1.** Let $G$ be a bipartite graph with $2N$ vertices. Then

1. $m^2 = Nq$ if and only if $G = NK_2$;
2. $m^2 = q$ if and only if $G$ is the direct sum of $h$ isolated vertices and a copy of a complete bipartite graph $K_{r,s}$ such that $rs = m$ and $h + r + s = 2N$;
(3) If $1 < m^2/q < N$ then $E(G) \leq \mathcal{E}(G)$, where

$$\mathcal{E}(G) = \frac{2}{\sqrt{N}} \left[ (m + \sqrt{(N-1)Q})^{1/2} + (N-1)(m - \sqrt{Q/(N-1)})^{1/2} \right]$$

and $Q := Nq - m^2$. The equality holds if $G$ is the graph of a symmetric BIBD. Conversely, if the equality holds and $G$ is regular then $G$ is the graph of a symmetric BIBD.

It is easy to show that the bound given in the theorem above improves McClelland’s inequality [10] restricted to bipartite graphs. In fact, we will prove that it improves the bound given in [11, inequality 51]. Also, we will compare $\mathcal{E}(G)$ with the nice bound given recently by Koolen and Moulton in a recent paper [9].

**Theorem 1.2.** Let $G$ be a bipartite graph with $2N$ vertices. Then

$$E(G) \geq 2m \sqrt{\frac{m}{q}}.$$  (4)

The equality holds if and only if $G = NK_2$ or $G$ is the direct sum of isolated vertices and complete bipartite graphs $K_{r_1,s_1}, \ldots, K_{r_j,s_j}$, such that $r_1s_1 = \cdots = r_j s_j$.

This theorem improves the results by McClelland [10] and [8, Corollary 7].

### 2. The basic inequalities

In this section, $m$ and $q$ denote positive real numbers and $M$ denotes the compact subset of $\mathbb{R}^N$, defined by the equations

$$x_1^2 + \cdots + x_N^2 = m, \quad x_1^4 + \cdots + x_N^4 = q.$$  (5)

We also define $Q = Nq - m^2$.

**Lemma 2.1.**

1. If $M \neq \emptyset$ then $Q \geq 0$ and the equality holds if and only if $M$ is the set of all points of the form $\sqrt{m/N}(\epsilon_1, \ldots, \epsilon_N)$, where $\epsilon_i = \pm 1$ for all $i$.
2. If $M \neq \emptyset$ then $m^2 \geq q$ and the equality holds if and only if $M$ consists of all points of the form $\pm \sqrt{m} \epsilon_i$ $(i = 1, 2, \ldots, N)$, where $\{\epsilon_1, \ldots, \epsilon_N\}$ is the canonical vector basis of $\mathbb{R}^N$.
3. If $1 < m^2/q < N$ is not an integer then $M$ is a submanifold of $\mathbb{R}^N$ of codimension two.
4. If $j := m^2/q$ is an integer and $1 < j < N$ then $M \setminus F$ is a submanifold of $\mathbb{R}^N$ of codimension two, where $F$ is the finite set consisting of all points $(x_1, \ldots, x_N)$ such that $\{x_1^2, \ldots, x_N^2\} \setminus \{0\}$ is the single set $\{m/j\}$. 

Proof. (1) If \((x_1, \ldots, x_N) \in M\) then, applying the Cauchy–Schwarz inequality to the vectors
\[
(x_1^2, \ldots, x_N^2) \text{ and } (1, \ldots, 1),
\]
we obtain \(Q := Nq - m^2 \geq 0\) and the equality holds if and only if \(x_1^2 = \cdots = x_N^2\).

(2) It follows from the fact that \(q \leq q + 2 \sum_{1 \leq k < l \leq N} x_k^2 x_l^2 = m^2\), where \((x_1, \ldots, x_N) \in M\).

(3) Define \(j = [m^2/q]\). It is easy to show that the system
\[
\xi^2 + j\eta^2 = m, \quad \xi^4 + j\eta^4 = q
\]
has a solution \((\xi, \eta)\) such that \(\xi > \eta > 0\). Let \(\bar{x} := (x_1, \ldots, x_N)\) be defined by \(x_1 = \xi\), \(x_i = \eta\) for \(2 \leq i \leq j + 1\) and \(x_i = 0\) otherwise. It is clear that \(\bar{x} \in M\) and so \(M\) is nonempty.

