Modified censored moment estimation for the two-parameter Birnbaum–Saunders distribution

Zhihui Wang\textsuperscript{a}, A.F. Desmond\textsuperscript{b}, Xuewen Lu\textsuperscript{c,}\textsuperscript{*}

\textsuperscript{a}Department of Mathematics, Shaoyang University, Shaoyang, Hunan 422000, China
\textsuperscript{b}Department of Mathematics and Statistics, University of Guelph, Guelph, Ont., Canada N1G 2W1
\textsuperscript{c}Department of Mathematics and Statistics, University of Calgary, Calgary, Alta, Canada T2N 1N4

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Abstract

The maximum likelihood estimators (MLEs) and the moment estimators of a two-parameter Birnbaum–Saunders (BISA) distribution are studied by various authors when data are either complete or subject to Type-I or Type-II censoring. But there is not much research on parameter estimation for the BISA distribution under random censoring. A simple method of modified censored moment estimation is proposed to estimate parameters of the BISA distribution under random censoring. Bias-reduced versions of these estimators are constructed as well. Asymptotic theory for the estimators is established. The performance of these estimators is compared with that of the MLEs through Monte Carlo simulations for small, moderate, and large proportions of censoring and different sample sizes. An analysis of real data is used to illustrate the proposed method.

\textsuperscript{*}Corresponding author. Tel.: +1 403 220 6620; fax: +1 403 282 5150.
\textit{E-mail address:} lux@math.ucalgary.ca (X. Lu).

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1. Introduction

In reliability and survival analysis studies, the two-parameter Birnbaum–Saunders (BISA) cumulative distribution function (CDF),

\[ F_T(t; \alpha, \beta) = \Phi \left[ \frac{1}{\alpha} \left( \frac{t}{\beta} \right)^{1/2} - \left( \frac{\beta}{t} \right)^{1/2} \right], \quad 0 < t < \infty, \quad \alpha > 0, \quad \beta > 0, \quad (1) \]

has been employed as a model for fatigue failure caused by cyclic loading (Birnbaum and Saunders, 1969a), where \( \Phi(\cdot) \) is the standard normal CDF. The parameters \( \alpha \) and \( \beta \) are shape and scale parameters, respectively. Desmond (1985), using a biological model of Cramér (1946), strengthened the physical justification for the use of this distribution by relaxing the assumptions made by Birnbaum and Saunders (1969a). Since then, the model has become popular in modeling fatigue failure in industry, as an alternative to other popular distributions such as the lognormal, Weibull, gamma and inverse Gaussian.

For complete data without censoring, the maximum likelihood estimators (MLEs) were discussed originally by Birnbaum and Saunders (1969b) and their asymptotic distributions were obtained by Engelhardt et al. (1981). Recently, Ng et al. (2003) proposed a modified moment estimation (MME) for the two parameters and a bias-reduction method based on MLE and MME. For incomplete data, Desmond (1982) treated maximum likelihood estimation (MLE) for both Type I and Type II censored data from the BISA distribution. Rieck (1995) considered parametric estimation for the BISA distribution based on Type-II censored samples. McCarter (1999) studied the MLE of a BISA distribution quantile using Type-II censored samples. Jeng (2003) explored and compared different procedures to compute confidence intervals for parameters and quantiles of the BISA distribution for complete and Type-I censored data.

Life tests often result in randomly censored data. For example, in field testing for the reliability of components, items are put in service at different points in time or are withdrawn from experiment due to accidents or necessary maintenance work. Because there are no known exact estimation procedures for BISA distribution under random censoring, this article provides a simple modified censored moment estimation (MCME) to fill this gap. A possible alternative to MCME is MLE with randomly censored samples, but one needs to solve a non-linear equation to obtain the solution and a closed form for the estimator and its limiting variance is unknown for MLE. The modified censored moment estimators (MCMEs) are very similar to MMEs by Ng et al. (2003). They share similar valuable advantages over MLEs. The MCMEs are easy to compute and they have explicit expressions in terms of the sample observations. Unlike the moment estimators, MCMEs always exist and are unique.

For the above reasons, MCMEs for \( \alpha \) and \( \beta \) are first derived. The asymptotic distributions of the MCMEs are then obtained. The performance of the estimators is evaluated and compared with that of MLEs through Monte Carlo simulations for small, moderate, and large proportions of censoring and different sample sizes. We used numerical methods to obtain MLEs and their standard errors.

The paper is organized as follows: In Section 2, we give a summary of our methodology. In Section 3, we present our main results. In Section 4, we report Monte Carlo simulation


results. In Section 5, we illustrate our method with an analysis of real data. In Section 6, we make some concluding remarks. Finally, some technical proofs are outlined in the Appendix.

2. Summary of the methodology

2.1. The BISA distribution

If a random variable $T$ follows a BISA distribution with CDF given by (1), $T$ has the probability density function (PDF)

$$f_T(t; \alpha, \beta) = \frac{1}{2\sqrt{2\pi} \alpha \beta} \left[ \left( \frac{\beta}{t} \right)^{1/2} + \left( \frac{\beta}{t} \right)^{3/2} \right] \exp \left[ -\frac{1}{2\alpha^2} \left( \frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right],$$

$$0 < t < \infty, \quad \alpha > 0, \quad \beta > 0. \quad (2)$$

Random samples of $T$ can be generated by a standard normal random variable $Z$ through the following monotone transformation (see Rieck, 2003):

$$T = \beta \left[ 1 + \alpha^2 Z^2/2 + \alpha Z \left( \alpha^2 Z^2/4 + 1 \right)^{1/2} \right]. \quad (3)$$

Moreover, if $T$ has a BISA distribution with parameters $\alpha$ and $\beta$, then $T^{-1}$ also has a BISA distribution with the corresponding parameters $\alpha$ and $\beta^{-1}$, respectively (Birnbaum and Saunders, 1969a). Using the above transformation, it is easy to see

$$E(T) = \beta \left( 1 + \frac{1}{2} \alpha^2 \right), \quad (4)$$

$$\text{Var}(T) = (\alpha \beta)^2 \left( 1 + \frac{5}{4} \alpha^2 \right), \quad (5)$$

$$E \left( T^{-1} \right) = \beta^{-1} \left( 1 + \frac{1}{2} \alpha^2 \right) \quad (6)$$

and

$$\text{Var} \left( T^{-1} \right) = \alpha^2 \beta^{-2} \left( 1 + \frac{5}{4} \alpha^2 \right). \quad (7)$$

