Speed gradient control of qubit state

Sergei Borisenok a,b, Alexander Fradkov c,d, Anton Proskurnikov c,d

a Dept. of Mathematics and Computer Science, Faculty of Science and Letters, Istanbul Kültür University, Bakırköy 34156, Istanbul, Turkey
s.borisenok@iku.edu.tr

b Abdus Salam School of Mathematical Sciences, Government College University, 68-B New Muslim Town, 54600 Lahore, Pakistan
borisenok@gmail.com

c Institute of Problems in Mechanical Engineering, Russian Academy of Sciences
61, Bolshoy, V.O., 199178 St.Petersburg, Russia

d Dept. Theoretical Cybernetics, St.Petersburg State University, 28 Universitetsky prospekt, 198504, Peterhof, St.Petersburg, Russia.
Alexander.Fradkov@gmail.com

Abstract: A closed-loop (feedback) based algorithm calculating the classical external field for efficient control of qubit state is proposed. The approach is illustrated with two-level atomic system (with decay) driven by classical optical field. It is shown analytically and by simulation that the proposed method allows one to achieve the desired state expressed as the difference of the density matrix diagonal elements.

1. INTRODUCTION

A broad spectrum of control methods for quantum systems has been developed recently D’Alessandro (2007). They have a number of differences with classical control approaches based on continuous observation of the system. In quantum case it is possible only for weakly measuring processes when the system is gradually collapsing to one of the measurement operator eigenstates. Also the controlled quantum system usually cannot be modelled in the terms of linear control, like in classics W. Wang and Yi (2009).

In general we can divide all the control methods into two main groups: feedforward (open-loop) and feedback (closed-loop) algorithms. The set of feedforward approaches, when the controlling field is known a priori, is much more wide. There are papers describing two-level quantum system (qubit) interacting in uncontrollable way with a quantum environment. The main accent here has been made on the qubit decoherence M. Thorwart and Hanggi (2010) and dephasing Yositake and Murakami (2005) by \( \pi \)-pulses. An important case is time optimal control (see for instance N. Khaneja and Glaser (2001), N. Khaneja and Glaser (2002) for spin systems). Time-optimal theory has been applied to the 2-level system governed by the Liouville equation to protect the system against noise Kallush and Kosloff (2006). As a rule the dissipation processes in this approaches are described by the Lindblad equation D. Sugny and Jauslin (2007)-S.G. Schirmer and Solomon (2001). The integrability of the model had been investigated in Bonnard and Sugny (2009).

Let us mention shortly some other results. An interesting effect is the so-called decay freezing under the application of a special set of control pulses Zhang and Zhuang (2009). In Zhang and Xu (2009) purity of a two-level atom was studied for the system driven by a classical field, interacting with a coherent field in a dissipative environment. Control landscapes (the functions of the control variables) for two-level quantum systems was discussed in A. Pechen and Rabitz (2008) by so-called Kraus mapping (completely positive trace preserving mapping). We should mention also the set of work where the control of several interacting qubits is applied to form the special kind of entangled states. The most attempts are focused around the Bell states X. Wang and Solomon (2001). Robust incoherent control of qubit systems you can find in D. Dong (2010).

The open-loop control approach is especially developed in quantum optics, where two-level atomic system interacting with classical laser field is a basement for the further models Allen and Eberly (1975). The recent developing of ultra-fast laser technologies gave new ways for many practical applications of open-loop methods Letokhov (2007). With femtosecond-scaled external fields we do not have much time for feedback design of the system. Thus, the problem of the effective choice for the control signal is especially important in laser control of atomic systems. Nevertheless, the set of control signal shapes is rather limited: they are usually periodic pulses Zhang and Zhuang...
spectra can be produced by \( \pi \)- and \( \pi/2 \)-pulses (Imoto, 1996), taking special non-constant shapes of external field A.Di. Piazza and Mittleman (2001) etc. Recently a control by specially designed pulses has been studied for two-level atomic system using simulating photon echo M. Tian and Babbitt (2004).

