Orthogonality Catastrophe and Decoherence in a Trapped-Fermion Environment

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The Fermi-edge singularity and the Anderson orthogonality catastrophe describe the universal physics which occurs when a Fermi sea is locally quenched by the sudden switching of a scattering potential, leading to a brutal disturbance of its ground state. We demonstrate that the effect can be seen in the controllable domain of ultracold trapped gases by providing an analytic description of the out-of-equilibrium response to an atomic impurity, both at zero and at finite temperature. Furthermore, we link the transient behavior of the gas to the decoherence of the impurity, and to the degree of the non-Markovian nature of its dynamics.

A Fermi gas may be shaken up by the switching of even a single, weakly interacting impurity, producing a complete rearrangement of the many-body wave function, which loses essentially any overlap with the initial, unperturbed one. This is the essence of Anderson’s orthogonality catastrophe [1,2], witnessed by the singular (edgelike) behavior of the excitation energy distribution. Such a many-body effect comes into play in x-ray photoemission spectra from most simple metals, where the expected sharp symmetric peak at the binding energy of a core level is converted into a power law singularity, as predicted by Mahan-Nozières–De Dominics (MND) theory [3,4]. Similar patterns have been observed in electron emission from carbon based nanomaterials [5] and for quantum dots [6]. Fermi-edge resonance and orthogonality catastrophe have also been revealed by nonequilibrium current fluctuations in nanoscale conductors [7] and enter prominently the physics of phenomena as diverse as the Kondo effect [2,8] and the scattering or sticking of low-energy atoms or ions on metal surfaces [9,10].

Recently, it has been proposed to observe this universal physics with ultracold atoms, probing the singular behavior either in the time domain, by Ramsey interference [11], or in the frequency domain, by radio-frequency spectroscopy [12]. However, an analytic framework for the case of a trapped Fermi gas is lacking. In this Letter, we provide such an analytic description and discuss the transient response of a harmonically trapped Fermi gas following the sudden switching of an embedded two-level atom excited by a fast pulse. The interaction with the impurity produces a local quench of the gas, giving rise to the Anderson catastrophe. We study the Fermi-edge physics at zero and finite temperature and both in the frequency domain, by looking at the excitation spectrum of the gas, and in the time domain, by analyzing the dynamics of the impurity. Thus, we link the Fermi-edge behavior of the excitation energy distribution to the decoherence of the impurity. In particular, we investigate the Loschmidt echo [13,14] and non-Markovian nature, using recently developed tools [15–19], employed so far to study open systems in different environments, ranging from spins [20] to Bose-Einstein condensates [21], and experimentally tested in optical setups [22,23]. We find that the non-Markovian nature of the decoherent dynamics of the impurity provides a novel interpretation of the essential physics of the shakeup process.

We consider a gas of noninteracting cold fermions confined by a one-dimensional trapping harmonic potential of frequency $\omega$, described by the Hamiltonian $\hat{H}_0 = 1/2 \sum_n \xi_n \hat{c}_n^\dagger \hat{c}_n$, with $\xi_n$ being the annihilation operator for the $n$th single-particle state of energy $\epsilon_n = \hbar \omega (n+1/2)$ and spin $\xi$. We add a two-level impurity (an atom of a different species from the trapped component), with internal states $|g\rangle$, $|e\rangle$ and Hamiltonian $\hat{H}_I = \sum_{i=0}^{2} \epsilon_i |i\rangle\langle i|$, trapped in an auxiliary potential and brought in contact with the Fermi gas. This can be achieved using a species selective dipole potential that has a frequency much greater than the trap which contains the gas, so that the impurity motion is essentially frozen. We assume that when the impurity is in the $|g\rangle$ state, it has a negligible scattering interaction with the gas; hence, the Hamiltonian of the composite system is given by $\hat{H} = \hat{H}_0 + \hat{H}_I + \hat{V} \otimes |e\rangle\langle e|$. With the fermions in their equilibrium configuration, set by $\hat{H}_0$, we suppose the impurity to be quickly excited; the gas then feels a sudden perturbation $\hat{V}(t) = V \delta(t)$, assumed to have an s-wave-like character.

At sufficiently low temperatures, the pseudopotential approximation is invoked, which amounts to replacing the complicated atomic interaction potential with an
effective short range potential of strength \( V_0 \), localized at the minimum of the harmonic well, which we scale with the trap length \( x_0 \) such that \( V(x) = \pi V_0 x_0 \delta(x) \). Because of the parity of the single-particle wave functions, only the fermions lying in even-parity states \(( n = 2r, \text{ with } r = 0, 1, 2, \ldots )\) feel the impurity and are involved in the shakeup process. Explicitly, the fermion-impurity interaction is given by \( V = \sum_{r,r'} e^{i x r} e^{-ir} V_{rr'} \), where \( V_{rr'} = V_0 (\cdot)^{r+r'} r^{1/2} \gamma_r \). We label the highest occupied level by \( n_F = 2r_F \), with \( r_F \) a positive integer, so that the Fermi energy reads \( \varepsilon_F = \hbar \omega (2r_F + 1/2) \).

A key quantity for the following is the vacuum persistence amplitude

\[
\nu_\beta(t > 0) = \langle e^{i/h} \hat{\rho}_0 e^{-i/h} \hat{\rho}_0 \rangle,
\]

with \( \langle \cdots \rangle \) denoting the grand canonical average over the unperturbed fermion state. \( \nu_\beta(t) \) is the probability amplitude that the gas will retrieve its equilibrium state at time \( t \), after the switching on of the perturbation, and its modulus gives the decoherence factor for the impurity (see below).

The Fourier transform \( \tilde{\nu}_\beta(E) \) gives the excitation spectrum of the gas. In the interaction picture, we get

\[
\tilde{\nu}_\beta(t) = (Te^{i/h} \int_0^t dt' \tilde{\nu}(t')), \quad \tilde{\nu}(t) = e^{i/h} \hat{\rho}_0 \tilde{\nu} e^{-i/h} \hat{\rho}_0,
\]

which, by virtue of the linked cluster theorem, reduces to an exponential sum of connected Feynman diagrams \( \nu_\beta(t) = e^{\Lambda_\beta(t)} \), with

\[
\Lambda_\beta(t) = \Lambda_\beta^1(t) + \Lambda_\beta^2(t) + \cdots.
\]

The closed graphs in \( \Lambda_\beta(t) \) contain products of vertices \((V_{rr})\) and lines \((G^\beta_{rr})\) representing the unperturbed propagators

\[
i \hbar G^\beta_{rr}(t) = e^{-irr' e_i} [\theta(t) f_r - \theta(-t) f_{r'}],
\]

where \( f_r^z = \left[ 1 + e^{\beta (\varepsilon_r - \mu)} \right]^{-1} \) are the particle-hole distributions, and \( \mu \) denotes the chemical potential [24].

