Medium Access Probability in Uniform Networks with General Propagation Models

Giancarlo Pastor¹,², Inmaculada Mora-Jiménez¹, Antonio J. Caamaño¹, and Riku Jäntti²
¹Department of Signal Theory and Communications, King Juan Carlos University, Madrid, Spain
²Department of Communications and Networking, Aalto University, Espoo, Finland

Abstract—In this paper, we consider a random network in which the nodes are deployed as a Poisson Point Process. Under this model, we find the medium access rate of nodes using a CSMA-type MAC protocol. For finite network areas this is a time-frequency parameter, while for infinite network areas it translates into a spatial parameter since stationarity can be assumed. The expressions found here make the definition and geometry of contenders explicit. Further, the expressions allow the use of general propagation models. We also present several practical calculations when the network coverage area is finite and regular and when different fading distributions are considered.

Index Terms—Medium access probability, stochastic geometry, random networks, Poisson point process.

I. INTRODUCTION

The theory of stochastic geometry provides useful tools for the analysis of wireless networks, given that most of the network performance metrics depend significantly on the spatial density of transmitters and receivers. This is mainly due to distance dependency of propagation models and interference. One main element of stochastic geometry is spatial point processes which can model the positions of nodes in a wireless network. In particular, for uniform deployments (e.g. sensor networks) a well accepted model is the Poisson point process [1] which, under a refinement of the density, allows the modeling of networks using MAC protocols [2], [3]. For an introduction to stochastic geometry and its applications to wireless networks refer to [4], [5].

In this paper, we focus on an ad-hoc network with uniformly deployed wireless nodes accessing the medium using a CSMA-type MAC protocol. We follow the approach of [3] and [6] where, for uniform and clustered networks, the spatial distribution of nodes with granted permission to transmit is characterized. However, unlike similar approaches assuming an infinite network, we are interested in finding practical expressions for finite network areas. Although this approach leads to a time-frequency parameter in the medium access probability, it translates into a spatial parameter for an infinite area network under the assumption of stationarity.

Our contributions are the following: (1) based on an arbitrary contention criterion, we present a general methodology to find the medium access probability of wireless nodes located independent and uniformly in finite network areas; (2) we find a very intuitive interpretation of the medium access probability which makes the definition and geometry of contenders explicit; further (3) our approach allows the inclusion of a set of active or priority nodes (e.g. primary users in cognitive networks) which cause a refinement of the medium access probability.

The remaining of this paper is organized as follows: Section II describes the system model and gives a brief theoretical introduction; Section III presents our methodology to find the medium access probability; Section IV gives examples using a regular area network, a commonly-used path loss function, and several fading distributions; Section V shows the numerical simulations; and Section VI summarizes the main results.

II. THEORETICAL BACKGROUND

This section presents the network and propagation models and briefly introduces two commonly-used elements of stochastic geometry.

A. Network model

Assume the nodes are independent and uniformly distributed. Then, we can model the network by a Poisson point process (PPP) $\Phi$ with intensity $\lambda$ (the density of nodes) [1]. The number of nodes inside a sector $S$ is an independent and identically distributed (i.i.d.) random variable (r.v.) given by

$$\Phi(S) \sim \text{Poisson}(\lambda|S|),$$

where $\Phi$ is the point measure and represents the network (i.e. set of nodes) and $|S|$ is the area of $S$. Then

$$P(\Phi(S) = k) = \frac{(\lambda|S|)^k}{k!} \exp(-\lambda|S|). \quad (1)$$

Under this model, nodes using a CSMA-type protocol can be modeled as nodes contending with its neighbors such that (s.t.) only the node with the smaller time mark achieves transmission.

In this paper, we will use the (conditional) probability generating functional (PGFL) which is defined as follows.

Definition 2.1 (PGFL of $\Phi$):

$$G_\Phi(w(.) = \mathbb{E} \left( \prod_{x \in \Phi} w(x) \right). \quad (2)$$

where function $w : x \in \Phi \mapsto [0,1]$, s.t. $1 - w$ is integrable. A dot in the argument of a function is a standard notation in stochastic geometry. It means that the function takes all the points $x$ of the point process of interest, e.g. of $\Phi$ in (2).
Definition 2.2 (Conditional PGFL of $\Phi$):

$$G_\Phi(w(.)) = E^i \left( \prod_{x \in \Phi} w(x) \right),$$

where $E^i$ is the expectation associated to the reduced Palm measure $P^i$. In particular, for the PPP both definitions match

$$G_\Phi(w(.)) = G_\Phi(w(.)) = \exp \left( -\lambda \int_{\mathbb{R}^2} (1 - w(x)) dx \right). \quad (3)$$

In what follows, variable $x$ sometimes refers to the node entities themselves, i.e. with no notion of coordinates (e.g. Sec. III), and sometimes to node locations (e.g. Sec. IV).