In the frame of feedback algorithm, the control signal itself is corrected by a measurement of the system state. As a rule, continuous quantum feedback is defined as the process to monitor and to control the dynamics of a quantum system using the temporal continuous measurement record. The most famous and fully developed is the approach based on master equations in its both main variants: the Markovian feedback model Wiseman and Milburn (1993)-Wiseman and Milburn (1994) and the so called Bayesian feedback model Doherty and Jacobs (1999)-A.C. Doherty and Tan (2000) (the later model was proposed by Wiseman in his comparative analysis of both these models in H.M. Wiseman and Wang (2002)). We should mention the homodyne control Xiao-Shu Liu and Gui (2005) and the Markovian control on geometric phase in dissipative system H. Y. Sun and Yi (2009), the last one has been formulated in pure quantum way for the Lindblad master equation. In H. Y. Sun and Yi (2009), in particular, has been proved that it is possible to control the phase in the open system even if its state can not be manipulated from an arbitrary initial one to an arbitrary final one.

In the paper Xiao-Shu Liu and Gui (2005) Markovian feedback of the white-noise measurement record via a Hamiltonian has been considered, and this model was applied to stabilize the state of two-level quantum system. Another approach of feedback is to construct the system of dynamical equations related to the qubit state, which includes the control field, and then to construct a reasonable algorithm that allows to find the control field via the dynamical variables. In this approach we don’t need to concentrate on the details of ‘monitoring’, like in Xiao-Shu Liu and Gui (2005)-Lloyd (2000), because the final shape of the control field can be accepted as the recommended form to achieve the control goal. The simple way to do it is a definition of a goal function expressing the desired meaning of the qubit state, and to demand that the control field must minimize the goal function. This idea leads us to speed gradient method A.L. Fradkov (1998); Fradkov (2007), that was applied in quantum optics in Saifullah (2008).

The feedback approach proposed in this paper is designed for qubit in a classical control field. The quantum system that possesses only a limited set of its eigenstates (two level atom in this paper). As a physical realization we take the “semiclassical approach” of the atom–field interaction, when a single quantum two-level atomic system (all other levels are neglected) is interacting with classical electromagnetic field. Our notation follows M.O. Scully (2006), but in our model the optical field plays the role of a control signal \( u(t) \) for closed-loop (feedback) control scheme Fradkov (2007). A similar case for the probability amplitudes (without decay) was formulated in Saifullah (2008), but the achievement of the control goal was not proved analytically.

Our model has a decay component, because it involves the effect of elastic collisions between atoms. In Section 2 we present our dynamical model with decay in generalized dimensionless form and then apply the speed gradient control procedure for the control field \( u(t) \), followed by the analysis of the goal achievement. Then we make numerical investigations of the model.

2. SPEED GRADIENT CONTROL OF QUBIT

2.1 Control model of two-level quantum system in classical field

We will discuss the model of controlled qubit in the terms of quantum optics, but it can be easily reformulated for other quantum systems. Let’s consider two-level atomic system in the classical optical field \( E(t) \).

![Fig. 1. Interaction of single two-level atom with classical optical field.](image)

Let \( |a\rangle \) and \( |b\rangle \) represent the upper and lower level states of the atom, i.e., they are eigenstates of the unperturbed part of the Hamiltonian \( H_0 \) with the eigenvalues: \( H_0|a\rangle = \hbar \omega_a|a\rangle \) and \( H_0|b\rangle = \hbar \omega_b|b\rangle \).

The equations of motion for the density matrix elements are given by M.O. Scully (2006):

\[
\dot{\rho}_{aa} = -\gamma_a \rho_{aa} + \frac{iE}{\hbar} (\varphi_{ab}\rho_{ba}e^{i\omega t} - \varphi_{ab}^*\rho_{ab}e^{-i\omega t}) \; ; \\
\dot{\rho}_{bb} = -\gamma_b \rho_{bb} - \frac{iE}{\hbar} (\varphi_{ab}\rho_{ba}e^{i\omega t} - \varphi_{ab}^*\rho_{ab}e^{-i\omega t}) \; ; \\
\dot{\rho}_{ab} = -\gamma_{ab} \rho_{ab} - \frac{iE}{\hbar} \varphi_{ab}(\rho_{aa} - \rho_{bb})e^{i\omega t},
\]

where \( \rho_{ba} = \rho_{ab}^* \), \( \varphi_{ab} \) is the matrix element of the electric dipole moment, \( \gamma_a \) and \( \gamma_b \) are the decay constants, \( \gamma_{ab} = (\gamma_a + \gamma_b)/2 + \gamma_{ph} \), \( \gamma_{ph} \) is a decay rate including elastic collisions between atoms, and \( \omega = \omega_a - \omega_b \) is the atomic transition frequency.