We focus on the lowest-order loops, namely,

\[
\hbar \Lambda_\beta^1(0) = -i \chi_x V_0 A^\beta_{00}(0),
\]

\[
\hbar^2 \Lambda_\beta^2(0) = -\chi_x V_0^2 \int_0^t dt' \int_0^t dt'' \Lambda^\beta_\beta(t') \Lambda^\beta(t''),
\]

with \( \chi_x = (2s + 1) \) accounting for the spin degeneracy and \( \Lambda^\beta_\beta(0) = \sum_{r=r_0}^{\infty} \gamma_r e^{2ir\omega t} f_r^z \).

This approximation will prove to accurately describe the singular response of the gas (contained in the two-vertex term) and to give the dominant contribution to the shakeup process if the interaction strength is small in the energy scale of the problem. The latter is set by both the level separation \( \hbar \omega \) and Fermi energy \( \varepsilon_F \), and we introduce \( \alpha = \chi_x V_0^2 / 2 \hbar \omega \varepsilon_F \) as a sensible interaction strength parameter.

The contribution (4) may be written as \( \hbar \Lambda_\beta^1(t) = -i t E^\beta_1 \). Here,

\[
E^\beta_1 = \sqrt{2 \chi_x \hbar \varepsilon_F \alpha \sum_{r=0}^{\infty} \gamma_r f_r^+}
\]

is the first-order shift to the gas energy, as provided by the Rayleigh-Schrödinger perturbation theory. The behavior of the unperturbed energy \( E^\beta_0 = \chi_x \sum \varepsilon_n f_n^+ \), and of its first and second-order corrections vs \( \varepsilon_F \) is shown in Fig. 1 for various temperatures. We notice that \( E^\beta_0 \) is 1 to 3 orders of magnitude larger than \( E^\beta_1 \), and that the Fermi energy plays an appreciable role in both \( E^\beta_0 \) and \( E^\beta_1 \) for \( \beta \hbar \omega \approx 0.05 \).

While \( \Lambda_\beta^1(t) \) only brings a phase factor to \( \nu_\beta(t) \), which corresponds to shifting the spectrum \( \tilde{\nu}_\beta(E) \) by \( E^\beta_1 \), the two-vertex connected graph gives the crucial contribution to the persistence amplitude. It can be split into three parts with well defined trends and physical meaning [24], i.e.,

\[
\Lambda_\beta^1(0) = \Lambda_{\beta 2}^1(0) + \Lambda_{\beta 2}^2(0) + \Lambda_{\beta 3}^2(0).
\]

These represent a (further) energy shift, a Gaussian envelope due to finite temperature effects, and periodic terms originating from the equal spacing of the unperturbed single-particle states, respectively, separately analyzed in Figs. 1(c), 2(a), and 2(b).

The first one \( \hbar \Lambda_{\beta 2}^2(0) = -i t E^\beta_2 \) provides the second-order correction to the energy of the gas (the \( n > 2 \)-vertex graphs would complete the perturbation series):

\[
E^\beta_2 = \alpha \varepsilon_F \sum_{r \neq r'} f_r^z \gamma_r \gamma_{r'} f_{r'}^z.
\]

Comparing Figs. 1(b) and 1(c), we notice that the chosen value of \( \alpha \) lets \( E^\beta_2 \) take absolute values smaller than \( E^\beta_1 \). However, \( E^\beta_2 \) is more sensitive to temperature than \( E^\beta_1 \) for \( \beta \hbar \omega < 0.05 \).

![FIG. 1 (color online). Equilibrium energy \( E^\beta_0 \) of a spin-1/2 gas into (a) a harmonic trap and perturbation corrections (b) \( E^\beta_1 \) [Eq. (6)] and (c) \( E^\beta_2 \) [Eq. (7)] due to the impurity potential \( V(x) \). All energy curves are reported in units of \( \hbar \omega \) vs \( \varepsilon_F / \hbar \omega \) for different values of \( \beta \hbar \omega \) and fixed coupling parameter \( \alpha = 0.4 \).](165303-2)
The second contribution \( \Lambda_{2p}^{\beta}(t) = -\delta_\beta \omega^2 r^2/2 \) produces a Gaussian damping in \( \nu_\beta(t) \) and, therefore, a Gaussian broadening in \( \tilde{\nu}_\beta(E) \) with standard deviation

\[
\delta_\beta = \sqrt{2\alpha g_\beta}, \quad g_\beta = \frac{\epsilon_\beta}{\hbar \omega} \sum_{r \neq r'} \gamma_r f_r^* f_{r'}.
\]  

The coefficient \( g_\beta \) is weakly influenced by the Fermi energy but strongly affected by temperature, changing by various orders of magnitude for \( \beta \hbar \omega \leq 0.5 \). No damping or broadening effects are present at the absolute zero, since \( \delta_\beta \to 0 \) for \( \beta \hbar \omega \to \infty \) [Fig. 2(a)].

The most important content of the second diagram, giving a nontrivial structure to \( \nu_\beta(t) \), arises from the third contribution [24]:

\[
\Lambda_{3p}^{\beta}(t) = -\frac{\alpha \epsilon_\beta}{2 \hbar \omega} \sum_{r \neq r'} \frac{1 - e^{2i(r-r')/\omega}}{(r-r')^2} \gamma_r f_r^* f_{r'}.
\]

Because of the harmonic form of the trapping potential, this is periodic in time with frequency \( 2\omega \); see Fig. 2(b). The zeroes of this subgraph (at \( \omega t = m \pi \) with \( m = 0, \pm 1, \pm 2 \)), when combined with the Gaussian damping (8), yield modulations in the vacuum persistence amplitude which, as discussed below, are a signature of non-Markovian dynamics of the impurity.

Leaving aside the shifts, the persistence amplitude is then

\[
\nu_\beta(t) = e^{-\delta_\beta \omega^2 t^2/2} e^{\Lambda_{3p}^{\beta}(t)}.
\]

Of particular interest for the discussion below is the behavior of \( |\nu_\beta(t)| \) exhibiting spikes at \( \omega t \approx \pi, 2\pi, \ldots \), which becomes more and more pronounced with increasing \( \beta \hbar \omega \); see the left panels in Fig. 3. The periodicity in the time domain is reflected in the excitation spectrum \( \tilde{\nu}_\beta(E) \) that offers an asymmetric, broadened signature of the singular behavior of the Fermi gas. The monotonic structure turns into a sequence of subpeaks, separated by \( 2\hbar \omega \) and related to even-level transitions in the gas as \( \beta \hbar \omega \) gets above \( \sim 0.5 \) [see Fig. 3(b)]. These features are observed for any \( r_F \) in the range of 5 to 100 [24].

