Spectral decay vs. correlation dimension of EEG

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Abstract

The paper relates to possible connection between spectrum power-law decay and correlation dimension estimation for electroencephalogram (EEG). EEG signals recorded during relaxed wakefulness were analysed. Power-law decay of about 2.28 prevailing over the exponential falling off was established when exponent from the whole EEG spectrum was taken. The correlation dimension was also estimated. The two measures proved to be highly correlated. Concurrent experimental applications of spectral decay and correlation dimension confirm comparability of their discriminative powers. However, estimating the spectral decay is clearly more effective than the data demanding and disputable computation of correlation dimension.

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1. Introduction

In the last decades, the growing need for a better understanding of large-scale brain dynamics has stimulated the expansion of research in this area. Besides linear approaches connected mostly to the Fourier analysis, there are many arguments for neuronal dynamics to be considered to behave in a non-linear manner. This fact has prompted the development of new analysis techniques, often referred to as non-linear methods.

The related theory of chaotic systems gave rise to the question whether it is possible to distinguish between random behaviour and seemingly irregular processes with deterministic origin. The corresponding algorithms are based on the concept of state space representations of the underlying dynamics. Problems with application of methods from chaotic systems to Electroencephalogram (EEG) analysis are connected with non-stationarity of signals, relatively high-noise component and other factors. A thorough review of powerful algorithms for the numerical analysis of complex time-series data can be found in [3]. Special emphasis is given to the issue whether these methods can be successfully applied in the case of human EEG.

In spite of the fact, that the non-linearity of brain responses have not been proved in a satisfactory manner, some non-linear measures turned out to be very successful in distinguishing between different brain states.

Among the various non-linear procedures available for experimental data, the calculation of the fractal complexity has probably received the widest attention. It has been mathematically established that, if we can measure any single variable of a dynamical system with sufficient accuracy, then it is possible to reconstruct a state portrait, topologically equivalent to the attractor of the original system. Complexity of the reconstructed attractor may provide important information about the system. The most popular tool to assess this complexity is the correlation dimension defined as

\[ D_2 = \lim_{\varepsilon \to 0} \frac{\ln \sum_{i=1}^{N(\varepsilon)} P_i^2}{\ln \varepsilon} = \lim_{\varepsilon \to 0} \frac{\ln C_2(\varepsilon)}{\ln \varepsilon}, \]

where \( N(\varepsilon) \) is the total number of hypercubes of side length \( \varepsilon \) that cover the attractor, and \( P_i \) is the probability of finding a point in the hypercube \( i \). As Grassberger and Procaccia [4] noticed the correlation integral \( C_2(\varepsilon) \) is approximately equal to the probability that the distance...
between two points \( x_m(i), x_m(j) \) of the attractor in \( m \)-dimensional state space is less than \( \varepsilon \):

\[
C_2(\varepsilon) \approx \frac{2}{N(N-1)} \sum_{i}^{N} \sum_{j>i}^{N} \Theta(\varepsilon - \| x_m(i) - x_m(j) \|),
\]

where \( \Theta \) is the Heaviside step function, and \( \| \cdot \| \) usually represents the maximum norm.

In order to estimate the correlation dimension, we plot \( \ln C_2(\varepsilon) \) as a function of \( \ln \varepsilon \) and follow the slope \( n(\varepsilon) \) of the obtained curve. This slope is called a correlation exponent, and the limit of it for vanishing \( \varepsilon \) represents the value of correlation dimension.

As it is visible in Fig. 1, the plot of \( n(\varepsilon) \) displays more regions of distinct types of behaviour. For small \( \varepsilon \), the graph depends on a statistically insufficient number of points, so for this part large differences in \( n(\varepsilon) \) are typical. Then the effect of noise in data leads to large values of \( n(\varepsilon) \) for some interval. If a clearly distinguishable flat part follows, then the value of this plateau is taken as an estimate of the searched correlation dimension. Finally, the slope approaches zero for close to the diameter of the sample set. In many experimental situations, noise may cover the hole scaling range and make dimension estimation impossible. As our data were recorded in electrically shielded room they do not contain much noise and small values of epsilon (about one percent of data set diameter) could be utilized in process of dimension estimation.

For more discussion on the correlation dimension estimation, see [7,16].

The relative simplicity of this technique, known as Grassberger and Procaccia (GP) method, ended in numerous applications for about last 20 years. The main idea behind the applications is that systems governed by stochastic processes are thought to be infinite dimensional whereas deterministic systems should have a finite value of dimension. Moreover, finite, non-integer value of the dimension was considered to be an indication of the presence of deterministic chaos. It is understandable that initially the GP-method was believed to offer an easy new tool to distinguish between random noise and low-dimensional determinism.