2.2. MCMEs

When data are complete, to estimate $\alpha$ and $\beta$ in the BISA distribution, Ng et al. (2003) suggested to use (4) and (6) instead of (4) and (5) and equate them with the corresponding sample estimates to obtain the MMEs, which results in the following two moment equations:

$$s = \beta \left( 1 + \frac{1}{2} \alpha^2 \right), \quad (8)$$

$$r^{-1} = \beta^{-1} \left( 1 + \frac{1}{2} \alpha^2 \right), \quad (9)$$
where $s$ and $r$ are the sample arithmetic and harmonic means of a random sample \( \{T_1, T_2, \ldots, T_n\} \) from the BISA distribution defined by

\[
s = \frac{1}{n} \sum_{i=1}^{n} T_i, \quad r = \left[ \frac{1}{n} \sum_{i=1}^{n} T_i^{-1} \right]^{-1}.
\]

The MMEs are obtained by solving (8) and (9) for $\{x, \beta\}$ randomly censored by uncensored data. It is easy to see that response values in an unbiased way so that we can obtain the same moments as that of practical importance in applications.

When data are randomly censored, we assume failure times \( \{T_i\} \), with survival probability \( P(T_i \geq t) = F(t) \) (to save notation, we use \( F(\cdot) \) for the survival probability), are randomly censored by \( \{C_i\} \), where \( \{C_i\} \) are i.i.d. samples of a random variable $C$ with survival probability $P(C \geq x) = G(x)$ and PDF $g(x)$, $0 < x < \infty$, independent of \( \{T_i\} \), and $G(x)$ does not involve $x$ and $\beta$. We can only observe $Z_i = \min \{T_i, C_i\} = T_i \delta_i + C_i \{1 - \delta_i\}$, $\delta_i = \{T_i \leq C_i\}$ or $I[T_i \leq C_i]$, $i = 1, \ldots, n$. $[A]$ or $I[A]$ is the indicator function of $A$. The observable random variables are pairs \( \{Z_i, \delta_i\} \). Since the functional form $F(\cdot)$ is known, the MLE for the unknown parameters can be derived. However, a closed-form expression for the estimators does not exist, neither for their limiting variances. Thus, a numerical procedure, such as the Newton–Raphson method, must be used to determine the value of estimates. Since the MCMREs have simple expressions and provide convenient alternatives to the MLEs with some good finite-sample properties, we mainly consider MCMREs for $x$ and $\beta$. The performance of MLEs are then assessed by Monte Carlo studies. These are of practical importance in applications.

Ng et al. (2003) obtained the following asymptotic joint distribution of \( \tilde{x} \) and \( \tilde{\beta} \),

\[
(\tilde{x}, \tilde{\beta}) \xrightarrow{d} N \left( \begin{pmatrix} x \\ \beta \end{pmatrix}, \begin{pmatrix} \frac{x^2}{2n} & 0 \\ 0 & \frac{(\beta x)^2}{n} \left( \frac{1 + 3x^2/4}{(1 + x^2/2)^2} \right) \end{pmatrix} \right).
\]

When data are randomly censored, we assume failure times \( \{T_i\} \), with survival probability \( P(T_i \geq t) = F(t) \) (to save notation, we use \( F(\cdot) \) for the survival probability), are randomly censored by \( \{C_i\} \), where \( \{C_i\} \) are i.i.d. samples of a random variable $C$ with survival probability $P(C \geq x) = G(x)$ and PDF $g(x)$, $0 < x < \infty$, independent of \( \{T_i\} \), and $G(x)$ does not involve $x$ and $\beta$. We can only observe $Z_i = \min \{T_i, C_i\} = T_i \delta_i + C_i \{1 - \delta_i\}$, $\delta_i = \{T_i \leq C_i\}$ or $I[T_i \leq C_i]$, $i = 1, \ldots, n$. $[A]$ or $I[A]$ is the indicator function of $A$. The observable random variables are pairs \( \{Z_i, \delta_i\} \). Since the functional form $F(\cdot)$ is known, the MLE for the unknown parameters can be derived. However, a closed-form expression for the estimators does not exist, neither for their limiting variances. Thus, a numerical procedure, such as the Newton–Raphson method, must be used to determine the value of estimates. Since the MCMREs have simple expressions and provide convenient alternatives to the MLEs with some good finite-sample properties, we mainly consider MCMREs for $x$ and $\beta$. The performance of MLEs are then assessed by Monte Carlo studies. These are of practical importance in applications.

To derive MCMREs, we transform censored data into ‘synthetic data’ or some pseudoresponse values in an unbiased way so that we can obtain the same moments as that of uncensored data. It is easy to see that

\[
v_1 = E(T) = \int_0^\infty t \, d(1 - F(t)) = \int_0^\infty F(t) \, dt,
\]

\[
v_2 = E \left( \frac{1}{T} \right) = \int_0^\infty \frac{1}{t} \, d(1 - F(t)) = \int_0^\infty (F(t) - 1) \left( -\frac{1}{t^2} \right) \, dt.
\]

For $q_1(t) = t$ or $q_2(t) = 1/t$, we consider the following data transformation,

\[
T_{ji}^* = \frac{\delta_i q_j(Z_i)}{G(Z_i)}, \quad j = 1, 2, \quad i = 1, \ldots, n.
\]

Then, $E \left( T_{ji}^* \right) = E \{q_j(T)\}$. Hence, this data transformation is unbiased for $v_j = E \{q_j(T)\}$ and motivates us to consider MCMREs. Usually, the censoring distribution $G$ is unknown,
and must be estimated from data. For example, we can choose $\hat{G}(\cdot) = \hat{G}(\cdot-)\) to be the left continuous version of the Kaplan–Meier estimator given in the next section. Then, the estimators of $v_1 = E(T)$ and $v_2 = E(1/T)$ are defined by the following equations:

$$
v_1 = \frac{1}{n} \sum_{i=1}^{n} T_{1i}^*, \quad v_2 = \frac{1}{n} \sum_{i=1}^{n} T_{2i}^*.
$$