B. Propagation model

Assume a general propagation model with two factors, a path loss or power attenuation function and a fading distribution. Further, assume that the power attenuation is normalized s.t. the transmission power and all scaling coefficients are gathered in the fading r.v. Then, we denote by $h_x$ the fading effect suffered by the receiver from a node located at point $x$ or at distance $x$. Finally, the distance dependence in the path loss function $l$ is drawn by an arbitrary, bounded and non-increasing function

$$l: (0, \infty) \to (0, \infty) : x \mapsto l(x).$$

III. MEDIUM ACCESS PROBABILITY

This section presents our main results. For a network with independent and uniformly distributed nodes using a CSMA-type protocol, we will find the medium access probability (MAP). The independence between the positions and time marks of nodes will allow the assumption of independence between all point processes defined by time marks.

For a given threshold operation power $\theta$, we define the set of contenders of node $x_i \in \Phi$ as

$$N_i = \{ x \in \Phi \setminus \{ x_i \} : h_x l(x) \geq \theta \}. \quad (4)$$

This is a very simple contention criterion since it does not consider interference. It is similar to so called protocol model [7] except that fading is considered.

Then, the event “node $x_j$ does not contend with node $x_i$,” can be written as

$$u^j_i = \{ x_j \notin N_i \} \quad (5)$$

while the complementary event is $u^j_i = \{ x_j \in N_i \}$, i.e. “node $x_j$ contends with node $x_i$.”

We have the following result.

Proposition 3.1: Let $\Phi$ be a network with independent and uniformly distributed nodes using a CSMA-type protocol. For each node, assume its time mark is independent and uniformly distributed in $[0, 1]$. Then, the time-frequency with which a node $x_i$ is in the set $\Psi \subset \Phi$ of nodes retained by the protocol (e.g. the nodes which are granted permission to transmit), given that $k-1$ nodes at positions $\{x_1, \ldots, x_{k-1}\}$ are simultaneously retained, is

$$\mathbb{P}(x_k \in \Psi|x_{[k-1]} \in \Psi) = \frac{1}{\lambda} \int_0^\lambda G_x(1 - \mathbb{P}(u^{[k-1]}), v^k)) ds,$$

where $\lambda$ is the density of the network $\Phi$, the collection $[k] = \{1, 2, \ldots, k\}$, function $w : \Phi \to [0, 1]$, and $G_x(.)$ is the PGFL of a PPP $\Phi_x$ with intensity $s$ (e.g. $\Phi = \Phi_x$).

Proof: Let $m(x_i) = t_i$ be the time mark of node $x_i$ and assume

$$0 = t_{k+1} \leq t_k \leq \cdots \leq t_1 \leq t_0 = 1.$$

For each $i \in [k]$, define the sets

$$\Phi_{[t_{i+1}, t_i]} = \{ x \in \Phi : m(x) \in [t_{i+1}, t_i) \}.$$

Then, $x_1, \ldots, x_k \in \Psi$ if the following $k-1$ conditions hold

$$C_i = \{ \Phi_{[t_{i+1}, t_i]} (\cup_j N_j) = 0 \}, \ i = 2, \ldots, k$$

which, for each $i$, represents the event that nodes $x_1, \ldots, x_i$ do not contend with nodes with smaller (hence better) marks. These conditions are independent since $\Phi_{[t_{i+1}, t_i]}$ and $x_i$ are independent PPPs and r.v.s, respectively. Then,

$$\mathbb{P}(C_i | t_{i+1}, t_i) = \mathbb{P}(u^i_2, \forall z \in \Phi_{[t_{i+1}, t_i]}) = \mathbb{P}(\Phi_{[t_{i+1}, t_i]}(N_{[i]})) = 0$$

$$\exp \left( -\lambda (t_i - t_{i+1}) \int_{\mathbb{R}^2} (1 - \mathbb{P}(u^i_2)) dz \right)$$

$$G_{\lambda (t_i - t_{i+1})}(b_i), \text{ with } b_i = \mathbb{P}(u^i_2). \quad (6)$$

Notice that

$$B_i = \int_{\mathbb{R}^2} (1 - \mathbb{P}(u^i_2)) dz \quad (7)$$

is the “measure” of neighbors of nodes $x_i$ when the density is normalized to 1. Now, let $T = \{ t_1, \ldots, t_k \}$. Then,

$$\mathbb{P}(x_k \in \Psi | T) = \mathbb{P}(\cap_{i \leq k} C_i | T)$$

and we just need decontrolling in the marks as follows.