EEG has been a matter of interest for the dimension computation since the first \( D_2 \) estimates for sleep cycles were made by Babloyantz et al. [1]. This attempt was inspired by the chaos hypothesis, i.e. it was assumed that the EEG could be described by a deterministic chaotic system and therefore the corresponding attractor could be characterized by the fractal dimension.

Similar pursuits are criticized for the following reasons: for a reliable dimension estimate a time series has to be

![Fig. 1](image_url)
stationary, long enough and the signal-to-noise ratio has to be acceptable [5]. It is hard to expect these properties to be preserved in the case of the EEG. While in some specific cases, e.g., epileptic seizure, the EEG does appear to exhibit low dimensionality, in general the brain is continually interacting with many other complex systems and EEG seems to be a mixture of noise, certain cyclic processes and possibly some random fractal signals. Each part of such a composition itself has been frequently reported to fool the algorithms used to detect chaotic dynamics as it came to light shortly after first applications of GP-method. Therefore, the correlation dimension estimates should be interpreted with extreme caution.

After all, even the understanding of low values of the correlation dimension as a sign of deterministic chaos has been questioned. Theiler [18] has shown that for data sets with long autocorrelation time the application of GP-algorithm leads to spuriously low estimates of dimension due to an anomalous shoulder in the graph of correlation integral. To reduce the effects of linear correlations, the author recommends taking much more data than the integral. To reduce the effects of linear correlations, the algorithm leads to spuriously low estimates of dimension with long autocorrelation time the application of GP-correlation dimension as a sign of deterministic chaos has interpreted with extreme caution.

Therefore, the correlation dimension estimates should be interpreted with extreme caution. Another remarkable paper questioning the view that stochastic time series lead to a non-convergence of the correlation dimension is that of Osborne and Provenzale [11]. They showed that noise exhibiting power-law spectra, may result in low values of the correlation dimension. These noises in fact are self-affine signals, which generate a fractal curve whose dimension is determined by the power spectrum. The GP-method cannot distinguish between fractal attractor of deterministic system and fractal random curve if their dimensions equal.

Despite the criticism, measures used for identifying low-dimensional chaotic systems, such as the correlation dimension, continue to be used for studying the EEG signals. For instance, some authors reported decreasing values of $D_2$ with deepening of the level of sleep or the level of relaxation. In [12] the mean values of $D_2$ are ranged between 5.0 for slow wave sleep (SWS) and 7.9 for waking. Kobayashi [6] reported a mean correlation dimension of 3.28 for SWS and 7.79 for vigilant state.

Now let us return to the most commonly used traditional technique for analysing EEG that is Fourier analysis. Regarding spectral properties of different types of data the next statements are generally accepted:

1. **Stochastic behaviour**: The power spectrum decays via a power law $P(f) \sim f^{-\gamma}$ [15]. As an example, white noise with $\gamma = 0$ or random walk time series with power spectrum that decreases as $1/f^2$ with increasing frequency can be mentioned. $\gamma$ came to be called spectral decay, fractal exponent, or power-law exponent.

2. **Periodic or quasiperiodic behaviour**: The power spectrum consists of discrete spikes corresponding to distinct frequencies. In the region of higher frequencies of experimental data the power spectrum falls polynomially due to presence of some form of noise.

3. **Chaotic behaviour**: The power spectrum falls exponentially at high frequencies [14]. Exponential decay of power spectrum is a decay of the form $P(f) \sim a e^{-b f}$, where $a$ and the exponent $b$ are constant. Since noise is always present in real systems, one can, in case of real chaotic systems, observe only a finite region of exponential decay. Then the spectrum settles into the power-law decay characteristic of noise.

The above summary shows that examination of the power spectrum can help us to answer the question, whether the observed erratic behaviour is essentially deterministic or stochastic [14,15].

As concerns character of EEG data, recent investigations indicate that although, EEG does lose information rapidly from second to second, there is a significant level of correlation that is evident after at least 5 s [21]. The non-trivial long-range correlation within the signal (indicated by the presence of very low frequencies) is characteristic of a fractal or scale-invariant processes. They are particularly relevant in the context of phase transition in physics, as scaling phenomena are known to occur in many physical systems near a critical state. Even though we do not ask whether the brain works in critical state, let us mention that some findings suggest that power-law scaling is characteristic of healthy cerebral activity and the breakdown of this scaling may lead to incapability of quick reorganization during processing demands [8,9].