Finally, if \(x \in (x_1, \ldots, x_N) \in M\) then \(x\) and \(x^3 := (x_1^3, \ldots, x_N^3)\) are linearly independent. Otherwise there exists a \(a \in \mathbb{R}\) such that \(x_i^3 = ax_i\) for all \(i\) and defining \(j\) as the cardinal of \(\{i: x_i^2 = a\}\), we find that \(ja = m\) and \(ja^2 = q\). Thus \(m^2/q = j\) is an integer and this contradiction proves (3).

(4) The point \(\bar{x}\) defined in the proof of part (3) belongs to \(M\) and \(\bar{x}\) and \(x^3\) are linearly independent. On the other hand, if \(x \in M\) then \(x\) and \(x^3\) are linearly dependent if and only if \(\{x_1^2, \ldots, x_N^2\} \setminus \{0\}\) is a single set. The proof follows easily. \(\square\)

From now on, we assume that \(q \leq m^2 \leq qN\) (and so \(M\) is nonempty). We define \(M^+\) as the set consisting of all points of \(M\) having nonnegative coordinates. Consider the differentiable function \(S: \mathbb{R}^N \to \mathbb{R}\) defined by
\[
S(x_1, \ldots, x_N) = x_1 + \cdots + x_N.
\]
Since \((|x_1|, \ldots, |x_N|) \in M\) if \((x_1, \ldots, x_N) \in M\), we conclude that
\[
\max(S|M) = \max(S|M^+).
\]
Moreover, using the above lemma, it is easy to show:

Remark 2.2.

(1) If \(m^2/q = N\) then \(\max(S|M) = \min(S|M^+) = \sqrt{mN}\).

(2) If \(m^2 = q\) then \(\max(S|M) = \min(S|M^+) = \sqrt{m}\).

(3) If \(q < m^2 < Nq\) then \(\max(S|M) > \min(S|M^+)\).

Lemma 2.3. If \(m^2/q\) is not an integer, then
\[
\sqrt{N} \max(S|M) = \left[m + \sqrt{(N-1)Q}\right]^{1/2} + (N-1)\left[m - \sqrt{Q/(N-1)}\right]^{1/2}.
\]
Moreover, if \(S(z) = \max(S|M)\) for some \(z = (z_1, \ldots, z_N) \in M\), then \(z_i > 0\) for all \(i\), \(\{z_1, \ldots, z_N\}\) is a two point set and \(x_* := \max|z_1, \ldots, z_N|\) has multiplicity one. By definition, this means that the set \(\{i: z_i = x_*\}\) is singular.
Proof. Fix \( z = (z_1, \ldots, z_N) \in M \) such that \( \max(S|M) = S(z) \). Since \((|z_1|, \ldots, |z_N|) \in M\), we conclude that \( z_i \geq 0 \) for all \( i \).

Using Lagrange’s multiplier rule, we find two constants \( \alpha, \beta \) such that

\[
1 = 2\alpha z_i + 4\beta z_i^3, \quad 1 \leq i \leq N.
\]

(6)

Claim 1. \( \beta \neq 0 \). Otherwise, \( z_1 = \cdots = z_N \) and hence \( m^2/q = N \). This contradiction proves the claim.

From this, each \( z_i \) is a solution of the cubic equation

\[
4\beta y^3 + 2\alpha y - 1 = 0.
\]

(7)

In particular, \( z_i > 0 \) for all \( i \).

Claim 2. Equation (7) has exactly two positive solutions. To show this, we first note that the sum of all solutions of this equation is equal to zero. Thus, (7) has at most two positive solutions. If this equation has a unique positive solution then \( z_1 = \cdots = z_N \) and hence \( m^2/q = N \). This contradiction proves the claim.