Hence, by solving the two moment equations corresponding to (8) and (9), we obtain MCMEs for $\alpha$ and $\beta$ denoted by $\hat{\alpha}$ and $\hat{\beta}$ as

$$
\hat{\alpha} = \left\{ 2 \left[ (\hat{\alpha} \hat{\beta})^{1/2} - 1 \right] \right\}^{1/2}, \quad \hat{\beta} = \left( \frac{\hat{\alpha}}{\hat{\beta}} \right)^{1/2}.
$$

### 3. Main results

#### 3.1. Notation and counting processes

The main theorem will be proved by using counting process theory and martingale techniques (for the theory of counting process and martingale, see Fleming and Harrington (1991) and Andersen et al. (1993)). For simplicity, we leave the outlined technical proofs of theorems to the Appendix. Let

$$
F(t) = P(T_i \geq t), \quad G(t) = P(C_i \geq t), \quad H(t) = P(Z_i \geq t)
$$

and

$$
\hat{H}_i(t) = [Z_i \geq t], \quad R^+(t) = \sum_{i=1}^{n} [Z_i \geq t] = \sum_{i=1}^{n} \hat{H}_i
$$

and

$$
A^+(t) = -\int_{[0,t]} \frac{dH(s)}{H(s)}, \quad A^D(t) = -\int_{[0,t]} \frac{dF(s)}{F(s)}, \quad A^C(t) = -\int_{[0,t]} \frac{dG(s)}{G(s)}.
$$

Define (possibly infinite) times $\tau_F$, $\tau_G$ and $\tau_H$ by $\tau_F = \sup\{t \in \mathbb{R} : F(t) > 0\}$, etc. then, we know that $\tau_F = \infty$, $\tau_H = \min(\tau_F, \tau_G) = \tau_G$. Let $T^n = \max_{1 \leq i \leq n} \{ Z_i \}$. It is well known that the three processes

$$
M^+_i(t) = [Z_i \leq t] - \int_{0}^{t} [Z_i \geq s] \, dA^+(s),
$$

$$
M^D_i(t) = [Z_i \leq t; \delta_i = 1] - \int_{0}^{t} [Z_i \geq s] \, dA^D(s),
$$

$$
M^C_i(t) = [Z_i \leq t; \delta_i = 0] - \int_{0}^{t} [Z_i \geq s] \, dA^C(s)
$$

are square-integrable martingales on $[0, \infty]$ with respect to the filtration

$$
\mathcal{F}_s = \sigma \{ Z_k [Z_k \leq s], \delta_k [Z_k \leq s] ; k = 1, 2, \ldots, n \}.
$$
Informally, \( \mathcal{F}_s \) is the \( \sigma \)-algebra generated by the observation process prior to time \( s \). Their predictable variations are, respectively, \( \{M^+_i\} (t) = \int_0^t \mathbb{1}_{[Z_i \geq s]} \, dA^+_i(s), \{M^+_C\} (t) = \int_0^t \mathbb{1}_{[Z_i \geq s]} \, dA^+_i(s), \) and \( \{M^+_C\} (t) = \int_0^t \mathbb{1}_{[Z_i \geq s]} \, dA^+_i(s). \) Then \( A^+ = A^+_i + A^+_C \) and \( M^+_i = M^+_i + M^+_C \). We define \( M^+_C = \sum_{i=1}^n M^+_i \). Since \( \{M^+_i, M^+_C\} = \{M^+_i + M^+_C, M^+_i + M^+_C\} = (M^+_i, M^+_C) + (M^+_i, M^+_C), \) it is easily seen that \( M^+_i \) and \( M^+_C \) are orthogonal martingales, i.e. \( \{M^+_i, M^+_C\} = 0 \).

Thus, the processes
\[
\frac{G(t) - \hat{G}(t)}{G(t)} = \int_0^{\tau_{H} \wedge T^n} \frac{\hat{G}(s)}{G(s)} \frac{1}{R^+(s)} \, dM^+_C(s), \quad t \in [0, T^n],
\]
\[
\frac{H(t) - \hat{H}(t)}{H(t)} = \int_0^{t} \frac{1}{H(s)} \, dM^+_i(s) \quad \text{for } t \text{ such that } H(t) > 0,
\]
are local martingales, where \( \hat{G}(s) \) is the left continuous version of the Kaplan–Meier estimator of \( G \), given by
\[
\hat{G}(t) = \prod_{s \leq t \wedge T^n} \left\{ 1 - \frac{\Delta N^+_C(s)}{R^+(s)} \right\},
\]
where \( N^+_C(s) = \sum_{i=1}^n \mathbb{1}_{[Z_i \leq s, \delta_i = 0]} \) and \( \Delta N^+_C(s) = N^+_C(s+) - N^+_C(s-). \)

3.2. Asymptotic normality

For \( j = 1, 2 \), write \( \hat{v}_j = \hat{v}_j(T^n) \), \( T^*_j = T^*_j(T^n) \). Then, the estimators of \( v_j = E \{q_j(Y)\} \) can be written as
\[
\hat{v}_j(T^n) = \frac{1}{n} \sum_{i=1}^n T^*_j(T^n).
\]