As the power of EEG spectrum is supposed to decrease polynomially, let us mention a result that advocates the use of the order of the polynomial falloff as a tool for dimension estimate. In [11] the authors found that for a stochastic signal with a $1/f^\gamma$ power spectrum the numerical estimate of correlation dimension is a small finite value $D_2 = 2/(\gamma-1)$ for $1 < \gamma < 3$. For $\gamma > 3$, the fractal properties disappear and the dimension becomes equal to the topological one. For $0 < \gamma < 1$, the dimension increases without limit, i.e. the signal actually generates a non-saturating dimension in this case. The authors found a good overall agreement between the theoretical relationship and numerical results. They confirmed that finite values of dimension need not be signs of low dimensionality of the system under study. Often they reflect the fractal nature of specific random processes.

To test these findings in the case of sleep EEG, Pereda et al. [12] investigated a correlation between the fractal exponent (spectral decay) and $D_2$. For the exponent computation, they selected the frequency range of 3–30 Hz arguing that spectra presented the clearest $1/f^\gamma$ dependence within it. The corresponding fractal exponent
ranged from 1.49 (awake) to 3.07 (SWS). The authors found negative linear correlation between $D_2$ and $\gamma$ present in all stages except during SWS.

On the other hand, Shen et al. [13] found significant non-linearity only in sleep stage 2. In their paper, the estimates of fractal exponent (computed for frequencies less then alpha activity) are between 0.98 (REM sleep) and 2.18 for SWS.

Looking at the above two papers we must call attention to the fact that their results do not fulfil exactly the relationship of $D_2$ and $\gamma$ derived in [11]. It may relate to findings of Theiler [19] who discussed the relevance of the relationship to the practical estimation of dimension from real-time series. He pointed to a need for introduction of frequency cut-offs. He derived the scaling of the correlation integral for Gaussian $1/f$ noise with explicit high and low frequency cut-offs. Since the data are stochastic, the correct scaling in $m$-dimensional space should be $C(N,r) \sim r^n$.

Actually, regardless of the cut-offs, this scaling is found for small $r$ and for very large $N$. In addition, an existence of next two scaling regimes was detected. When there is a high frequency cut-off, the trajectory locally looks one-dimensional. This can be reflected as a scaling of $C(N,r) \sim r^n$ over some range of $r$. Eventually, the anomalous scaling of $r^2(\gamma-1)$ considered in [11] is observed only for large $r$ and relatively small $N$.

It is therefore clear that the discrepancies between theoretical prediction and empirical results reflect the substantial effect of the number of data and other parameters on the value of the actual estimate of the (false) dimension.

Even though the question of the exact relationship between spectral decay and dimension is not addressed in [12,13] the authors clearly show that there is a negative linear correlation between the two measures. Then the values and variations of dimension estimates in the course of night can to a large extent be explained by linear properties of the EEG time series.

In this paper, our aim is to verify declarations about $D_2$ and spectral decay in the case of relaxing EEG. At first, we check the presence of exponential or power-law decay in the power spectra. As there seems to be no consensus regarding the choice of the regions of power-law decay in the literature, we are going to study the whole spectrum of EEG to find the sections of clear power-law behaviour. Thereafter, we look for regions that lead to maximal mutual information between correlation dimension and spectral decay. Finally, we are going to think over the discriminating power of $D_2$ and $\gamma$, to be able to recommend one of them as preferable characteristic of changes in EEG.

2. Materials and methods

Eight healthy volunteers (3 females and 5 males) took part in EEG recording. Participants ranged in age from 24 to 39 years, with a mean of 25.5 years, s.d. 5.1 years. They did not have any known neurological deficit and were not taking any drugs known to affect the EEG. All the experiments were carried out with an adequate understanding and written consent of the subjects. They attended two measurements per each of 25 days. Every time data of 3-min length were recorded. During recording subjects were lying in a darkened, electrically shielded room. They were instructed to keep their eyes closed and relax both physically and mentally.

Monopolar EEG montage comprised eight channels with electrodes placed on F3, F4, C3, C4, P3, P4, O1, O2 locations according the International 10–20 system. A standard cap system with Ag–AgCl electrodes was employed. The reference electrode was located at Cz and the ground electrode at Fpz point. Thereafter six difference signals F3–C3, F4–C4, C3–P3, C4–P4, P3–O1, and P4–O2 were derived. In order to prevent signal distortions, scalp-electrode impedances were kept below 5 k\(\Omega\), and balanced within 1 k\(\Omega\) of each other.