Let \( x^* = y^* \) be the positive solutions of (7) and let \( s^* = t^* \) be, respectively, the cardinal numbers of the sets \( \{i: z_i = x^*\} \) and \( \{i: z_i = y^*\} \). Then, \( s^* + t^* = N \) and

\[
s^* x^{\frac{2}{s}} + t^* y^{\frac{2}{t}} = m, \quad s^* x^\frac{4}{s} + t^* y^\frac{4}{t} = q.
\]

(8)

Since \( x^* > y^* > 0 \), we have

\[
x^2 = \frac{m + \sqrt{Q/s}}{N}, \quad y^2 = \frac{m - \sqrt{Q/s}}{N}
\]

and \( m > \sqrt{Q/s} \), which is equivalent to say that \( s^* < m^2/q \). For all real \( 0 < s < m^2/q < N \), we define positive real numbers \( X(s) > Y(s) \) by the relations

\[
X(s)^2 = \frac{m + \sqrt{Q(N - s)/s}}{N}, \quad Y(s)^2 = \frac{m - \sqrt{Q(N - s)} / s}{N}.
\]

Let \( 1 \leq j < m^2/q \) be an integer. Then the point \( \xi(j) := (x_1, \ldots, x_N) \in \mathbb{R}^N \), defined by \( x_i = X(j) \) for \( 1 \leq i \leq j \) and \( x_i = Y(j) \) for \( j < i \leq N \), belongs to \( M \) and \( z = \xi(s^*) \).

Let us define \( f(s) = sX(s) + (N - s)Y(s) \) and note that \( S(\xi(j)) = f(j) \). By direct calculation we have

\[
f'(s) = X(s) - Y(s) - \frac{N \sqrt{Q}}{4 \sqrt{Q(s(N - s))}} \frac{X(s) + Y(s)}{X(s)Y(s)}
\]

and hence

\[
\sqrt{s(N-s)}[X(s) + Y(s)] f'(s) = N \sqrt{Q} \left[ 1 - \frac{(X(s) + Y(s))^2}{4X(s)Y(s)} \right].
\]

Consequently, \( f'(s) < 0 \), since \( X(s) \neq Y(s) \). Finally, since the point \( \xi(1) \) belongs to \( M \), we deduce

\[
f(1) = S(\xi(1)) \leq S(z) = f(s^*) \leq f(1).
\]

Therefore, \( s^* = 1 \) and \( S(z) = f(1) \). The proof is complete. \( \square \)
Lemma 2.4. The following inequality holds:

\[ \min(S|M^+) \geq m\sqrt{m/q}. \]

Moreover, if \( S(v) = \min(S|M^+) \) for some \( v = (v_1, \ldots, v_N) \in M^+ \), then \( \{v_1^2, \ldots, v_N^2\} \setminus \{0\} \) is a single set. In this case, \( m^2/q \) is the cardinal number of \( \{i : v_i \neq 0\} \).

Proof. Let us fix \( u = (u_1, \ldots, u_N) \in M^+ \) such that \( S(u) = \min(S|M^+) \). Since \( u_\sigma : = (u_{\sigma(1)}, \ldots, u_{\sigma(N)}) \in M^+ \) for all permutations \( \sigma \) of \( 1, \ldots, N \) and \( S(u_\sigma) = S(u) \), we can assume that \( u_1 \geq \cdots \geq u_N \). Define now \( j = N \) if \( u_N > 0 \) and \( j = \min\{i : u_{j+1} = 0\} \) if \( u_N = 0 \). Then, \( u_i > 0 \) for \( 1 \leq i \leq j \) and \( u_i = 0 \) for all \( i > j \). Let \( L \) be the subset of \( \mathbb{R}^j \) defined by the equations

\[ y_1^2 + \cdots + y_j^2 = m, \quad y_1^2 + \cdots + y_j^2 = q, \]  

and define \( T : \mathbb{R}^j \to \mathbb{R}^j \) by \( T(y_1, \ldots, y_j) = y_1 + \cdots + y_j \). Since \( \bar{u} := (u_1, \ldots, u_j) \in L^+ \) then, using the natural inclusion

\[ L^+ \ni (y_1, \ldots, y_j) \to (y_1, \ldots, y_j, 0, \ldots, 0), \]

we conclude that

\[ \min(T|L^+) = T(\bar{u}) = S(u) = \min(S|M^+), \]

where \( L^+ \) is the subset of \( L \) of all points having nonnegative coordinates.