The coefficient (8) of the Gaussian power law and the periodic contribution \( \Lambda_{3p}^{\beta}(t) \) can be approximated as [24]

\[
g_\beta = 2 \sum_{m=1}^{\infty} (-1)^m m e^{\beta \hbar \omega m^2/2} e^{|\beta \hbar \omega m^2|/4} \]  

and

\[
\Lambda_{3p}^{\beta}(t) = \alpha \sum_{m=-\infty}^{\infty} \ln \left( e^{2\tau_0 \omega - 1} - 1 \right) e^{2\tau_0 \omega \omega}. 
\]

At low temperatures, the leading behavior of the Gaussian standard deviation is \( \delta_\beta = 2\alpha^{1/2} e^{-\beta \hbar \omega/4} \) for thermal energies \( \beta \hbar \omega \geq 6 \) [see Fig. 2(a)]. On the other hand, Eq. (12) contains a singularity at the absolute zero that we regularized by introducing a cutoff parameter \( \tau_0 \). This regularization is needed to remove a zero temperature indefiniteness of the analytic approximation, whereas the numerical evaluation of the vacuum persistence amplitude does not suffer from divergence problems. As shown below and as detailed in Ref. [24], a similar parameter enters the original MND theory, and we can interpret it as the time scale over which transitions occur in the gas. On the other
hand, thermal fluctuations introduce other characteristic times \( \tau_m = m \beta \hbar \).

Taking \( g_\beta \) and \( \Lambda_{2\text{e}}(t) \) as in Eqs. (11) and (12) and using them in Eq. (10) gives an accurate approximation to the numerical results for \( \beta \hbar \omega \approx 0.1 \), a number of particles larger than 10, and for suitable values of the cutoff parameter, say, \( \omega \tau_0 < 0.02 \) [see Fig. 4(a) and Ref. [24]]. In particular, at \( T = 0 \), the vacuum persistence amplitude takes the form

\[
\nu^{\alpha \rightarrow \mu}_{\beta \rightarrow \omega}(t) = \left[ \frac{e^{2\tau_0 \omega} - 1}{e^{2\omega(\tau_0 + it)} - 1} \right]^\alpha. \tag{13}
\]

To compare our findings to the one-dimensional free-fermion theory, one needs to fix \( \alpha \) and let the harmonic frequency go to zero by keeping the number of particles in the gas \( (2\pi_F = \omega_F/\hbar \omega) \) finite. No Gaussian damping occurs in this case, and the two-vertex graph tends to

\[
\Lambda_{\text{NMD}}(t) = -\alpha \ln(\tau_0 + 1), \tag{14}
\]

yielding the Nozières–De Dominicis propagator \( \nu_{\text{NMD}}(t) = (it/\tau_0 + 1)^{-\alpha} \), originally calculated for a suddenly switched-on core hole in a free electron gas [4]. Equation (14) was obtained by a long-time limit solution of the generalized Dyson equation for the electron Green’s function in a constant window potential of width \( \hbar/\tau_0 \). For this reason, the MND spectrum lacks formal justification away from the threshold. In the present derivation, we have taken into account the full perturbation at an arbitrary time \( t > 0 \), retaining only the first nonadiabatic contribution in the linked cluster expansion [25]. We expect the effect of higher-order diagrams to be mainly concerned with the adiabatic correction to the equilibrium energy and some additional broadening of the excitation peaks. The latter should provide a renormalization to the critical parameter. Nevertheless, in the investigated ranges of temperatures and particle numbers, our definition of \( \alpha \) produces a markedly singular response with the same range of criticality as the MND edge response parameter \( \alpha = 0 \rightarrow 1 \).

From this comparison with the free-gas case, we learn that the trapping frequency \( \omega \) enters crucially the physics of the shakeup process. Indeed, it modifies the long-time response of the gas, as all single-particle excitations involve energy exchanges which are now even multiples of \( \hbar \omega \). This gives rise to the periodic part of the fermion response and to the corresponding spectral peaks broadened at finite temperatures due to the Gaussian envelope, the latter being a typical effect of suddenly switched perturbations [2]. Up to now, we treated the response of the Fermi gas without any reference to the dynamics of the impurity that has just been assumed in the excited states for \( t > 0 \). If, instead, the two-level atom is subject to (say) a fast \( \pi/2 \) pulse and quickly prepared in the superposition \( (|g\rangle + |e\rangle)/\sqrt{2} \), it experiences a purely dephasing dynamics due to the coupling with the gas, and its state at later times is \( \rho_{\text{deh}}(t) = (|g\rangle\langle g| + |e\rangle\langle e| + \nu_{\beta}(t)|g\rangle \times \langle e| + \text{H.c.})/2 \). The decoherence factor entering the off-diagonal elements is just the persistence amplitude obtained before, going to zero at long times due to the orthogonality catastrophe. In the theory of open systems, one typically uses the so-called Loschmidt echo \( L(t) = |\nu_{\beta}(t)|^2 \), which gives a measure of the environmental response to the perturbation induced by the system [13,14,26] and which is linked with the non-Markovian nature of the open system dynamics [19]. The degree of the non-Markovian nature of a dynamical map can be evaluated in different manners [15–18], which are essentially equivalent for a purely dephasing quantum channel [19,27]. By adopting the information flow approach of Ref. [15], one finds

\[
\mathcal{N} = \sum_n |\nu_{\beta}(t_{\text{max},n})| - |\nu_{\beta}(t_{\text{min},n})|, \tag{15}
\]

where the summation is performed over all maxima and minima of \( |\nu_{\beta}(t)| \), occurring at \( t_{\text{max},n} \) and \( t_{\text{min},n} \), respectively. Using our previous results for the amplitude, we obtain the non-Markovian nature of the dynamics of a two-level system in a trapped Fermi environment. The results are shown in Fig. 4b, where we see that \( \mathcal{N} \) depends on the temperature and on the critical parameter \( \alpha \). In particular, it has a maximum at small \( \alpha \) increasing with low temperatures, and goes to zero both for large temperatures (as thermal fluctuations suppress oscillations in the persistence amplitude) and for \( \alpha > 1 \). In the latter case, excitations are generated at every energy scale in the gas, as witnessed by the fact that the spectrum becomes structureless. This implies that the gas becomes more and more stiff (in the sense that it is not able to react on the impurity any more) and explains why \( \mathcal{N} \) is zero: the open system does not receive information back, its Loschmidt echo decays monotonically and the dynamics is Markovian. A non-Markovian dynamics, then, can be characterized in our
case by the appearance of specific spectral features in the excitation energy distribution [28].