The resolution of the A/D converter was 16 bits. A digital high-pass FIR filter with cut-off at 0.75 Hz, with the width of 3000 data points, and with a Blackman window was utilized. To minimize problems with undesirable effects of low pass filtering, the measuring device fully covered frequency band from 1 to 100 Hz. For the purpose of this study, 2300 EEGs were analysed. The EEG measures were computed from 3-min epochs digitized at 500 Hz. Following the filtering the first and the last 1500 points were omitted and 87000 data point EEGs remained. They can be regarded as relatively stationary with respect to subject’s quiet behavioural state and sufficiently long to be used for $D_2$ estimation.

3. Results

In order to calculate the correlation dimension, the data (87000 values in one series) were embedded into $m$-dimensional spaces ($m = 1, 2, \ldots, 7$). Following the proposal of Takens [17] vectors built-up from delay coordinates were used for the reconstruction. The vectors were constructed with a time lag $\tau = 10$, which corresponds to 20 ms. This choice of delay was based on analysis of mutual information between the original signal and its shifted versions. According to the most common practice the first minimum of the mutual information was selected to ensure that the components of the reconstruction vectors would be relatively independent [3].

Then $D_2$ was calculated using the GP-algorithm. To evaluate the plateau in the graph of correlation exponent $\nu(e)$ we used the method which searches for the first wide flat interval in the plot. The procedure is based on two parameters: the minimum width required for a scaling region and the maximum amplitude of allowed variations of $\nu(e)$ in the scaling region. The method was tested and approved in [2]. The obtained estimates of $D_2$ “saturated”, i.e. they approached a constant value as embedding dimensions increased above $m = 5$. The average of the
estimates stated for embedding dimensions 5, 6, and 7 were taken as the resultant estimate of $D_2$. Use of higher embedding dimensions would not be applicable, due to the limitation of the number of data points [7,16].

As a result a significant indication of relatively low values of $D_2$ estimates (between 3 and 6) with the mean of 4.35 was found.

Alternatively we applied the GP-algorithm for only one tenth of the data (8700 data points). The reason was testing the critical dependence of the dimension estimates on the length of input time series [7,16]. In our case, decreasing the number of points resulted in larger fluctuations of the correlation exponent $\nu(c)$. Even though the plateau was still clearly visible, its mean value went down to 2.96. See Fig. 1.

As a next step, we looked for the presence of exponential or power-law decay in the power spectra of EEG. The spectrum was computed using standard FFT with frequency step of 0.029 Hz and variance reduction factor of 10. Exponential decay is expected to manifest as a straight line when spectra are plotted in a log-linear coordinate system, and the slope of this line would give the corresponding exponent. Power-law decay can be obtained accordingly as a slope of linear regression applied to the power spectrum in log–log coordinate system. In our case peak of about 10 Hz corresponding to brain alpha activity was apparent both in log-linear graph of spectrum and in log–log graph of spectrum. However, contrary to log-linear graph, in the case of log–log presentation frequencies before and after alpha peak fell linearly indicating a presence of power-law behaviour. Fig. 2 shows a typical spectrum displayed in log-log graph. Power-law model proved to be preferable over exponential model in 99% of frequency ranges both before alpha activity and following alpha activity.

To test the dependence of $\gamma$ estimates on the length of input series we computed $\gamma$ using only one tenth (8700 data points) and one hundredth (870 data points) of the whole time series. In contrast to correlation dimension these estimates do not differ much from the estimates obtained with longer time series.

Consequently, different regions of the whole power spectra (including alpha range) were taken to assess the efficiency of power-law model in comparison to exponential model. The best (in the least-squares sense) of both types of models were compared. Differences were considered significant when $P < 0.05$. By automatic routine intervals of fixed left frequency limit to ascending right limits as well as segments of descending left limits to fixed right bound were systematically investigated. Fig. 3 illustrates the efficiency (in percentage) of power-law model in contrast to exponential model for particular parts of spectra. As one can see, power-law decay is clearly prevailing over an exponential falling off except that the investigated interval is restricted to a neighbourhood of alpha frequencies.

Our next task was to verify declarations about the relation between $D_2$ and power-law decay. Therefore, we computed both linear correlation between these measures and also mutual information as a general dependency measure that can potentially detect non-linear relationship between $D_2$ and $\gamma$.