Case 1. \( (u_1, \ldots, u_j) \) and \( (u_1^2, \ldots, u_j^2) \) are linearly dependent. In this case, \( u_1^2 = \cdots = u_j^2 \), since \( u_i > 0 \) for \( 1 \leq i \leq j \). In particular, \( qj = m^2 \) and

\[ \min(S|M^+) = j\sqrt{m/j} = jm = m\sqrt{m/q}. \]

Case 2. \( (u_1, \ldots, u_j) \) and \( (u_1^2, \ldots, u_j^2) \) are linearly independent. In this case, \( L \) is a continuously differentiable manifold about \( u \) and \( T \) attains a local minimum at \( (u_1, \ldots, u_j) \), since this point is interior to \( L^+ \). Applying the argument of Lemma 2.3 to \( (T, \bar{u}) \), we conclude that \( (u_1, \ldots, u_j) \) is a two-point set \( \{v_*, w_*\} \), where \( v_* > w_* \) and

\[ s_*v_*^2 + t_*w_*^2 = m, \quad s_*v_*^2 + t_*w_*^2 = q \]

for some positive integers \( s_*, t_* \) such that \( s_* + t_* = j \). In particular,

\[ v_*^2 = m + \sqrt{Q_j/t_*}/s_*, \quad w_*^2 = m - \sqrt{Q_j/s_*}/t_*, \]

where \( Q_j := qj - m^2 \). Note that \( Q_j > 0 \) since \( (u_1, \ldots, u_j) \) and \( (u_1^2, \ldots, u_j^2) \) are linearly independent.

For all real \( 0 < s \leq m^2/q \) let us define nonnegative real numbers \( V(s), W(s) \) by the relations

\[ V(s)^2 = \frac{m + \sqrt{Q_j(s - j)/s}}{j}, \quad W(s)^2 = \frac{m - \sqrt{Q_j(j - s)/s}}{j}, \]

and note that \( V(s) > 0 \) for \( 0 < s < m^2/q \) and \( W(s) = 0 \) if and only if \( s = m^2/q \). By the arguments in Lemma 2.3, we conclude that if \( s < m^2/q \) is an integer, then the point

\[ \left( \frac{V(s), \ldots, V(s)}{j - s}, \frac{W(s), \ldots, W(s)}{j - s} \right) \]
belongs to \( L^+ \). Similarly, if we define \( f(s) = sV(s) + (j - s)W(s) \), we prove that \( f'(s) < 0 \) and so,
\[
\min(T|L^+) = f(s_*) > f(m^2/q) = m\sqrt{m/q}.
\]
Note that condition \( w_* > 0 \) implies \( s_* < m^2/q \). Thus, the proof is complete. □

**Remark 2.5.** The lemma above can be rephrased as follows. Given real numbers \( x_1, \ldots, x_N \), we have
\[
(x_1^2 + \cdots + x_N^2)^3 \leq (|x_1| + \cdots + |x_N|)^2(x_1^4 + \cdots + x_N^4)
\]
and the equality holds if \( \{x_1^2, \ldots, x_N^2\} \setminus \{0\} \) is a single set. In this case, if \( x_i \neq 0 \) for some \( i \), then \( (x_1^2 + \cdots + x_N^2)^2/(x_1^4 + \cdots + x_N^4) \) is the cardinal number of \( \{i: x_i \neq 0\} \).

Next we show that the condition “\( m^2/q \) is not an integer” can be omitted from the hypothesis of Lemma 2.3.

**Theorem 2.6.** If \( q < m^2 < Nq \) then the conclusion of Lemma 2.3 holds.

**Proof.** Fix \( z = (z_1, \ldots, z_N) \in M \) such that \( S(z) = \max(S|M) \).

**Claim.** \( z \) and \( z^3 \) are linearly independent. To show this, assume on the contrary that there exists \( a \in \mathbb{R} \) such that \( z_i^3 = az_i \) for all \( i \). If we denote by \( j \) the cardinal number of \( \{i: z_i^2 = a\} \), then by Lemma 2.4 and Remark 2.2, we conclude that
\[
\max(S|M) = S(z) = j\sqrt{a} = m\sqrt{m/q} \leq \min(S|M^+) < \max(S|M^+),
\]
and this contradiction proves the claim.

By the above claim, \( M \) is a manifold about \( z \) and the proof follows as in Lemma 2.3. □

### 3. Upper and lower bounds for the energy of bipartite graphs

In this section we use the notation from the introduction. Recall that \( m \) is the number of edges, \( q = Mq/2 \) and so \( \mu = (\mu_1, \ldots, \mu_N) \in M^+ \), where \( \mu_i (i = 1, \ldots, N) \) are the nonnegative eigenvalues of \( G \) satisfying (1).