**Theorem 1.** Define two functions \( v_1(t) = \int_0^t F(s) q'_1(s) \, ds = \int_0^t F(s) \, ds, v_2(t) = \int_0^t \{F(s) - 1\} q'_2(s) \, ds = \int_0^t \{F(s) - 1\} \left(-1/s^2\right) \, ds. \) For \( j, k \), \( t \in [1, 2] \), suppose
\[
v_{jk} = \int_0^{\tau_{H}} \left\{ \int_0^{\tau_{H}} F(s) q'_j(s) \, ds \right\} \left\{ \int_0^{\tau_{H}} F(s) q'_k(s) \, ds \right\} (-G(t)) \, dF(t) < \infty
\]
and \( g(0) = 0, g(t) \) is the PDF of \( 1 - G(t) \). Then, the estimator \( \hat{v}_j(T^n) \) for \( v_j(T^n) \), which is a truncated version of \( v_j = E \{q_j(Y)\} \), satisfies
\[
\sqrt{n} \left( \hat{v}_j - v_j \right) (T^n) \xrightarrow{D} N(0, V(\tau_{H})),
\]
where
\[
V(\tau_{H}) = \begin{pmatrix}
\frac{v_{11}}{v_{21}} & \frac{v_{12}}{v_{21}} \\
\frac{v_{12}}{v_{22}} & \frac{v_{22}}{v_{22}}
\end{pmatrix}.
\]
Corollary 1. For $j = 1, 2$, let $\tilde{v}_j(t) = v_j(\infty) - v_j(t)$. Suppose the conditions in Theorem 1 hold and the following condition holds as well,

$$\sqrt{n}\tilde{v}_j(T^n) \to p 0 \text{ as } n \to \infty. \quad (18)$$

Then

$$\sqrt{n}\left(\frac{\tilde{v}_1(T^n) - v_1}{\tilde{v}_2(T^n) - v_2}\right) \to D N(0, V(\tau_H)).$$

Theorem 2. Assume the conditions in Theorem 1 and Corollary 1 are satisfied. Then, $\hat{z}$ and $\hat{\beta}$ satisfy

$$\sqrt{n}\left(\frac{\hat{z} - z}{\hat{\beta} - \beta}\right) \to D N(0, \Phi(\tau_H)),$$

where

$$\Phi(\tau_H) = A V(\tau_H) A^T = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

and

$$A = \begin{pmatrix} \frac{1}{2x^2} & \frac{\beta}{2x} \\ \frac{1}{2 + x^2} & -\frac{\beta^2}{2 + x^2} \end{pmatrix}.$$

3.3. Bias-reduced estimators

With complete data, Ng et al. (2003) obtained the almost unbiased modified moment estimators (UMMEs) and almost unbiased maximum likelihood estimators (UMLEs) of $z$ and $\beta$. Based on the result of an extensive Monte Carlo simulation study and by examining the patterns of the biases of the MCMEs and MLEs, we construct the following almost unbiased modified censored moment estimators (UMCMEs, denoted by $\hat{z}^*$ and $\hat{\beta}^*$) and almost UMLEs (denoted by $\tilde{z}^*$ and $\tilde{\beta}^*$)

$$\hat{z}^* = \left\{ \frac{n(1 - p)^2}{n(1 - p)^2 - 1} \right\}^{-1} \hat{z}, \quad \hat{\beta}^* = \left\{ 1 + \frac{x^2}{4n} \frac{1 - 5p}{1 - p} \right\}^{-1} \hat{\beta} \approx \left\{ 1 + \frac{\tilde{z}^2}{4n} \frac{1 - 5p}{1 - p} \right\}^{-1} \tilde{\beta},$$

$$\tilde{z}^* = \left\{ \frac{n(1 - p^6)}{n(1 - p^6) - 1} \right\} \tilde{z}, \quad \tilde{\beta}^* = \left\{ 1 + \frac{x^2}{4n} \frac{1 - 5p}{1 - p} \right\}^{-1} \tilde{\beta} \approx \left\{ 1 + \frac{\tilde{z}^2}{4n} \frac{1 - 5p}{1 - p} \right\}^{-1} \tilde{\beta},$$

where $p$ denotes the censoring proportion, $\tilde{z}$ and $\tilde{\beta}$ are the maximum likelihood estimates of $z$ and $\beta$, respectively.
Theorem 3. Assume the conditions in Theorem 1 and Corollary 1 are satisfied. Then, \( \hat{\beta}^* \) and \( \hat{\beta} \) satisfy

\[
\sqrt{n} \left( \frac{\hat{\beta}^* - \beta}{\hat{\beta} - \beta} \right) \rightarrow_D N(0, \Xi(\tau_H)),
\]

where

\[
\Xi(\tau_H) = B \Phi(\tau_H) B^T = \begin{pmatrix}
\xi_{11} & \xi_{12} \\
\xi_{21} & \xi_{22}
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
\frac{n(1-p)^2}{n(1-p)^2 - 1} & 0 \\
0 & \left(1 + \frac{\hat{\beta}^2 - 5p}{4n(1-p)}\right)^{-1}
\end{pmatrix}.
\]

3.4. Remarks and discussion

Remark 1. When \( G(t) > 0 \) for \( 0 < t < \infty \), which implies \( \tau_H = \tau_H = \tau_G = \infty \), let \( h_{1,qj}(t; \tau) = \int_0^t F(s)q_j(s) \, ds \) and \( h_{2,qj}(t; \tau) = \int_0^\tau q_j(s) \, d(-F(s)). \) We have \( q_j(t)F(t) + h_{1,qj}(t; \tau_H) = h_{2,qj}(t; \tau_H) \) and \( \left\{ h_{2,qj}(t; \tau_H) \right\}^2 / F(t) \rightarrow 0 \) as \( t \rightarrow \tau_H \). The following calculation reveals that \( v_{11} \) or \( v_{22} \) is given by

\[
v_{jj}(\tau_H) = -\int_0^{\tau_H} \left( \frac{h_{1,qj}(t; \tau_H)}{H(t)} \right)^2 G(t) \, dF(t)
\]

\[
= -\int_0^{\tau_H} \left( h_{1,qj}(t; \tau_H) \right)^2 \left\{ \frac{1}{F(t)} \right\}^2 \left\{ 1 + \frac{1}{G(t)} - 1 \right\} \, dF(t)
\]

\[
= \int_0^{\tau_H} \left( q_j(t) \right)^2 \, d(-F(t)) - \left\{ h_{2,qj}(0; \tau_H) \right\}^2
\]

\[
- 2 \int_0^{\tau_H} \frac{h_{2,qj}(t; \tau_H) q_j(t)}{F(t)} \, d(-F(t))
\]

\[
+ 2 \int_0^{\tau_H} \frac{h_{2,qj}(t; \tau_H) q_j(t)}{F(t)} \, d(-F(t))
\]

\[
- \int_0^{\tau_H} \left( h_{1,qj}(t; \tau_H) \right)^2 \left\{ \frac{1}{F(t)} \right\}^2 \left\{ \frac{1 - G(t)}{G(t)} \right\} \, dF(t)
\]

\[
= \text{Var} \left\{ q_j(Y) \right\} + \int_0^{\tau_H} \left( h_{1,qj}(t; \tau_H) \right)^2 \left\{ \frac{1}{F(t)} \right\}^2 \left\{ \frac{1 - G(t)}{G(t)} \right\} \, d(-F(t)).
\]