Mutual information $MI$ for measurements $X_i$ and $Y_j$ drawn from sets $X$ and $Y$ was estimated by the algorithm proposed by Moddemeijer [10] making use of

\begin{align*}
\text{MI} & = \sum_{X, Y} p(X, Y) \log \frac{p(X, Y)}{p(X)p(Y)}
\end{align*}

\[ p(X, Y) = p(X|Y)p(Y) = p(Y|X)p(X) \]

**Fig. 2.** Typical graph of spectral exponent estimation for EEG frequency range before alpha activity and following alpha activity.

**Fig. 3.** Efficiency in % of power-law model in contrast to exponential model: (a) computed for 2300 power spectra from fixed 5 Hz to ascending right limit (values of horizontal axes) of the spectral range; (b) computed for 2300 power spectra from fixed 250 Hz to descending left limit of the spectral range.
Mutual information (MI) is a measure of the amount of information one random variable contains about another. It is defined as:

$$MI(X, Y) = \sum_{x_i,y_j} P_{XY}(x_i,y_j) \log_2 \left( \frac{P_{XY}(x_i,y_j)}{P_X(x_i)P_Y(y_j)} \right),$$

where $P_X(x_i)$ and $P_Y(y_j)$ denote prior probabilities of observed values $x_i$ and $y_j$, and $P_{XY}(x_i,y_j)$ represents the joint probability of pairs of observed values. $MI = 0$ for independent time series and otherwise it takes positive values with a maximum for identical signals [3,5].

To estimate the distributions, normalized two-dimensional histograms were calculated using uniform binning of 45-45 bins. This partition corresponds to the cubic root of the number of data and is suggested by the author of the algorithm as a default value when no prior knowledge about the data is provided [10]. According to our tests, this choice of binning turned out to be superior for the analysed EEG data as well.

The linear correlation coefficient $r$ for two time series $X$ and $Y$ was also computed.

$$r = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum (X_i - \bar{X})^2 \sum (Y_i - \bar{Y})^2}},$$

where $\bar{X}$ and $\bar{Y}$ are means of time series $X$ and $Y$, respectively. The value of $r$ lies between $-1$ (complete negative correlation) and $1$ (complete positive correlation). A value of $r$ near zero indicates that time series $X$ and $Y$ are uncorrelated.

As there was no consensus in the choice of frequency interval for power-law exponent in the literature, we compared $D_2$ with exponents of different intervals of spectrum. The result was unexpectedly clear: the highest mutual information and also the strongest negative correlation of -0.8 between the dimension and spectral exponent were attained when exponent from the whole spectrum was taken. The average value of $\gamma$ established from the whole EEG spectrum (in our case from 5 to 250 Hz) is about 2.28. As an illustration, Fig. 4 shows linear correlation and mutual information between $D_2$ and $\gamma$. Computed from intervals of fixed 5 Hz to ascending right limit (values of horizontal axes) of the spectral range.

Fig. 5 compares visually an evaluation of the two measures—power-law decay (spectral range from 5 to 250 Hz) and correlation dimension. Evidently their discriminating power is highly comparable.

4. Discussion

Nowadays ascending number of experts believes that there is little evidence for any role of deterministic chaos in brain dynamics. In the case of observing low values of $D_2$ for EEG, the hypothesis of presence of scale-invariant fractal-like structures is preferred rather than the suggestion of deterministic chaos. The authors begin to refer to their dimension estimates not as an absolute measure of an attractor complexity, but as a valuable relative measure, which, estimated with highest caution, remain usable as one of invariants of underlying system.

In this study, power-law decay prevailing over the exponential falling off was established from the relaxing EEG spectrum. Consequently, excellent correspondence was found between the spectral decay computed from the whole EEG spectrum and the estimate of the correlation dimension. As both measures seem to reflect the same information, it is obvious that in this case the dimension estimate by GP-algorithm only mirrors the spectral features of signal.

Moreover, algorithms of correlation dimension computation are time consuming, data demanding, and connected with difficult evaluation and interpretation of results. On the other hand, estimates of $\gamma$ are easily obtained by standard procedures and they are quite stable regarding the number of data used for computation. For a reliable number of points (say larger than a few hundreds) a very good estimate of spectral exponent can be obtained, while for the same number of data it is impossible to investigate the correlation dimension.
Considering all the above arguments, for EEG as an example of a real system with power-law spectrum the use of spectral decay can be recommended as more effective than correlation dimension computation. Our experiments regarding the sleep onset, which are going to be published soon, confirmed the above findings: the spectral exponent was much more successful to discriminate between states of relaxation and states of light sleep than correlation dimension. Revisions of other studies using $D_2$ to distinguish different psycho-physiological states are expected to manifest superior discriminative power of the spectral decay as well.

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References