We conclude with two remarks. First, the spectral distribution of energy excitations obtained here coincides with the so-called work distribution function, which is a central quantity in nonequilibrium processes [29,30]. In the setup described above, it is simple enough to conceive a “reverse” protocol, with the Fermi gas brought to thermal equilibrium in the presence of the impurity (i.e., with the two-level atom in the excited state) which is then switched off. The comparison of the work distribution functions in the direct and reverse protocols would lead to a direct experimental test of the Crooks relation in the quantum regime [31]. The second remark is on the experimental realization of the model that we have described. Many experiments have recently dealt with impurities in trapped Fermi gas [32], and state-dependent scattering lengths have been discussed [33]. This would lead to a direct test of our theory. Another viable candidate could be a gas of hard-core bosons in one dimension, where the Loschmidt echo is equivalent to that of the corresponding Fermi gas [34] and in which impurities have recently been experimentally generated [35].

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I. IMPURITY POTENTIAL

As explained in the main text, the non-interacting fermions in the harmonic trap lie in their equilibrium configuration, set by $H_0$, until the impurity is excited and the sudden perturbation $V(t) = V\theta(t)$ is felt by the gas. To mimic a very strong difference in scattering length depending on the internal state $|e\rangle$ of the impurity, we have modelled it by a spatially localized potential, activated by the population of the excited state, with the structure-less form $V(x) = \pi V_0 x_0 \delta(x)$. For mathematical simplicity we have placed the impurity at the minimum of the harmonic potential. Thus, the coupling matrix elements between two unperturbed one-fermion states, 

$$\int dx \psi_n^*(x)V(x)\psi_{n'}(x) = \pi V_0 x_0 \psi_n^*(0)\psi_{n'}(0),$$

involve the Harmonic oscillator wave-functions at $x = 0$. These have the usual expression

$$\psi_n(x) = \frac{x^{-1/2}\pi^{-1/4}}{2^{n/2}n!^{1/2}} H_n\left(\frac{x}{x_0}\right) e^{-x^2/2x_0^2},$$

in terms of the Hermite polynomials $H_n$ with $x_0$ being the characteristic oscillator length. By the parity of the Hermite polynomials, $H_n(-x/x_0) = (-1)^n H_n(x/x_0)$, we have $H_n(0) = 0$ for odd-$n$, i.e., $n = 2r+1$, with $r = 0, 1, \cdots, \infty$. Therefore, the impurity potential induces excitations which connect only unperturbed one-fermion states labeled by even numbers $n = 2r$, with $r = 0, 1, \cdots, \infty$. The representation of the Hermite polynomials with even numbers in power series,

$$H_{2r}\left(\frac{x}{x_0}\right) = \sum_{k=0}^{r} \frac{4^k(2r)!}{(2k)!}(\frac{-1}{(r-k)!})\left(\frac{x}{x_0}\right)^{2k},$$

allow us to write $H_{2r}(0) = (-1)^r(2r)!/r!$, which leads to the matrix elements

$$V_{rr'} = \int dx \psi_{2r}^*(x)V(x)\psi_{2r'}(x)$$

$$=\sqrt{\pi} V_0 \left(\frac{(2r)!^{1/2}(2r')!^{1/2}}{2^{r+r'} r!^{1/2} r'!^{1/2}}\right),$$

appearing in the second quantized representation of the impurity potential in the harmonic oscillator basis. Using supported in the derivation presented in the main text, which has been concerned with the sudden response of a trapped Fermi gas to an excited impurity. Arguments are organized as follows: in Sec. I, we discuss the properties of the model potential for the impurity atom. In Sec. II, we focus on the Dyson-Wick perturbation series and the Wick theorem, selecting equal-time and two-time contractions to obtain the linked cluster expansion for the vacuum persistence amplitude, which is reported in the main text and truncated at the 2-nd order. In Sec. III, we report on some convergence and stability tests with the numerical methods used to compute the one- and two-vertex connected diagrams entering the decoherence factor and the excitation spectrum of the gas. In Sec. IV, we deal with the analytical approximations for the vacuum persistence amplitude at low thermal energies and we provide a link to the Nozières-De Dominicis approach to the Fermi edge singular behavior of a free electron metal. Finally, in Sec. V we provide the details on how to calculate the non-markovianity measure for the impurity dynamics. Equations and figures from the main text are referenced as (M-1)-(M-15) and Fig. 1-Fig. 4, respectively. New equations and figures appearing in what follows will be labeled (S-1)-(S-55) and Fig. 5-Fig. 11, respectively.

These auxiliary notes are meant to provide some additional arguments and mathematical details for the derivation presented in the main text, which has been concerned with the sudden response of a trapped Fermi gas to an excited impurity. Arguments are organized as follows: in Sec. I, we discuss the properties of the model potential for the impurity atom. In Sec. II, we focus on the Dyson-Wick perturbation series and the Wick theorem, selecting equal-time and two-time contractions to obtain the linked cluster expansion for the vacuum persistence amplitude, which is reported in the main text and truncated at the 2-nd order.
the identity

\[ \gamma_r = \frac{\Gamma(r + 1/2)}{\Gamma(r + 1)} = 2^{-r/2} \pi^{r/2} \frac{(2r)!}{r!}, \]

where \( \Gamma \) is the Euler gamma function, we get \( V_{rr'} = V_0(-1)^{r+r'} \gamma_r^{1/2} \gamma_{r'}^{1/2} \). The diagonal matrix elements \( V_{rr} = V_0 \gamma_r \), shown in Fig. 5A, entirely determine the one-vertex graph of Eq. (M4), representing first order corrections to the single-particle energies \( \varepsilon_{2r} = \hbar \omega (2r+1/2) \). On the other hand, both diagonal and off-diagonal matrix elements of \( V(x) \), shown in Fig. 5B, appear as absolute squares, \( |V_{rr'}|^2 = V_0^2 \gamma_r \gamma_{r'} \), in the two-vertex graph given in Eq. (M5).