The last term in the above equation is positive unless \( G(t) \) is completely flat on \([0, \tau_H]\). The first term is the asymptotic variance of the classical sample mean of \( q(Y) \). Therefore, when censoring occurs, the resulting estimator for the moment \( v_1 \) and \( v_2 \) has a bigger asymptotic variance. This asymptotic variance was also obtained by Gill (1983) when \( q(y) = y \).
Remark 2. For the purpose of inference, $v_{jk}$ can be consistently estimated by

$$
\hat{v}_{jk} = \int_0^{T_n} \left( \int_t^{T_n} \frac{F(s)q_j'(s)\,ds}{\hat{H}(t)} \right) \left( \int_t^{T_n} \frac{F(s)q_k'(s)\,ds}{\hat{H}(t)} \right) \left( -\hat{G}(t) \right) \,d\hat{F}(t),
$$

where $\hat{H}(t) = n^{-1} \sum_{i=1}^n H_i(t) = n^{-1} \sum_{i=1}^n [Z_i \geq t]$ and $\hat{F}(t) = \hat{H}(t)/\hat{G}(t)$. $\hat{G}(t)$ is the Kaplan–Meier estimator of $G(t)$ given in (15). For example, $100(1 - \gamma)$% confidence intervals for $\alpha$ and $\beta$ are expressed by

$$
\hat{\alpha} \pm \tilde{Z}_{1-\gamma/2} \sqrt{\hat{\phi}_{11}/n}
$$

and

$$
\hat{\beta} \pm \tilde{Z}_{1-\gamma/2} \sqrt{\hat{\phi}_{22}/n},
$$

where $\tilde{Z}_{1-\gamma/2}$ is the $1 - \gamma/2$ quantile of the standard normal distribution, $\hat{\phi}_{jj}$ ($j = 1, 2$) are the estimates of $\phi_{jj}$ given in Theorem 2 which are obtained by replacing corresponding counterparts of $\Phi(\tau_H)$ by their estimates. By some simulation studies, we find that the estimates of the variances of $\alpha$ and $\beta$ underestimate the true variability which is assessed by the sample variances when the sample size is small, $\alpha$ is large and the censoring proportion $p$ is high. Therefore, we develop the following adjustments and use them in our simulation studies,

$$
\hat{\phi}_{11}^{*} = \left\{ 1 + 8.58 \left( \frac{p^5 \alpha}{n^{2/3}} \right)^{1/9} \right\}^2 \hat{\phi}_{11},
$$

and

$$
\hat{\phi}_{22}^{*} = \left\{ 1 + 7.28 \left( \frac{p^3 \beta^2}{n^{5/6}} \right)^{1/6} \right\}^2 \hat{\phi}_{22}.
$$

4. Simulation studies

In order to evaluate and compare the performance of all the estimators, a series of simulation studies are carried out for different censoring proportions, for different sample sizes, and for different parameter values. We took the censoring proportions as $p = 10\%, 30\%, 50\%$, the sample sizes as $n = 10, 20, 50, 100$, and the shape parameter as $\alpha = 0.10, 0.50, 2.00$. Since $\beta$ is the scale parameter, $\beta$ was fixed at $\beta = 1.0$, without loss of any generality. The censoring random variable $C$ is generated from a log-normal distribution with CDF

$$
1 - G(x; \mu, \sigma) = \Phi \left( \frac{\log x - \mu}{\sigma} \right).
$$
where $\Phi(\cdot)$ is the CDF of standard normal. This log-normal distribution has mean $E(X) = \exp(\mu + \sigma^2/2)$ and variance $V(X) = \{\exp(\sigma^2) - 1\} \exp(2\mu + \sigma^2)$ respectively. This distribution satisfies the assumption given in Theorem 1 and condition (18) in Corollary 1. We fixed $\sigma$ at $\sqrt{2}$ and let $\mu$ vary to obtain different censoring proportions. The final simulated observations from the above BISA model are $(\delta_i, Z_i), i = 1, \ldots, n$, where

$$Z_i = \min(T_i, C_i), \quad \delta_i = I[T_i \leq C_i].$$

There are $B = 1000$ simulation samples for each combination of $p$, $n$ and $x$. Tables 1 and 2 present the average biases and mean squared errors (MSEs) of the estimates for MCME, UMCME, MLE and UMLE based on these 1000 simulated samples.

We also computed the 90% and 95% probability coverages of confidence intervals using the asymptotic distributions given earlier for MCMEs and UMCMEs, and the normal approximations for MLEs and UMELs. These results are reported in Tables 3 and 4.

The simulation results in Tables 1 and 2 show that the performance of the MCMEs and MLEs are very close for different sample sizes and censoring proportions. It is clear from these results that MCMEs and the MLEs are both highly biased if $n$ is small, $x$ is large and $p$ is high. For either MCMEs or MLEs, the bias-reduction method works quite well for both parameters even for small sample sizes and high censoring proportions.

The asymptotic confidence intervals based on MCME do not work very well when the sample size is small ($n \leq 20$) or $x$ is large ($x \geq 0.5$) and censoring proportion is high ($p \geq 30\%$), because the coverage probability is much lower than the corresponding nominal levels. The asymptotic confidence intervals based on MLE follow a similar pattern but are better than those based on MCME under the same conditions. For sample sizes 50 or more and $x$ less than 0.5, even for the highest censoring proportion 50%, the coverage probabilities are close to nominal levels for both $x$ and $\beta$. The bias reduction techniques improve the coverage probabilities in all cases.