$$\mathbb{P}(x_k \in \Psi) = \int_{[0, 1]^k} \mathbb{P}(\cap_{i \leq k} C_i | T) d\rho_{i \leq k}$$

\[6\]

$$\int_{[0, 1]^k} \prod_{i \leq k} G_{\lambda (t_i - t_{i+1})}(b_i) dt_i$$

$$\int_{[0, 1]^k} \prod_{i \leq k} \exp(-\lambda (t_i - t_{i+1}) B_i) dt_i$$

$$= \int_{[0, 1]^k} \prod_{i \leq k} \exp(-s(B_i - B_{i-1})) dt_i \quad B_0 = 0$$

$$= \int_{[0, 1]^k} \prod_{i \leq k} \exp (-s_i (B_i - B_{i-1})) \frac{ds_i}{\lambda} \quad \lambda t_i = s_i$$

$$= \prod_{i \leq k} \frac{1}{\lambda} \int_0^\lambda \exp(-s(B_i - B_{i-1})) ds$$

$$= \prod_{i \leq k} \frac{1}{\lambda} \int_0^\lambda \exp(-s \int_{\mathbb{R}^2} [\mathbb{P}(u^{[i-1]}_2, v^i_2)] ds)$$

$$= \prod_{i \leq k} \frac{1}{\lambda} \int_0^\lambda G_x(1 - \mathbb{P}(u^{[i-1]}_2, v^i_2)) ds \quad \mathbb{P}(x_i \in \Psi | x_{[i]} \in \Psi) \]
This result is very intuitive since it makes the well-known chain rule of probability explicit, e.g. when \( k = 2 \) we have
\[
P(x_1, x_2 \in \Psi) = P(x_2 \in \Psi | x_1 \in \Psi) P(x_1 \in \Psi).
\]
Note also that when nodes \( x_1, \ldots, x_k \) are far distant from each other the event “node \( x_j \) does not contend with nodes \( x_{[i-1]} \) and contends with node \( x_i \),” tends to “node \( x_j \) contends with node \( x_i \),” i.e.
\[
\{u_{i-1}, v_j\} \rightarrow \{v_j\},
\]
e.g. when \( k = 2 \) we have that
\[
P(x_2 \in \Psi | x_1 \in \Psi) \rightarrow P(x_2 \in \Psi), \text{ as } |x_2 - x_1| \rightarrow \infty.
\]
Proposition 3.1 can be written in the more compact form as follows.

**Proposition 3.2:** Under the conditions of Proposition 3.1,
\[
P(x_k \in \Psi | x_{[k-1]} \in \Psi) = \frac{P(|\mathcal{N}_k \setminus \mathcal{N}_{[k-1]}| > 0)}{E[|\mathcal{N}_k \setminus \mathcal{N}_{[k-1]}|]},
\]
where the RHS is the probability that node \( x_k \) contends for the medium, divided by the expected effective number of potential contenders, i.e. the neighborhood-dependency of the medium access probability is shown now explicitly.

**Proof:** From Proposition 3.1, simple calculations show that
\[
P(x_k \in \Psi | x_{[k-1]} \in \Psi) = \frac{1 - \exp(-\lambda c_k)}{\lambda c_k},
\]
where
\[
\lambda c_k = \int_{\mathbb{R}^2} P(u_{[k-1]}, v_k^k) d\lambda = E[|\mathcal{N}_k \setminus \mathcal{N}_{[k-1]}|]
\]
is the expected number of nodes which are in the neighborhood of node \( x_k \) and are not in the neighborhoods of nodes \( x_{[k-1]} \). It can also be understood as the expected number of remaining contenders of node \( x_k \), i.e. those nodes which have not lost the contention with nodes \( x_{[k-1]} \) in the current MAC frame.

Now, since \( |\mathcal{N}_k \setminus \mathcal{N}_{[k-1]}| \) is a Poisson r.v. [8] we have that
\[
P(|\mathcal{N}_k \setminus \mathcal{N}_{[k-1]}| = 0) \equiv \exp(-\lambda c_k),
\]
and we have the result.