Let’s denote \( \varphi_{ab} = |\varphi_{ab}|e^{i\phi} \) and

\[
\rho_+ \equiv \rho_{ba}e^{i(\omega t + \phi)} + \rho_{ab}e^{-i(\omega t + \phi)} \; ; \\
\rho_- \equiv i \left[ \rho_{ba}e^{i(\omega t + \phi)} - \rho_{ab}e^{-i(\omega t + \phi)} \right].
\]
By (2) the system (1) is converted to the real one:

\[ \dot{\rho}_{aa} = -\gamma_a \rho_{aa} + \frac{|\psi_{ab}| E}{\hbar} \cdot \rho_- ; \]
\[ \dot{\rho}_{ab} = -\gamma_b \rho_{ab} - \frac{|\psi_{ab}| E}{\hbar} \cdot \rho_- ; \]
\[ \dot{\rho}_+ = -\gamma_a \rho_+ + \omega \rho_- ; \]
\[ \dot{\rho}_- = -\gamma_b \rho_- - \omega \rho_+ - \frac{2|\psi_{ab}| E}{\hbar} \cdot (\rho_{aa} - \rho_{bb}) . \]

For further calculations we put \( \gamma_a = \gamma_b = \gamma \). Then

\[ (\rho_{aa} + \rho_{bb})(t) = e^{-\gamma t}(\rho_{aa} + \rho_{bb})(0) . \]

The first two equations of the system (3) can be combined together. We put:

\[ \rho_{aa}(t) - \rho_{bb}(t) = e^{-\gamma t}x(t) ; \]
\[ \rho_+(t) = e^{-\gamma t}y(t) ; \]
\[ \rho_-(t) = e^{-\gamma t}z(t) . \]

By substitution of (5) in (3) we can eliminate the decay \( \gamma \)-containing terms. Finally, rescaling the time by \( \omega : \tau = \omega t, \) and denoting the dimensionless control signal by \( u(t) \equiv 2|\psi_{ab}| E(t)/\hbar \omega \) and \( \epsilon = \gamma_{ph}/\omega \), we get the simplified system

\[ \dot{x} = u \cdot z ; \]
\[ \dot{y} = -\epsilon \cdot y + z ; \]
\[ \dot{z} = -\epsilon \cdot z - y - u \cdot x . \]

Here the dot stands the derivative with respect to the new dimensionless time \( \tau \). Remind that \( x \in [-1, 1], \) since \((\rho_{aa} - \rho_{bb}) \in [-1, 1], \) and \((\rho_{aa} - \rho_{bb}) \to 0 \) as \( t \to \infty \). Thus, the real variable \( x = \) by Eqs (4)-(5) unambiguously defines the state of our qubit. To control an arbitrary qubit state in the frame of model (6) means to control \( x \).

If we put the notation \( r = (x^2 + y^2 + z^2)^{1/2} \), by (6) we get:

\[ \dot{r} = -\epsilon \cdot r + \epsilon \cdot \frac{x^2}{r} . \]

If the decay \( \gamma_{ph} \) is absent (i.e. \( \epsilon = 0 \), the evolution of the system (6) is reduced to the movement on the surface of so-called Bloch sphere \( \tau(\tau) = r(0) = 1 \). Its ‘south’ (bottom) pole corresponds to the pure state \(|b>\), and the ‘north’ (top) pole – to the state \(|a>\). The control \( u \) cannot remove the system state out of the Bloch surface. If the decay \( \epsilon \) occurs, the Bloch sphere degenerates with a time in its radial \( y \)- and \( z \)-directions.

### 2.2 Speed gradient control method

Speed-gradient method was first introduced in control engineering and then was successfully used for many control problems for physical systems A.L. Fradkov (1998); Fradkov (2007). To construct a control algorithm according to the SG-method, one needs first to choose a goal function \( Q(x) \) which small values correspond to achievement of the control goal. Then the speed of changing \( Q(x) \) along trajectories of the controlled system is evaluated:

\[ \frac{dQ}{d\tau} = \dot{Q} = \nabla_x Q(x)^T F(x, u) , \]

where \( F(x, u) \) is the vector of the right-hand-sides of the controlled system dynamical model. Finally, the gradient of the speed \( \dot{Q}(x) \) w.r.t. controlling variables is calculated and the control algorithms is chosen as follows:

\[ u(\tau) = -\Gamma \nabla_x \dot{Q}(x) , \]

where \( \Gamma > 0 \) is the gain (design parameter). The algorithm (7) suggests to change control variables along the gradient of the speed \( \dot{Q}(x) \). Such a choice allows in many cases to change controlling variables in the direction ensuring decrease of \( \dot{Q}(x) \). Eventually, \( \dot{Q}(x) \) may become negative, \( Q(x) \) itself may start decreasing and the goal may be achieved. Exact conditions ensuring achievement of the control goal can be found in A.L. Fradkov (1998); Fradkov (2007).

### 2.3 Speed gradient based state stabilization

Let us stabilize the state of our qubit \( x(\tau) \) at some desired goal level \( x_s \) as \( \tau \to \infty \) by means of SG-method. In order to design the SG-algorithm, choose the goal function as follows:

\[ Q(x) = \frac{1}{2} (x - x_s)^2 . \]

Apparently, the greater is value of the goal function \( Q \), the larger is the distance between the current state \( x \) and the goal state \( x_s \). According to the SG-method, the control field \( u \) is defined as follows:

\[ u = -\Gamma \cdot \nabla_x \dot{Q} , \]

where \( \Gamma > 0 \). In the case of multidimensional \( u \) the symbol \( \nabla_x \) denotes the gradient in the control signal space, but in the model (6) the control signal is one-dimensional, and the gradient reduces to the partial derivative in \( u \). The positive constant \( \Gamma \) is an arbitrary positive number — parameter of SG-control algorithm.

Taking the time derivative yields:

\[ \dot{Q} = (x - x_s) \cdot \dot{x} = (x - x_s) z \cdot u , \]

where \( \dot{x} \) is substituted from (6). Thus, it is \( u \)-depended. Evaluating the partial derivative in \( u \), we arrive at the following expression for the control field (9):

\[ u = -\Gamma (x - x_s) z . \]

Thus the closed loop control system is described by (6) and (11). The feedback here stands in the algorithm itself: we cannot, like in the open-loop (feedback) approach, first define the control field \( u \), and then calculate the set of variables \( (x, y, z) \). According to the SG-method the state variable and the control field value at every time instant can be found only together. Let us analyse properties of the control system (6), (11).

### 2.4 Achievement of the control goal

Specific features of the controlled system model (6) do not allow one to apply existing conditions A.L. Fradkov (1998); Fradkov (2007) ensuring achievement of the control goal and the problem needs special consideration. To
find conditions for achievement of the control goal and to evaluate system accuracy, let us calculate velocity of changing the goal function (8) along trajectories of the closed loop system (6), (11):

$$\dot{Q} = -2\Gamma Q \cdot |z(\tau)|^2.$$  \hspace{1cm} (12)

The linear differential equation (12) is readily integrable:

$$Q(x(\tau)) = Q(x(0)) \exp(-2\Gamma \int_0^{\tau} z^2(s)ds).$$ \hspace{1cm} (13)

Since \(Q(x(\tau))\) is monotonically decreasing, there exists the limit \(Q_\infty = \lim_{\tau \to +\infty} Q(x(\tau))\) estimating the limit error of the control system. However, the value \(Q_\infty\) cannot be estimated directly from (13), since \(z(\tau)\) is not known in advance. Let us evaluate the limit set of the system by means of LaSalle theorem A.L. Fradkov (1998). It claims that if a system possesses a smooth non-negative function \(Q\) such that \(\dot{Q} \leq 0\) and has only bounded trajectories, then its limit set coincides with the maximum invariant set \(M\) contained in the set \(\{(x, y, z) : \dot{Q} = 0\}\). Let the trajectory \(X(\tau) = (x(\tau), y(\tau), z(\tau))\) belongs to the set \(M\) for all \(0 \leq \tau < \infty\). It is seen from (12) and identity \(\dot{Q} \equiv 0\) that either \(Q(x(\tau)) \equiv 0\), or \(z(\tau) \equiv 0\). Let \(z(\tau) \equiv 0\). Then \(z(\tau) \equiv 0\) and, from the last equation of (6) it follows that \(y(\tau) \equiv 0\). Since \(X(\tau)\) belongs to the Bloch sphere we derive that \(x(\tau)^2 \equiv 1\), i.e. \(X(\tau)\) coincides with some pure state. Therefore the set \(M\) on the Bloch sphere consists of two pieces: the goal set (circle) \(x = x_*\) and set of two points \((1, 0, 0), (-1, 0, 0)\) (poles or pure states). We evolve at an alternative: either the control goal is achieved, or the trajectory converges to a pure state. Apparently, since \(Q(\tau)\) is non-increasing, if there are no pure states in the initial layer then there will be no pure states in the set \(M\). Therefore, if the inequalities