**Proof of Theorem 1.1.** (1) Assume \( Nq = m^2 \). Then, by part (1) of Lemma 2.1, \( \mu_i = \sqrt{m/N} \) for all \( i = 1, \ldots, N \), and so by [3] (see also [1, Theorem 6.4]), we conclude that \( G = NK_2 \).

(2) If \( m^2 = q \) then, by part (2) of Lemma 2.1, \( \mu_1 = \sqrt{m} \) and \( \mu_i = 0 \) for all \( i \geq 2 \). Consequently, part (2) of Theorem 1.1 follows from [1, Theorem 6.5].

(3) Assume now \( q < m^2 < Nq \). Then, by Theorem 2.6,
\[
\sqrt{N}S(\mu) \leq \sqrt{N} \max(S|M) = [m + \sqrt{(N - 1)Q}]^{1/2} + (N - 1)[m - \sqrt{Q}/(N - 1)]^{1/2}.
\]
Since \( E(G) = 2S(\mu) \), we immediately deduce that \( E(G) \leq E(G) \).
If \( E(G) = \tilde{E}(G) \) and \( G \) is regular then \( S(\mu) = \max(S|M) \). By Theorem 2.6, \( G \) has exactly two positive eigenvalues: \( \mu_1 \) has multiplicity one and \( \mu_2 = \cdots = \mu_N \) has multiplicity \( N - 1 \). From [3], we conclude that \( G \) is the graph of a symmetric BIBD.

Finally, suppose that \( G \) is the graph of a symmetric BIBD. Then there exist integers \( l < k < N \) such that \( l(N - 1) = k(k - 1) \) and the positive eigenvalues of \( G \) are \( k \) of multiplicity one and \( \sqrt{k - l} \) of multiplicity \( N - 1 \). In particular, \( m = k^2 + (N - 1)(k - 1) = Nk, q = k^4 + (N - 1)(k - 1)^2 \) and by a proper calculation, \( Q = (N - 1)N^2 \). Now it is easy to show that \( E(G) = \tilde{E}(G) \) and the proof of Theorem 1.1 is complete. \( \blacksquare \)

**Proof of Theorem 1.2.** It follows from Lemma 2.4 that
\[
E(G) = 2S(\mu) \geq 2 \min(S|M^+) \geq 2m \frac{m}{\sqrt{q}}.
\]
If \( E(G) = 2m \frac{m}{\sqrt{q}} \) then clearly \( S(\mu) = \min(S|M^+) \). Therefore, by Lemma 2.4, there exists \( 1 \leq j \leq N \) such that \( \mu_1 = \cdots = \mu_j \) and \( \mu_i = 0 \) for all \( i > j \). The result follows from [1, Theorems 6.4 and 6.5]. \( \blacksquare \)

Assume that \( G \) has \( n = 2N + 1 \) vertices. Then the eigenvalues of \( G \) can be enumerated as \( \pm \mu_1, \ldots, \pm \mu_N, 0 \), with \( \mu_1 \geq \cdots \geq \mu_N > 0 \). Define \( Q \) as above.

**Theorem 3.1.** Let \( G \) be a bipartite graph with \( 2N + 1 \) vertices. Then

1. \( Q \geq 0 \) and the equality holds if and only if \( G \) is the direct sum of an isolated vertex with \( NK \); and
2. Inequality in part (3) of Theorem 1.1 remains true if \( q < m^2 < Nq \), and the equality holds if \( G \) consists of an isolated vertex and a copy of the graph of a symmetric BIBD.

**Proof.** By the Cauchy–Schwarz inequality, we have \( Q \geq 0 \) and the equality holds if and only if \( \mu_1 = \cdots = \mu_N \). From this, \( \pm \sqrt{m/N} \) are eigenvalues of \( G \) of multiplicity \( N \) and \( \mu = 0 \) is an eigenvalue of multiplicity one. By [1, Theorem 6.5], \( G \) is the direct sum of \( N \) complete bipartite graphs \( K_{r_i,s_i} \) such that \( r_i s_i = m/N \) for \( i = 1, \ldots, N \) and \( h \) isolated vertices where \( h = 1 + 2N - [(r_1 + s_1) + \cdots + (r_N + s_N)] \). Since \( r_i, s_i \geq 1 \), we conclude that \( h \leq 1 \). Now it is easy to show that \( h = 1 \) and the proof of (1) is complete. The proof of part (2) follows as in Theorem 1.1. \( \blacksquare \)

A similar argument as in the proof of Theorem 1.2 gives the following result.