A brief remark on the definition of the set of contenders. Definition (4) is a very simple criterion. To further consider interference and noise, these variables should be introduced in the definition as follows. For a given threshold operation SNR \( \Theta \), we define the set of contenders of node \( x_i \in \Phi \) as
\[
\mathcal{N}_i = \left\{ x \in \Phi \setminus \{x_i\} : \frac{h(x)}{I_i + N_i} \geq \Theta \right\},
\]
where r.v.s \( I_i \) and \( N_i \) represent the interference and noise, respectively, suffered at position \( x_i \). Now, since these two variables also depend locally we should be able to characterize (8) in terms of the distribution and geometry of neighbors. However, the expressions for the statistics of the aggregate interference lack tractability in both the finite and infinite network area [9], [10]. Hence a better approach is to use an interference control technique [11] which allows to bound both the interference and noise variables so that we can continue using the simple contention criterion (4).

**IV. PRACTICAL APPLICATIONS**

This section presents two types of practical applications in the form of examples. Notice that the probabilities found in the previous section are time-frequency parameters since the contention process repeats in each MAC frame. However, for infinite network areas we can assume stationarity and then the results easily translate into spatial parameters.

**Density of the set \( \Psi \) of nodes retained by the protocol**

**Example 1 \((k = 1)\):** The time-frequency for a node \( x_1 \) to be granted permission to transmit is (cf. equation (5) in [3])
\[
\lambda_1 = \frac{1 - \exp(-\lambda c_1)}{\lambda c_1} = \frac{P(|\mathcal{N}_1| > 0)}{E[|\mathcal{N}_1|]},
\]
where \( c_1 = \int_{\mathbb{R}^2} \mathbb{P}(u_1^i) dz \).

Then, the point process \( \Psi \) can be understood as the thinning of PPP \( \Phi \) with constant retention function \( \lambda c_1 \), i.e. \( \Psi \) is approximated by a PPP with intensity \( \lambda c_1 \) (the modified Matérn thinning type II).

**Example 2 \((k = 2)\):** The time-frequency for a node \( x_2 \) to be granted permission to transmit given that a node \( x_1 \) has already been granted permission is (cf. equation (6) in [3])
\[
\lambda_2(x_1) = \frac{1 - \exp(-\lambda c_2)}{\lambda c_2} = \frac{P(|\mathcal{N}_2 \setminus \mathcal{N}_1| > 0)}{E[|\mathcal{N}_2 \setminus \mathcal{N}_1|]},
\]
where \( c_2 = \int_{\mathbb{R}^2} \mathbb{P}(u_2^i, v_2^i) dz \).

Then, the point process \( \Psi \) can be understood as the thinning of PPP \( \Phi \) with \( x_1 \)-dependent retention function \( \lambda_2(x_1) \), i.e. \( \Psi \) is approximated by an inhomogeneous PPP viewed by a tagged receiver \( x_2 \) with intensity \( \lambda c_2(x_1) \).

Hence by recursion we obtain refined versions of the density of the point process \( \Psi \) of retained nodes with respect to one tagged node \( x_k \) (tagged in the sense that it should be retained) given a collection of nodes \( x_1, \ldots, x_{k-1} \) retained by the protocol.

A similar result is presented in [6] for the Random Sequential Adsorption (RSA) model or Matérn type III PP with infinite network area. Further, under some conditions of continuity, their results extend to a Poisson-type point process, i.e. a point process, say \( \Psi \), obtained from a non-necessarily homogeneous thinning of PPP \( \Phi \) with density \( \lambda \). Here we state their result using our notation (5).

**Proposition 4.1:** Let \( \Phi \) be a network in which the nodes are deployed as a RSA point process using a CSMA-type protocol. Then, the time-frequency with which a node \( x_1 \) is in the set \( \Psi \subset \Phi \) of nodes retained by the protocol is
\[
P(\mathcal{N}_{\Psi} = \mathcal{N}_1) = \frac{1}{\lambda} \int_0^\infty G_\Psi(\mathbb{P}(u_1^i)) ds,
\]
where \( \lambda \) is the density of the network \( \Phi \), and \( G_\Psi \) is the PGFL of a Poisson-type point process \( \Psi \).

Note that the above is exactly Proposition 3.1 when \( k = 1 \), except for a small correction, where \( u_1^i \) should be replaced by \( u_1^i \) in the argument function of the PGFL \( G_\Psi \).
The circular sector network area scenario

Since the set of contenders defines the integration domain of the PGFL, closed-form expressions for the medium access probability can be obtained when regular area networks are considered. Thus, these expressions will show a dependence on the position of each node and hence on its neighborhood. Now it is straightforward to evaluate the effect of general path loss functions and fading distributions.