II. LINKED CLUSTER EXPANSION OF THE VACUUM PERSISTENCE AMPLITUDE

The vacuum persistence amplitude, introduced in Eqs. (M1) and (M2), may be expanded by the Dyson-Wick series

\[ \nu_\beta(t) = 1 + \sum_{m=1}^{\infty} \nu_\beta^{(m)}(t), \]

whose coefficients account for processes where the gas retrieves its equilibrium unperturbed configuration after \( m = 1, 2, \cdots \) ‘scatterings’ with the impurity potential:

\[ \nu_\beta^{(m)}(t) = \frac{(-i)^m}{\hbar^m m!} \int_0^t dt_1 \cdots \int_0^t dt_m \langle \hat{T} \hat{V}(t_1) \cdots \hat{V}(t_m) \rangle. \]

In the interaction picture, with

\[ \hat{V}(t) = \sum_{r,r',\xi} V_{rr'} \hat{c}_{2r\xi}^\dagger(t) \hat{c}_{2r'\xi}(t), \]

the time-evolution of creation and annihilation operators is that of the undisturbed Harmonic oscillator, i.e.,

\[ \hat{c}_{2r\xi}^\dagger(t) = e^{i\omega(2r+1/2)t} \hat{c}_{2r\xi}, \hat{c}_{2r\xi}(t) = e^{-i\omega(2r+1/2)t} \hat{c}_{2r\xi}. \]

Accordingly,

\[ \langle \hat{T} \hat{V}(t_1) \cdots \hat{V}(t_m) \rangle = \sum_{r_1, r_1', \xi_1} V_{r_1 r_1'} \cdots \sum_{r_m, r_m', \xi_m} V_{r_m r_m'} \times \langle \hat{T} \hat{c}_{2r_1\xi_1}^\dagger(t_1) \hat{c}_{2r_1'\xi_1}(t_1) \cdots \hat{c}_{2r_m\xi_m}^\dagger(t_m) \hat{c}_{2r_m'\xi_m}(t_m) \hat{c}_{2r_2\xi}(t_2) \cdots \hat{c}_{2r_m'\xi_m}(t_m) \hat{c}_{2r_1'\xi}(t_1) \cdots \hat{c}_{2r_m\xi_m}^\dagger(t_m) \hat{c}_{2r_1\xi_1}(t_1) \cdots \hat{c}_{2r_m\xi_m}^\dagger(t_m) \hat{c}_{2r_1\xi_1}(t_1) \cdots \hat{c}_{2r_m\xi_m}^\dagger(t_m) \hat{c}_{2r_1\xi_1}(t_1) \cdots \rangle. \]

Here, the time-ordered average at the right-hand side may be decomposed using the Wick’s theorem into sums of products where each factor is a contracted pairs of creation/annihilation operators. The central approximation of the work is to retain terms that contain only equal time and two-time contractions, namely

\[ \langle \cdots \hat{c}_{2r_j\xi_j}^\dagger(t_j) \hat{c}_{2r_j'\xi_j}(t_j) \cdots \rangle = \cdots f_{r_j r_j'}^{++} \delta_{r_j r_j'}, \]

and

\[ \langle \cdots \hat{c}_{2r_j\xi_j}^\dagger(t_j) \hat{c}_{2r_j'\xi_j}(t_j) \cdots \rangle = -\hbar^2 \delta_{\xi_j \xi_j'} G_{r_j}^\beta(t_j - t_j) \delta_{r_j r_j'} G_{r_j}^\beta(t_j - t_j) \delta_{r_j r_j'}, \]

where the Fermion occupation numbers \( f_{r}^\pm \) and the unperturbed Fermion propagators \( G_{r}^\beta(t) \) have been introduced in the main text (see Eq. (M3)). When these expressions are summed over even level numbers and integrated over time variables, we are left with products including either

\[ A_1^\beta(t) = -\frac{i(2s + 1)}{\hbar} \sum_{r} V_{rr} f_{r}^+, \]

or

\[ A_2^\beta(t) = -\frac{(2s + 1)}{2} \sum_{r,r'} |V_{rr'}|^2 \times \int_0^t dt' \int_0^t dt'' G_{r'}^\beta(t' - t'') G_{r'}^\beta(t'' - t'), \]

which are just the connected diagrams reported in Eqs. (M4) and (M5). Each product equals \( A_1^\beta(t) A_2^\beta(t)^{m-j} \), obtained by \( \binom{m}{j} \) distinct contractions for some \( j \) between 0 and \( m \).

This means that

\[ \nu_\beta^{(m)}(t) \approx \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} A_1^\beta(t)^j A_2^\beta(t)^{m-j}. \]

so that the Dyson-Wick series for the vacuum persistence amplitude takes the exponential form

\[ \nu_\beta(t) = \sum_{m=0}^{\infty} \frac{[A_1^\beta(t) + A_2^\beta(t)]^m}{m!} = e^{A_1^\beta(t) + A_2^\beta(t)}. \]

The single vertex graph (M4) gives rise to the first order energy shift discussed in the main text, while the two-vertex connected graph has a more involved structure. Performing the time-ordered integrals in (M5) we rewrite it as

\[ A_2^\beta(t) = -\frac{\alpha_{\xi'}}{\hbar} \sum_{r,r'=0}^{\infty} \left[ \frac{i\varphi_{rr'} + \psi_{rr'}(t)}{2\omega} \right] f_{r}^+ f_{r'}^-, \]

in which

\[ \varphi_{rr'} = \gamma_r \gamma_{r'}, \quad \psi_{rr'}(t) = \varphi_{rr'} \frac{1 - e^{2i(r-r')\omega t}}{r - r'}. \]

Here, we may separate the sums over even-state labels \( (r,r') \), so that the two-vertex connected graph can then be split into three contributions:

\[ A_2^\beta(t) = A_2^{\lambda_0}(t) + A_2^{\lambda_2}(t) + A_2^{\lambda_4}(t), \]

where the subscripts stand for Shift, Gaussian and Periodic, respectively. In particular:
(i) the off-diagonal summands in \((S-16)\) that multiply \(\varphi_{rr'}\) give rise to \(\Lambda_{2r}(t) = -iE_2^\beta/\hbar\), where
\[
E_2^\beta = \alpha \varepsilon_f \sum_{r \neq r'} f^+_r \varphi_{rr'} f^-_{r'}, \tag{S-17}
\]
is the energy correction reported in Eq. (M7)

(ii) the diagonal elements of Eq. \((S-16)\) yield the quadratic power law in \(\Lambda_{2r}(t)\) [see Eq. (M8)];

(iii) the remaining terms of the series in Eq. \((S-16)\) give the time periodic sub-diagram
\[
\Lambda_{2r}(t) = -\frac{\alpha \varepsilon_f}{2\hbar \omega} \sum_{r \neq r'} f^+_r \psi_{rr'} f^-_{r'}, \tag{S-18}
\]
also reported in Eq. (M9).