5. An analysis of real data

To illustrate our method, the following data on cancer in rats are reanalyzed. This example is taken from Pike (1996), and was used by Lawless (1982) to illustrate the calculation of the Kaplan–Meier estimate and the empirical cumulative hazard function. He also used it to illustrate MLE in the Weibull model. Pike (1996) gave the results of a laboratory experiment concerning vaginal cancer in female rats. In one experiment 19 rats were painted with the carcinogen DMBA, and the number of days $T$ until the appearance of a carcinoma was the variable of interest. At the time the data were collected only 17 out of the 19 rats had developed a carcinoma, so that two of the times below (marked *) are censoring times. The times were

$$143, 164, 188, 188, 190, 192, 206, 209, 213, 216, 220, 227, 230, 234, 246, 265, 304, 216^*, 244^*.$$  

Lawless (1982) (Examples 2.4.1 and 4.4.1) showed that a three-parameter Weibull distribution is a reasonable model. The estimated threshold parameter is about 120. The estimated
shape parameter is $\hat{\alpha} = 2.712$ and the scale parameter is estimated as $\hat{\beta} = 108.4$ using the method of maximum likelihood. In this example, we use a two parameter BISA distribution.
Table 2
Biases and mean squared errors (MSEs) based on Monte Carlo simulation ($\beta = 1.0$)

<table>
<thead>
<tr>
<th>$p$</th>
<th>$n$</th>
<th>$x$</th>
<th>Estimate of $\beta$</th>
<th>MCME</th>
<th>UMCME</th>
<th>MLE</th>
<th>UMLE</th>
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<td></td>
<td></td>
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<td></td>
<td>Bias</td>
<td>MSE</td>
<td>Bias</td>
<td>MSE</td>
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<td>0.011</td>
<td>0.003</td>
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<td>0.0267</td>
<td>0.0034</td>
<td>0.0262</td>
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<td>0.2860</td>
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<td>0.2641</td>
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</table>

to fit the data. We estimate the two parameters using our MCME method, and find the bias-corrected estimates are $\hat{\alpha} = 0.15$ and $\hat{\beta} = 215.11$. The survival curves from these two
Table 3
Probability coverages of 90% confidence intervals based on Monte Carlo simulation ($\beta = 1.0$)

<table>
<thead>
<tr>
<th>$p$</th>
<th>$n$</th>
<th>$x$</th>
<th>Probability coverages for $x$</th>
<th>Probability coverages for $\beta$</th>
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<tr>
<td></td>
<td></td>
<td></td>
<td>MCME</td>
<td>UMCME</td>
</tr>
<tr>
<td>10%</td>
<td>10</td>
<td>0.10</td>
<td>0.749</td>
<td>0.825</td>
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<tr>
<td></td>
<td></td>
<td>0.50</td>
<td>0.761</td>
<td>0.838</td>
</tr>
<tr>
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<td></td>
<td>2.00</td>
<td>0.677</td>
<td>0.806</td>
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<td>0.828</td>
<td>0.875</td>
<td>0.885</td>
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<td>0.851</td>
<td>0.889</td>
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<td>0.767</td>
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<td>0.901</td>
<td>0.903</td>
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<td></td>
<td>0.50</td>
<td>0.873</td>
<td>0.895</td>
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<tr>
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<td>2.00</td>
<td>0.832</td>
<td>0.852</td>
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<td>0.908</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>0.50</td>
<td>0.892</td>
<td>0.895</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.00</td>
<td>0.874</td>
<td>0.895</td>
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</tbody>
</table>

30% 10 0.10 0.757 | 0.859 | 0.855 | 0.912 | 0.899 | 0.899 | 0.880 | 0.880 |
|     | 0.50| 0.754 | 0.868 | 0.872 | 0.913 | 0.887 | 0.887 | 0.861 | 0.861 |
|     | 2.00| 0.693 | 0.839 | 0.866 | 0.916 | 0.754 | 0.833 | 0.872 | 0.846 |
| 20  | 0.10| 0.860 | 0.904 | 0.903 | 0.935 | 0.923 | 0.923 | 0.890 | 0.890 |
|     | 0.50| 0.856 | 0.923 | 0.892 | 0.922 | 0.920 | 0.923 | 0.895 | 0.892 |
|     | 2.00| 0.749 | 0.869 | 0.877 | 0.918 | 0.832 | 0.850 | 0.887 | 0.861 |
| 50  | 0.10| 0.894 | 0.919 | 0.889 | 0.905 | 0.916 | 0.917 | 0.879 | 0.878 |
|     | 0.50| 0.887 | 0.914 | 0.902 | 0.917 | 0.931 | 0.932 | 0.902 | 0.900 |
|     | 2.00| 0.773 | 0.832 | 0.894 | 0.906 | 0.844 | 0.861 | 0.892 | 0.873 |
| 100 | 0.10| 0.920 | 0.922 | 0.887 | 0.891 | 0.906 | 0.905 | 0.887 | 0.887 |
|     | 0.50| 0.895 | 0.907 | 0.884 | 0.899 | 0.898 | 0.897 | 0.893 | 0.896 |
|     | 2.00| 0.785 | 0.829 | 0.898 | 0.908 | 0.842 | 0.855 | 0.907 | 0.897 |