$$|x(0) - x_*| < \min\{|x_* - 1|, |x_* + 1|\}$$  \hspace{1cm} (14)

hold, then achievement of the control goal is guaranteed. For \(x_*\) from the interval \([-1, 1]\) inequality (14) implies:

$$\min\{|x_* - 1|, |x_* + 1|\} = 1 - |x_*|. \hspace{1cm} (15)$$

Using the inequality \(|a| - |b| \leq |a - b| \leq |a| + |b|\), we obtain

$$|x(0)| - |x_*| \leq |x(0) - x_*| \leq 1 - |x_*|. \hspace{1cm} (16)$$

If \(|x(0)| > |x_*|\), the inequality (16) becomes \(|x(0)| < 1\), that is always true in our model. If \(|x(0)| < |x_*|\), then

$$|x(0)| < |x_*| < \frac{1}{2}(1 + |x(0)|) = |x(0)| + \frac{1}{2}(1 - |x(0)|). \hspace{1cm} (17)$$

A more subtle analysis based on the central manifold theory shows that the convergence to a pure state is an exceptional event: measure of the set of initial conditions such that trajectory converges to a pure state is equal to zero. More information about behavior of the closed loop system trajectories can be obtained by computer simulation.

2.5 Simulation results

The method is illustrated by numerical simulations below. Let us suppose that \(x_* = 0.8\), the decay rate \(\epsilon = 0.05\), the control constant \(\Gamma = 10\). The typical trajectory achieving asymptotically the control goal along the spiral in the space \((x, y, z)\) is plotted in Fig.2. The value \((x^2(0) + y^2(0) + z^2(0))^{1/2} = 1\) is chosen. The corresponding plot for \(x(t)\) versus time can be seen in Fig.3.

Fig. 2. The typical phase trajectory for \(x(0) = -0.9, y(0) = 0.4358, z(0) = 0.01\).

Fig. 3. The asymptotic achievement of the goal \(x_* = 0.8\) for \(x(0) = -0.9\).

In the process of numerical simulation we must satisfy the inequalities (14).

An important note is that in this paper we analyzed and simulated the controlled system assuming that all the necessary signals are measured and the feedback law (11) is implemented as is. However it is well known that feedback in the quantum system is hard to be implemented directly. The key approach to implementation is in that, during simulation the control signal is generated as a function of time \(u(t)\). Then, for the real time implementation of control the designed function of time \(u(t)\) is applied to the real system in the open loop mode, i.e. without
measurements and feedback. To justify an open loop implementation one should ensure weak dependence of the process characteristics under open loop control on initial conditions (robustness with respect to initial conditions). Simulation results demonstrating required robustness are presented in Fig. 4. It is seen that 10% variation of initial conditions does not influence significantly the process dynamics.

3. CONCLUSIONS

The qubit, which state is different from the classical states 0 or 1, can be controlled efficiently by the application of speed gradient method. It means that by the application of the classical external field designed by (7) in the closed-loop algorithm we can convert easily the qubit state \( x(t) = e^{\gamma t}[\rho_{aa}(t) - \rho_{bb}(t)] \) to another desired final state \( x_* \). The asymptotic convergency of the approach is proved by analytical and numerical investigations.

It is worth to notice that the choice of \( \Gamma \) is not arbitrary. As we can learn from (13) the meaning of the constant \( \Gamma \) must be proportional to the initial \( |z(0)|^{-2} \) to provide the reasonable speed of monotonically decreasing of the goal function \( Q \).

Simulation results demonstrate robustness of the open loop implementation of of control with respect to initial conditions: 10% variation of initial conditions does not influence significantly the process dynamics.

The proposed algorithm can be applied also to multi-level quantum systems (qutrits etc), if the number of fields controlling the inter-level transitions is extended to stimulate the corresponding passages between the levels.

REFERENCES