III. NUMERICAL COMPUTATIONS

As shown in the main text, the real and imaginary parts of the connected graphs \(\Lambda_{s}^\beta(t)\) [Eqs. (M4), Fig. 1B] and \(\Lambda_{2}^\beta(t)\) [Eq. (M5), Fig. 1C, Fig. 2] combine in the vacuum persistence amplitude to give:
\[
\nu_\beta(t) = e^{-\frac{\mu}{\alpha \varepsilon_f}(E^\beta_1 + E^\beta_2)} e^{-\alpha g_\beta \omega \alpha^2 t^2} e^{\Lambda_{s}^\beta(t)}. \tag{S-19}
\]
The knowledge of \(\nu_\beta(t)\), allows us to determine the decoherence factor \(|\nu_\beta(t)|\) (Fig. 3 A, C, E), the shake up spectrum \(|\tilde{\nu}_\beta(E)|\) (Fig. 3B, D, F, Fig 4A), and the non-Markovianity measure \(\mathcal{N}\) [Eq. (M15), Fig 4B]. Then, the basic quantities in our calculations are:

(i) the first and second order corrections, \(E_1^\beta\) [Eq. (M6), Fig. 1B] and \(E_2^\beta\) [Eqs. (M7), Fig. 1C], to the equilibrium energy \(E_0^\beta\) (Fig. 1A);

(ii) the coefficient \(g_\beta\) determining the standard deviation \(\delta_\beta\) [Eq. (M8), Fig. 2A] of the Gaussian sub-diagram \(\Lambda_{2s}^\beta\);

(iii) the shake-up sub-diagram \(\Lambda_{2r}^\beta\) [Eqs. (M9), Fig. 2B].

These contributions contain summations running over all one-fermion eigenstates of the trap weighted by the fermi factors \(f^+_r, f^-_{r'}\). These are expressed as \(f^+_r = \left[1 + e^{2\beta \hbar \omega (r-r_r)}\right]^{-1}\) and \(f^-_{r'} = \left[1 + e^{-2\beta \hbar \omega (r-r_{r'})}\right]^{-1}\), using the parametrization \(\mu = \hbar \omega (2r_\mu + 1/2)\) for the chemical potential. The index \(r_\mu\) tends to \(r_F + 1/4\) for \(\beta \to \infty\), so that the \(\mu\) lies in the middle between the highest occupied (\(\varepsilon_F = \varepsilon_{2r_F}\)) and the lowest unoccupied one-fermion levels (\(\varepsilon_{2r+1}\)) of the gas. For finite \(\beta\) we determined \(r_\mu\) by numerically constraining the conservation of particle number:
\[
2r_F + 1 = \sum_r \left(f^+_r + f^+_{r+1/2}\right),
\]
As shown in Fig. 6, \(r_\mu\) and hence \(\mu\), reach their maximum values at the absolute zero (\(\beta \to \infty\)). Both these quantities decrease with decreasing \(\beta\) and take largely negative values for \(\beta \to 0\) where the classical limit applies. Interestingly enough, they are almost independent on temperature for \(\beta \hbar \omega \gtrsim 0.4\) and \(r_F = 5 - 500\).

With the computed \(f^+_r, f^-_{r'}\) distributions, we ran numerical computations of the basic quantities (i)-(iii) using a high energy cut-off \(\varepsilon_{\text{cut}} \gg \varepsilon_F\). To fix \(\varepsilon_{\text{cut}} = \hbar \omega (2r_{\text{cut}} + 1/2)\), we performed convergency tests by changing \(r_{\text{cut}}\) in order to have a maximum instability error below 0.1%.

In our applications to a spin 1/2 gas, we observed that accurate estimations of \(E^\beta_2\) and \(g_\beta\) require values of \(r_{\text{cut}}\) of the order of \(10^4\) for \(r_F\) below \(\sim 200\) and \(\beta \hbar \omega\) larger than \(\sim 10^{-5}\). In particular, the plots of Fig. 1C and Fig. 2A of the main text were generated with a energy cut-off of \(10 - 10^3\) \(\varepsilon_F\). As for \(E^\beta_0, E^\beta_1,\) and \(\Lambda_{2r}^\beta\), we found out a cut-off of \(\sim 10 \varepsilon_F\) to be sufficient in the investigated ranges of fermion numbers and temperatures. Accordingly, we included up to \(10^3\) one-fermion states in Fig. 1A-B and Fig. 2C of the main text.

To provide a more complete picture of the basic quantities involved in the numerical calculations, in Fig. 7A-C we report the behavior of the equilibrium energy and the two energy shifts vs \(\beta \hbar \omega\) for gases made of different fermion numbers. We remark that both \(E^\beta_1\) and \(E^\beta_2\) are indeed small corrections to the unperturbed value \(E^\beta_0\) of a spin-1/2 gas, for values of the critical exponent \(\alpha \lesssim 1\). In addition, \(E^\beta_1\) is generally larger that \(E^\beta_2\). These energies are strongly affected by the number of particles in the gas and weakly dependent on temperature for \(\beta \hbar \omega \gtrsim 0.4\).

On the other hand the Gaussian damping/broadening brought by \(\Lambda_{2r}^\beta\), with standard deviation \(\delta_\beta\), is almost entirely dependent on the thermal energy \(\beta \hbar \omega\) and the critical exponent \(\alpha\) (Fig. 7D). This contribution leads to a smearing of the shake up response of the system in way that resembles the Anderson-Yuval approach to the Kondo problem [2]. Indeed, \(\delta_\beta\) decreases exponentially
to zero with increasing $\beta\hbar\omega$, following the limiting trend
\[ \delta_\beta \approx 2^{3/2} \alpha^{1/2} e^{-\beta\omega \hbar/4} \]  
(S-19)
for $\beta\hbar\omega \gtrsim 7 - 8$ (Figs. 2A and 7B).

FIG. 7. (color on line) Equilibrium energy $E_0^\beta$ of a spin-1/2 gas into a harmonic trap (panel A) compared with first and second order corrections, $E_1^\beta$ [Eq. (M6), panel B] and $E_2^\beta$ [Eq. (M7), panel C], due to the sudden impurity potential $V(x)$ with $\alpha = 0.1, 0.4$ (see also Fig. 1 of the main text). All Energy values are expressed in units of $\hbar\omega$ for different Fermi numbers/energies and temperatures. Significant changes are induced by $r_F$, whereas negligible differences are observed in the energy curves for $\beta\hbar\omega > 0.4$. Gaussian standard deviation $\delta_\beta$ (panel D) appearing in the two-vertex contribution $\Lambda_{2v}^\beta$, and reported in units of $\omega$ for the same values of $r_F$ and $\alpha$ used in the plots of panels (A-C).