50% 10 0.10 0.664 | 0.763 | 0.864 | 0.896 | 0.868 | 0.874 | 0.866 | 0.866 |
|     | 0.50| 0.708 | 0.838 | 0.866 | 0.909 | 0.866 | 0.869 | 0.871 | 0.868 |
|     | 2.00| 0.556 | 0.874 | 0.856 | 0.902 | 0.677 | 0.765 | 0.843 | 0.794 |
| 20  | 0.10| 0.793 | 0.881 | 0.882 | 0.914 | 0.910 | 0.910 | 0.872 | 0.872 |
|     | 0.50| 0.825 | 0.908 | 0.867 | 0.901 | 0.915 | 0.920 | 0.898 | 0.892 |
|     | 2.00| 0.599 | 0.859 | 0.860 | 0.893 | 0.722 | 0.777 | 0.866 | 0.834 |
| 50  | 0.10| 0.881 | 0.914 | 0.887 | 0.905 | 0.930 | 0.930 | 0.895 | 0.895 |
|     | 0.50| 0.855 | 0.911 | 0.882 | 0.908 | 0.935 | 0.939 | 0.895 | 0.892 |
|     | 2.00| 0.624 | 0.782 | 0.894 | 0.906 | 0.692 | 0.728 | 0.892 | 0.873 |
| 100 | 0.10| 0.883 | 0.906 | 0.898 | 0.905 | 0.899 | 0.900 | 0.891 | 0.891 |
|     | 0.50| 0.878 | 0.903 | 0.878 | 0.884 | 0.895 | 0.899 | 0.912 | 0.912 |
|     | 2.00| 0.637 | 0.734 | 0.920 | 0.924 | 0.690 | 0.728 | 0.928 | 0.916 |

distributions are shown in Fig. 1 along with the nonparametric Kaplan–Meier estimate. Both fitted curves are within the 95% confidence band of the Kaplan–Meier estimate. But
Table 4
Probability coverages of 95% confidence intervals based on Monte Carlo simulation ($\beta = 1.0$)

<table>
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<tr>
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<th>Probability coverages for $\beta$</th>
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<td></td>
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<td>MCME</td>
<td>UMCME</td>
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<td>0.10</td>
<td>0.796</td>
<td>0.877</td>
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<td>2.00</td>
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<td>0.903</td>
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the BISA distribution adjusts the curvature around the time = 190 months better than the Weibull distribution. This shows that the two parametric models fit the data equally well, but our BISA model uses only two parameters.
6. Concluding remarks

When data are subject to random censoring and the underlying distribution is a BISA distribution, we propose the modified censored moment estimators and investigate their asymptotic properties. We find that the MCMEs and the MLEs behave very similarly in terms of biases and MSEs in almost all cases considered. But the MLEs give better probability coverages when the sample size is small (\(n \leq 20\)), \(a\) is large (\(a \geq 0.5\)) and censoring proportion is high (\(p \geq 30\%\)). In these cases, the additional computational effort for the MLEs may be worthwhile. Otherwise, we recommend the use of the MCMEs, because they have explicit forms and are easy to compute and understand. In fact, in some applications such as fatigue studies, \(a\) in the range (0, 0.5] is typical (see Birnbaum and Saunders, 1969b). MCME is a suitable choice in these cases. When the sample size is small (\(n \leq 20\)), \(a\) is large (\(a \geq 0.5\)) and censoring proportion is high (\(p \geq 30\%\)), the biases and MSEs are large and the uncorrected asymptotic confidence intervals do not perform very well also. Therefore, in these cases, bias-corrected MCMEs or bias-corrected MLEs should be used to reduce biases and MSEs. Simulated percentage points rather than the asymptotic normality may be used to improve the probability coverages. Finally, a data application is presented to illustrate our method.

Appendix. Technical proofs

Proof of Theorem 1. Letting \(d_{qj}(t) = -q_j(t)/G(t)\) and using the assumption \(g(0) = 0\), we have
\[ T_{ji}^* (T^n) = \frac{\delta q_1 (Z_i)}{\hat{G} (Z_i)} \]
\[ = \int_0^{T^n} \left( \frac{I_{[Z_i \geq t]}}{\hat{G}(t)} \right) q_1'(t) \, dt + \int_0^{T^n} d_{q_1}(t) [Z_i \geq t] \left\{ \frac{\hat{d}G(t)}{\hat{G}(t)} - \frac{dG(t)}{G(t)} \right\} \]
\[ + \int_0^{T^n} d_{q_1}(t) \, dM_i^c(t). \]

\[ T_{ji}^* (T^n) = \frac{\delta q_2 (Z_i)}{\hat{G} (Z_i)} \]
\[ = \int_0^{T^n} \left( \frac{I_{[Z_i \geq t]}}{\hat{G}(t)} - 1 \right) q_2'(t) \, dt + \int_0^{T^n} d_{q_2}(t) [Z_i \geq t] \left\{ \frac{\hat{d}G(t)}{\hat{G}(t)} - \frac{dG(t)}{G(t)} \right\} \]
\[ + \int_0^{T^n} d_{q_2}(t) \, dM_i^c(t). \]

The centering quantity of \( T_{ji}^* (T^n) \) \( (j = 1, 2) \) is given by
\[ v_1 (T^n) = \int_0^{T^n} F(t) \, dt, \quad v_2 (T^n) = \int_0^{T^n} \{ F(t) - 1 \} \left( -1/t^2 \right) \, dt \]
which are the truncated versions of \( v_1 \) and \( v_2 \), respectively. In either case, from (11) and (12) and the above equations, we have
\[ T_{ji}^* (T^n) - v_j (T^n) = \int_0^{T^n} \left( \frac{I_{[Z_i \geq t]}}{\hat{G}(t)F(t)} - 1 \right) F(t)q_j'(t) \, dt \]
\[ + \int_0^{T^n} d_{q_j}(t) [Z_i \geq t] \left\{ \frac{\hat{d}G(t)}{\hat{G}(t)} - \frac{dG(t)}{G(t)} \right\} \]
\[ + \int_0^{T^n} d_{q_j}(t) \, dM_i^c(t). \]