**Theorem 3.2.** Let \( G \) be a bipartite graph with \( 2N + 1 \) vertices. Then inequality (4) remains true. Moreover, the equality holds if and only if \( G \) is the direct sum of isolated vertices and complete bipartite graphs \( K_{r_1,s_1}, \ldots, K_{r_j,s_j} \) such that \( r_1 s_1 = \cdots = r_j s_j \).

We next compare our bound \( \tilde{E}(G) \) given in Theorem 1.1, with several bounds that have appeared in the literature.
In [9], it was shown that $E(G) \leq E^*(N, m)$, if $n = 2N$ and $m \geq N$, where

$$E^*(N, m) = 2m^2 N + 2 \sqrt{(N - 1) \left[ m - \left( \frac{m}{N} \right)^2 \right] }.$$ 

**Theorem 3.3.** If $N^3 q \leq m^4$ then $E(G) \leq E^*(N, m)$.

**Proof.** Let $E_0(N, m, q) = (\sqrt{N}/2)E(G)$. If we take the derivative of $E_0$ with respect to $q$, we find that this function is strictly decreasing in $q$. On the other hand, 

$$m = \mu^2_1 + \sum_{i \geq 1} \mu^2_i \leq \mu^2_1 + \sqrt{(N - 1) \sum_{i \geq 1} \mu^4_i} = \mu^2_1 + \sqrt{(N - 1)(q - \mu^4_1)}.$$ 

Arguing as in [9], the function

$$f(x) = x^2 + \sqrt{(N - 1)(q - x^4)}$$

is decreasing in the interval $[\sqrt[4]{q/N}, \sqrt{q}]$ and since $N^3 q \leq m^4$, we have

$$\sqrt[4]{q/N} \leq \frac{m}{N} \leq \mu_1 \leq \sqrt{q}.$$ 

It follows that

$$m \leq \left( \frac{m}{N} \right)^2 + \sqrt{(N - 1) \left[ q - \left( \frac{m}{N} \right)^2 \right] },$$

and so

$$q_0 := \left[ \frac{m - (m/N)^2}{N - 1} \right]^2 + \left( \frac{m}{N} \right)^4 \leq q.$$ 

Now it is easy to show by direct calculation, that $2N^{-1/2}E_0(N, m, q_0) = E^*(N, m)$.  

**Remark 3.4.** Condition $N^3 q \leq m^4$ is not too restrictive. For example, it is verified by hexagonal systems and for a large family of bipartite complete graphs.

On the other hand, the bound $E(G)$ improves [11, inequality 51], restricted to bipartite graphs. More specifically, if $n = 2N$ (the case $n = 2N + 1$ is analogous) then

$$E(G) \leq \left[ 2m(2N - 1) + 2N \left( \frac{4m^2 - 2q}{2N(2N - 1)} \right)^{1/2} \right]^{1/2}.$$ 

This can be shown as follows: by squaring both sides of the inequality and by proper manipulations we obtain
\[ m(3N - 4) + N^2 \left( \frac{2m^2 - q}{N(2N - 1)} \right)^{1/2} + 2\sqrt{Q} \sqrt{N - 1} (N - 2) \geq 2(N - 1) \left( m^2 - Q + \frac{(N - 2)\sqrt{Q}m}{\sqrt{N - 1}} \right)^{1/2}. \]

Since
\[ \left[ m(3N - 4) + N^2 \left( \frac{2m^2 - q}{N(2N - 1)} \right)^{1/2} + 2\sqrt{Q} \sqrt{N - 1} (N - 2) \right]^2 \geq \left[ m(3N - 4) \right]^2 + 2\left[ m(3N - 4) \right] \left[ 2\sqrt{Q} \sqrt{N - 1} (N - 2) \right], \]

the problem reduces to show that
\[ \left[ m(3N - 4) \right]^2 + 2\left[ m(3N - 4) \right] \left[ 2\sqrt{Q} \sqrt{N - 1} (N - 2) \right] \geq 4(N - 1)^2 \left( m^2 - Q + \frac{(N - 2)\sqrt{Q}m}{\sqrt{N - 1}} \right), \]

which can be easily verified.

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