In addition, as evident by comparing Fig. 2B of the main text with the plots of Fig. 8, the sub-diagram $\Lambda_{2p}^\beta$ has a time period of $\pi/\omega$. Its modulus $|\Lambda_{2p}^\beta|$ presents zeroes at $\omega t = m\pi$ and maxima at $\omega t = m\pi/2$, with $m = 0, \pm 1, \pm 2, \cdots$. The intensities of such maxima (Fig. 8A-C) increase with increasing the Fermi number (${2r_F}$), the critical exponent ($\alpha$), and the thermal energy ($\beta^{-1}$). The phase of $\Lambda_{2p}^\beta$ is discontinuous at the extremes of $|\Lambda_{2p}^\beta|$ and less dependent on these parameters (Fig. 8D-F).

IV. LOW THERMAL ENERGY APPROXIMATION

We work at sufficiently low temperatures such that the chemical potential is well approximated by its maximum value
\[ \mu \rightarrow \hbar\omega(2r_F + 1), \]  
(S-20)
corresponding to
\[ r_\mu \rightarrow r_F + 1/4, \]  
(S-21)

FIG. 8. (color online): Modulus and phase of the periodic component $\Lambda_{2p}^\beta$ (M9) vs $\omega t$, for $\beta\hbar\omega = 0.01 - 5$ and $\alpha = 0.1$. Several fermion numbers are tested, i.e., $r_F = 5$ (A,D), $r_F = 25$ (B,E), and $r_F = 500$ (C,F).

as shown in Fig. 6. Then, we consider systems with a relatively large number of particles ($r_F \gtrsim 10$) and focus on the sub-diagrams $\Lambda_{2c}^\beta(t)$ and $\Lambda_{2p}^\beta(t)$. In this way we provide the formal justifications for the analytic approximations introduced in Eqs. (M11) and (M12), which determine the ‘unshifted’ amplitude $\nu_\beta'(t)$ given in Eq. (M10) and the excitation spectrum $\nu_\beta'(E)$, shown in Fig 4A.

Using the power series
\[ f_+^* f_+^- \equiv \sum_{m=1}^\infty (-1)^{m+1} m e^{\pm \beta m(\omega t - \mu)}, \]
we rewrite the Gaussian damping parameter, reported in Eq. (M8), as
\[ g_\beta = \sum_{m=1}^\infty (-1)^{m+1} m g_m^{\beta}. \]

The coefficients of this expansion read
\[ g_m^{\beta} = \frac{\varepsilon_F}{\hbar\omega} \sum_{r < r_F} \gamma_r^2 e^{\beta m(\omega t - \mu)} + \frac{\varepsilon_F}{\hbar\omega} \sum_{r > r_F} \gamma_r^2 e^{-\beta m(\omega t - \mu)} \]
\[ = \frac{\varepsilon_F}{\hbar\omega} \sum_{r < r_F} \gamma_r^2 e^{-2\omega\tau_m(r_F - r)} + \frac{\varepsilon_F}{\hbar\omega} \sum_{r > r_F} \gamma_r^2 e^{-2\omega\tau_m(r_F - r)}, \]
with $\tau_m$ being the characteristic times $\tau_m = m\beta\hbar$, induced by thermal fluctuations. Then, we use Eq. (S-21)
and perform a change of summation indices to write
\[ g^\beta_m = \frac{\xi_F}{h\omega} e^{-2\omega \tau_m (r_m - r_r)} \sum_{r=0}^{r_r} \gamma^2_{r_r-r} e^{-2\omega \tau_m r} + \frac{\xi_F}{h\omega} e^{2\omega \tau_m (r_m - r_r)} e^{2\omega \tau_m r}. \]

The transformed summations in this last line are dominated by low \( r \) terms. In a many fermion environment, we may use the asymptotic relation \( \gamma^2_{r \pm r} \approx \gamma^2_{r \approx r} \approx r^{-1} \) and obtain
\[ g^\beta_m \approx 2 e^{2\omega \tau_m (r_m - r_r)} + e^{2\omega \tau_m (r_m - r_{\alpha + 1})} - e^{-2\omega \tau_m r_r}. \]

Then, using the asymptotic form (S-21) and neglecting \( e^{-2\omega \tau_m \alpha} \), we find
\[ g^\beta_m \approx 2 e^{\omega \tau_m / 2} e^{2\omega \tau_m - 1}, \]

which leads to Eq. (M11), i.e.,
\[ g^\beta \approx 2 \sum_{m=1}^{m_{cut}} (-1)^m m e^{\omega \tau_m / 2} e^{2\omega \tau_m - 1}, \]

and let us approximate the standard deviation with
\[ \delta^\beta \approx 2 \alpha^{1/2} \left[ \sum_{m=1}^{m_{cut}} (-1)^m m e^{\omega \tau_m / 2} e^{2\omega \tau_m - 1} \right]^{1/2}. \]  

Eq. (S-22), plotted in Fig. 9A for \( m_{cut} = 1,100 \), is independent of the number of particles in the gas. We have verified that the truncated series for \( m_{cut} = 100 \) works extremely well for \( r_r = 5 - 500 \) and \( \beta \hbar \omega \gtrsim 0.4 \). The asymptotic form of \( \delta^\beta \) leads to the result reported in Fig. 2A of the main text.

As for the Fermi-edge component \( \Lambda^\beta_{2p}(t) \), we consider the auxiliary functions \( \lambda^\beta_{\pm}(t) \) introduced in the main text, which enter the connected graph \( \Lambda^\beta_{2}(t) \) (see Eq. (M5)). We replace the particle-hole distributions \( f^\pm_r \) with the power series expansion
\[ f^\pm_r = \sum_{m=0}^{\infty} (-1)^m e^{\pm \beta m (2r - \mu)} \quad \varepsilon_{2r} \lesssim \mu \]

and let us write
\[ \lambda^\beta_{\pm}(t) = \sum_{m=0}^{\infty} (-1)^m \lambda^\beta_{m,\pm}(t). \]

Here, the coefficients \( \lambda^\beta_{m,\pm}(t) \) may be computed exactly by the finite summations
\[ \sum_{r=r_1}^{r_2} \gamma_r z^r = z^{r_1} 2 \hat{F}_1(r_1, z) - z^{r_2+1} 2 \hat{F}_1(r_2 + 1, z), \]

holding for any \( z \neq 1 \), in which \( 2 \hat{F}_1 \) is the regularized Hypergeometric function
\[ 2 \hat{F}_1(r, z) = \gamma_{r+1} 2 \hat{F}_1(1, 1/2 + r, 1 + r; z) \]