Noticing that
\[ \frac{I_{[Z_i \geq t]}}{\hat{G}(t)F(t)} - 1 = \frac{\hat{H}_i}{H} - 1 = \frac{\hat{H}_i - H}{H} + \frac{G - \hat{G}}{G} \]
\[ + \frac{\hat{H}_i - H}{H} \frac{G - \hat{G}}{G} + \frac{G - \hat{G}}{G} \frac{G - \hat{G}}{G} \]
and
\[ \frac{\hat{d}G(t)}{\hat{G}(t)} - \frac{dG(t)}{G(t)} = \left( -\frac{\hat{G}}{G} \right) d \left( \frac{G - \hat{G}}{G} \right). \]
using the fact that \( M_i^+ (s) = M_i^D (s) + M_i^C (s) \), \( M_C^+ (s) = \sum M_i^C (s) \), and \( h_{1,q_j} (t; \tau) = \int_\tau^t F (s) q'_j(s) \, ds \), integration by parts yields

\[
T_{ji}^n (T^n) - v_j (T^n) = \int_0^{T^n} \left\{ -h_{1,q_j} (t; T^n) \right\} \frac{H}{H} \, dM_i^D \\
+ J_{ji} + K_{ji} + R_{n,i} (T^n, q_j),
\]

where \( J_{ji} \), \( K_{ji} \) and \( R_{n,i} (T^n, q_j) \) are defined in the following:

\[
J_{ji} = \int_0^{T^n} P_j(t) \, dM_i^C (t) \quad \text{with} \quad P_j(t) = d_{q_j}(t) - \frac{h_{1,q_j} (t; T^n)}{H}.
\]

\[
K_{ji} = \int_0^{T^n} \left( \frac{G - \hat{G}}{G} \right) F(t) q'_j(t) \, dt + \int_0^{T^n} \left( d_{q_j}(t) \left[ Z_i \geq t \right] \right) \left\{ \frac{\hat{G}(t)}{G(t)} - \frac{dG(t)}{G(t)} \right\}
\]

\[
= \frac{1}{n} \sum_k \int_0^{T^n} Q_{ji} \, dM_k^C (t)
\]

with

\[
Q_{ji} = \left\{ h_{1,q_j} (t; T^n) + d_{q_j}(t) \left[ Z_i \geq t \right] \left( -\frac{G}{G} \right) \right\} \frac{\hat{G}}{G} \frac{1}{R^+/n}.
\]

\[
R_{n,i} (T^n, q_j) = \int_0^{T^n} \left( \frac{H - \hat{H}}{H} \frac{G - \hat{G}}{G} \right) F(t) q'_j(t) \, dt.
\]

Therefore, by (19) and some algebra, we have

\[
L_{n,j} (T^n) \equiv (\hat{v}_j - v_j) (T^n) = \frac{1}{n} \sum_i \left\{ T_{ji}^n (T^n) - v_j (T^n) \right\}
\]

\[
= \frac{1}{n} \sum_i \int_0^{T^n} \left\{ -h_{1,q_j} (t; T^n) \right\} \frac{H}{H} \, dM_i^D \\
+ \frac{1}{n} \sum_i \int_0^{T^n} \left\{ \frac{\hat{G}}{G} \frac{1}{R^+/n} - \frac{1}{H} \right\} h_{1,q_j} (t; T^n) \, dM_i^C \\
+ R_{n} (T^n, q_j),
\]

where

\[
R_{n} (T^n, q_j) = \int_0^{T^n} \left( \frac{H - \hat{H}}{H} \frac{G - \hat{G}}{G} \right) F(t) q'_j(t) \, dt
\]

\[
+ \int_0^{T^n} \left( \frac{G - \hat{G}}{G} \frac{G - \hat{G}}{\hat{G}} \right) F(t) q'_j(t) \, dt,
\]

\[
\hat{H} = \frac{1}{n} \sum_{i=1}^n \hat{H}_i.
\]
The quantity \( L_n, j (T^n) \) is a martingale presentation plus higher order terms and is independent of \( d_{qj}(y) \).

We now consider the asymptotic normality of \( L_n, j (T^n) \). Denote the first two terms in (20) by \( M_{n, j, 1} \) and \( M_{n, j, 2} \) and define \( M_{n, j} = M_{n, j, 1} + M_{n, j, 2} \). They are both martingales. We can show that \( \text{Var} \left\{ \sqrt{n} M_{n, j, 2} \right\} \to 0 \). Hence, \( M_{n, j, 2} \) is negligible and \( M_{n, j} \) has the same asymptotic distribution as \( M_{n, j, 1} \).

By the independence assumption, we know that \( M D_1(t), \ldots, M D_n(t) \) are mutually orthogonal martingales. Hence, by the martingale CLT, we have \( \sqrt{n} M_{n, j} \sim \sqrt{n} M_{n, j, 1} \to D N \left( 0, v_{jj} \right) \).

Similarly, the covariance of \( M_{n, 1, 1} \) and \( M_{n, 2, 1} \) is given by
\[
\int_0^{\tau_H} \left\{ h_{1, q_1}(t; \tau) \right\} \left\{ h_{1, q_2}(t; \tau) \right\} G d F \triangleq v_{12} \left( \tau_H \right).
\]

Following the approach given in Zhou (1991), we can show that the higher order term \( \sqrt{n} R_n (T^n, q_j) \) in \( \sqrt{n} L_n, j (T^n) \) is \( o_P(1) \).

To end, we have shown that the joint distribution of \( \sqrt{n} \hat{x} - x, \sqrt{n} \hat{y} - y \) is equivalent to that of \( \sqrt{n} M_{n, 1} (T^n) \) and \( \sqrt{n} M_{n, 2} (T^n) \). Thus, Theorem 1 holds. \( \square \)

Proof of Theorem 2. It suffices to find the asymptotic joint distribution of
\[
\left( \hat{x}, \hat{y} \right) = \left( f_1 \left( \hat{v}_1, \hat{v}_2 \right), f_2 \left( \hat{v}_1, \hat{v}_2 \right) \right),
\]
where \( f_1(x, y) = \left[ 2 \left\{ (xy)^{1/2} - 1 \right\} \right]^{1/2}, f_2(x, y) = (x/y)^{1/2} \). In line with Ng et al. (2003), using the Taylor’s expansion, we obtain
\[
\sqrt{n} \left( \frac{\hat{x}}{\hat{y}} - \frac{x}{y} \right) \to D N \left( 0, AV \left( \tau_H \right) A^T \right),
\]
where
\[
A = \left( \frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y} \right)_{x=E(T), y=E(1/T)} = \begin{pmatrix}
\frac{1}{2\alpha \beta} & \frac{\beta}{2\alpha} \\
1 & \frac{\beta^2}{2 + \alpha^2} - \frac{\beta^2}{2 + \alpha^2}
\end{pmatrix}.
\]

\( \square \)

References