In what follows, let $k = 1$, and assume that the node $x_1$ is located in the center of a circular sector $S$, with radius $R$ and opening angle $\rho$ (in radians), which contains the set of potential contenders of node $x_1$ (see Fig. 1). Further, consider the simplified attenuation function $l(x) = |x|^{-\alpha}$.

**Example 3 (No fading):** Assume a no fading scenario, i.e. transmitted powers are not affected by fading effects. Then,

$$c_1 = \int_S P(h_x|x|^{-\alpha} \geq \theta)dx$$

$$= \rho \int_0^R 1_{(0,\infty)}(r) r dr$$

$$= \frac{1}{2} \rho R^2,$$

which is just the area of the circular sector $S$.

**Example 4 (Rayleigh):** Assume r.v. $h$ is exponentially distributed with inverse mean parameter $\eta$, i.e. $h \sim \text{Exponential}(\eta)$. Then,

$$\hat{c}_1 = \int_S P(h_x|x|^{-\alpha} \geq \theta)dx$$

$$= \rho \int_0^R \exp(-\eta \theta r^\alpha) r dr, \quad \text{and}$$

$$\frac{\hat{c}_1}{c_1} = \left( -\frac{2}{\alpha} \right) E_{\alpha-2}(\eta \theta R^\alpha),$$

where $E_n(z)$ is the exponential integral function.

**Example 5 (Nakagami-m):** Assume r.v. $h$ is gamma distributed with unit mean, shape parameter $m \geq \frac{1}{2}$ and hence scale parameter $\frac{1}{m}$, i.e. $h \sim \Gamma(m,\frac{1}{m})$. Then,

$$\hat{c}_1 = \int_S P(h_x|x|^{-\alpha} \geq \theta)dx$$

$$= \rho \int_0^R Q(m, m \theta r^\alpha) r dr, \quad \text{and}$$

$$\frac{\hat{c}_1}{c_1} = Q(m, m \theta R^\alpha) + \psi_0 \psi_1 Q(m + \frac{2}{\alpha}, m \theta R^\alpha), \quad (9)$$

where $\psi_0 = \frac{1}{R^2 \theta \pi}, \quad \psi_1 = \frac{\Gamma(m+\frac{2}{\alpha})}{m \pi \Gamma(m)}$, $Q(n,z) = \frac{\Gamma(n,z)}{\Gamma(n)}$ (resp., $P(n,z) = 1 - Q(n,z)$) is the regularized upper (resp., lower) incomplete gamma function, $\Gamma(n,z)$ is the incomplete upper gamma function, and $\Gamma(z)$ is the gamma function.

Recall that in the Example 5, if we take $m = 1$ we have the Rayleigh fading (with parameter $\eta = 1$). Further, if we take $\alpha = 4$ in the Example 4, as some $\theta, \eta, R \rightarrow 0$ we have that

$$2 \frac{\hat{c}_1}{c_1} = \sqrt{\frac{\pi}{2}} \frac{\text{erf}(R^2 \sqrt{\pi} \theta)}{R^2 \sqrt{\pi} \theta} \approx \psi_0 \sqrt{\pi} P(\frac{1}{2}, R^4 \eta \theta) \rightarrow 2.$$
A brief description of Fig. 2.

- In Fig. 2 (a) we plot the MAP when we have no fading for different values of the radius $R$. As expected, the MAP decreases as the number of potential contenders increases with $R$.

- In Fig. 2 (b) we plot the MAP when we have Rayleigh fading for different values of the inverse mean parameter $\eta$. As expected, the MAP increases as the mean transmitted power decreases, i.e., the effect of contenders vanishes when $\eta$ increases.

- In Fig. 2 (c) we plot the MAP when we have Nakagami-$m$ fading for different values of the parameter $m$. It is shown that the MAP is not affected by changes in the parameter $m$. This normalization behavior can be explained by the unit mean assumption in r.v. $h$.

- In Fig. 2 (d) we plot the MAP when we have log-normal fading for different values of the parameters $\mu$ and $\sigma$. It is shown that the MAP is not affected by changes in the parameters $\mu$ or $\sigma$. Now, this normalization behavior can be explained by the relation between the error and regularized incomplete gamma functions.

VI. CONCLUSIONS

In this paper, we considered a random network using a CSMA-type protocol. The independent and uniformly deployed nodes were modeled as a Poisson point process, and the time marks were modeled as independent and uniformly random variables. Under this model, we found the medium access probability. The expressions found here made the effect of the definition and geometry of contenders explicit. These expressions are based on the probability generating functional, where the integration domain is defined by the area of contenders. Finally, we presented several examples for a regular area network, a commonly-used path loss function and different fading distributions.

REFERENCES