Finally, we may perform the \( t' \) and \( t'' \)-integrals, excluding terms proportional to \( t \) and \( t^2 \). What is left is a combina-
tion of logarithmic and polylogarithmic functions, dominated by the Fermi-edge term reported in Eq. (M12). Indeed, as shown in Fig. 9, the zero temperature form

\[
\Lambda_0^\infty(t) \approx \ln \left[ \frac{e^{2\tau_0\omega} - 1}{e^{2\omega(\tau_0 + i\alpha)} - 1} \right] \alpha\]

\(= \alpha \ln \left[ \frac{e^{2\tau_0\omega} - 1}{\sqrt{-2e^{-2\tau_0\omega} \cos(2\omega t) + e^{-4\tau_0\omega} + 1}} + i\alpha \tan^{-1} \left( \frac{e^{2\tau_0\omega} - \cos(2\omega t)}{e^{2\tau_0\omega} - 1} \right) \right]
\]

obtained from this procedure is in excellent agreement with the numerical calculations reported in Fig. 2B for \(r_f = 100\) and \(\beta \hbar \omega\) larger than \(~1\). The reliability of such an approximation is also attested by the comparison in Fig 4A. We will see in the following appendix that Eq. (S-27) provides an accurate description of the singular response of a Fermi gas with low numbers of particles.

In the main text, we observed that \(\Lambda_0^\infty(t)\) correctly tends to the MND form given by Eq. (M14) when the harmonic trap frequency is lowered to zero, keeping the number of particles in the gas finite. The singularity index is proportional to the height of the impurity potential barrier. Nozières and De Dominicis [4] calculated the propagator for a free electron in the transient potential activated by a structure-less core-hole, by solving the associated Dyson equation in the long-time limit with Muskhelishvili techniques for singular integral equations. They assumed a constant potential of arbitrary height and width \(\hbar/\tau_0\) around the Fermi level of the gas and found the singularity index to depend of the phase-shifts of this potential. Therefore they provided an asymptotic expression for all closed loops \(\Lambda^\infty(t) = \sum_n \Lambda_n^\infty(t)\), which may be written as Eq. (M14). In our derivation, we have given a model to the impurity potential, being described by non constant matrix elements \(V_{rr'}\) (see appendix I). In addition we have found an analytical form for two vertex graph, Eq. (M12), which accurately describes \(\Lambda_0^\infty(t)\) at any time \(t\) for a sufficiently wide range of temperatures and particle number. We expect the non trivial part of the higher order contributions \(\Lambda^\beta_{n>2}(t)\) to change the value of the \(\alpha\)-parameter in such a way that it will depend on the phase shifts of the impurity potential.

V. DECOHERENCE FACTOR, EXCITATION SPECTRUM AND NON MARKOVIANITY MEASURE

With the arguments given in the previous section, the decoherence factor takes the analytical approximation

\[|\nu^\beta(t)| = e^{-2\omega^2 t^2 \sum_{m=1} \alpha^m m e^{-\beta \hbar \omega m/2}} \times \prod_{m=-\infty}^{\infty} \left( \frac{e^{2\tau_m \omega} - 1}{e^{2(i\alpha t + \tau_m)\omega} - 1} \right)^\alpha,\]

and the excitation spectrum relative to the perturbed equilibrium energy of the gas may be written

\[\nu^\beta(E) = \int_{0}^{\infty} \frac{dt}{2\pi \hbar} e^{\frac{\hbar}{\pi}(E + E^3 + E^2)} \nu^\beta(t).\]

We performed numerical calculations of both \(|\nu^\beta(t)|\) and \(\nu^\beta(E)\) by selecting different fermion numbers \((r_f = 5 - 100)\), coupling parameters \((\alpha = 0.1 - 1.05)\), and thermal energies \((\beta \hbar \omega = 0.001 - 10)\). Then, we evaluated the maxima/minima of \(|\nu^\beta(t)|\) in a finite time-window with \(t = 0 - 100\delta_t^{-1}\) and plugged the differences \(|\nu^\beta(t_{\max,n})| - |\nu^\beta(t_{\min,n})|\) into Eq. (M15) to estimate the non-Markovianity \(N\) of the two-level impurity in the gas.

FIG. 10. (color online): Absolute value of the decoherence factor \(|\nu^\beta(t)|\) (left panels, A,C) and excitation spectrum \(\nu^\beta(E)\) (right panels, B,D), calculated from Eq. (M10) by numerically computing the Gaussian damping (M8) and the periodic sub-diagram (M9), for \(\beta \hbar \omega = 1.3, r_f = 5\), and \(\alpha = 0.1 - 1.05\).

FIG. 11. (color online): (A) Absorption spectrum \(\nu^\beta(E)\), calculated numerically from Eq. (M10) with \(\beta \hbar \omega = 0.1 - \infty, r_f = 5\), and \(\alpha = 0.1\), and analytical approximation \(\nu^\beta_\infty(E)\) obtained from Eqs. (M11) and (M12) with \(\hbar \omega \beta = 10\) and \(\omega \tau_0 = 0.016\); (B) Non-Markovianity measure as a function of the critical parameter \(\alpha\) for various temperatures.

In Fig. 3 and 4A, we have presented an application to a Fermi gas of 402 particles \((r_f = 100)\) where the Fermi-edge behavior, superimposed on a Gaussian damping
trend, appears as a sequence of spikes in $|\nu_\beta(t)| = |\nu'_\beta(t)|$ and an asymmetric peak structure in $\tilde{\nu}'_\beta(E)$. Such features becoming more and more marked with decreasing temperature, which reduces the effect of the Gaussian damping $\delta_\beta$ (see Fig. 2A), correspond to a sharp peak of $\mathcal{N}$ at $\alpha < 0.2$ (Fig. 4B). Similar considerations hold for environments containing low fermion numbers (i.e., for $r_f = 5$ in Fig. 10 and Fig. 11A) where shake-up effects are even more visible because of the decreasing of $\delta_\beta$ with decreasing $\varepsilon_f$, leading to a sharp peaks in $\mathcal{N}$ at $\alpha < 0.5$ (Fig. 11B). We finally notice that the analytical approximation given by Eqs. (M11) and (M12) works extremely well with environments containing both low (Fig. 4A) and large (Fig. 11A) fermion numbers.