Separable Monte Carlo combined with importance sampling for variance reduction

Anirban Chaudhuri* and Raphael T. Haftka

Department of Mechanical & Aerospace Engineering,
University of Florida,
Gainesville, FL 32611, USA
Email: anirban.chaudhuri01@gmail.com
Email: haftka@ufl.edu
*Corresponding author

Abstract: Monte Carlo (MC) methods are often used to carry out reliability based design of structures. Methods that improve the accuracy of MC simulation include Separable Monte Carlo (Separable MC), Markov Chain Monte Carlo and importance sampling. We explore the utility of combining Separable MC and importance sampling for improving accuracy. The accuracy of the estimates is compared for Standard MC, Separable MC, importance sampling and combined method for a composite plate example and a tuned mass damper example. For these examples Separable MC and importance sampling reduced the error individually by factors of 2–5, and the combination reduced it further by about a factor of 2. The results were also compared with the First Order Reliability Method (FORM). FORM was grossly inaccurate for the tuned mass-damper example which has a failure region bounded by safe regions on either side.

Keywords: Monte Carlo simulation; importance sampling; separable MC; SMC; separable Monte Carlo; variance reduction technique; reliability analysis; probability of failure; ImpSMC.

Reference to this paper should be made as follows: Chaudhuri, A. and Haftka, R.T. (2013) 'Separable Monte Carlo combined with importance sampling for variance reduction', *Int. J. Reliability and Safety*, Vol. 7, No. 3, pp.201–215.

Biographical notes: Anirban Chaudhuri is currently enrolled as a Doctoral student in the University of Florida. He received his Bachelor of Technology in Mechanical Engineering from Malaviya National Institute of Technology, Jaipur, India in 2008. He earned his Master of Science in Mechanical Engineering from University of Florida in 2010 with research on uncertainty analysis of spur gear response and life cycle of a spur gear based on wear propagation. His current research topics are reliability analysis, surrogate based global optimisation and surrogate based experimental optimisation.

Raphael T. Haftka is a Distinguished Professor of Mechanical and Aerospace Engineering in the University of Florida. He received a PhD in aerospace engineering in 1971 from the University of California at San Diego. He has also taught at Technion, Israel, Illinois Institute of Technology and Virginia Tech. He served as president of International Society for Structural and Multidisciplinary Optimisation (ISSMO) from 1995–1999. His research area is structural and multidisciplinary optimisation with emphasis on design under uncertainty.

This paper is a revised and expanded version of a paper entitled 'Combining separable Monte Carlo with importance sampling for improved accuracy' presented at the 'ASME 2011 IDETC/CIE Conference', Washington DC, 28–31 August 2011.

1 Introduction

Reliability-based design of structures is often carried out through Monte Carlo (MC) methods. The general Standard Monte Carlo (Standard MC) method is not accurate for small probabilities of failure when there is a limit on the number of samples because of expensive simulations. Methods that improve the accuracy of reliability analysis include Separable Monte Carlo (Separable MC) (Smarslok et al. 2006; Ravishankar et al., 2010; Smarslok et al., 2010), importance sampling (Glynn and Iglehart, 1989; Melchers, 1989; Hurtado and Barbat, 1998; Melchers, 1999; Dawson and Hall, 2006; Kalos and Whitlock, 2008; Zhang et al., 2009; Kuczera et al., 2010; Swiler and West, 2010; Nikolaidis and Mourelatos, 2011) and Markov chain Monte Carlo methods (Geyer, 1992; Smith and Roberts, 1993; Gilks, 2005), as well as response surface approximations (Zheng and Das, 2000) and tail modelling (Ramu et al., 2006). Au and Beck (1999) proposed a method in which they used Markov chain in order to define the importance function in an adaptive importance sampling scheme. Several other adaptive importance sampling strategies have also been developed along the years showing better performance than ordinary importance sampling (Dawson and Hall, 2006; Oh and Berger, 1992). Melchers (1989) and Schueller and Stix (1987) proposed methods to handle multiple limit states while using importance sampling. We propose here a combination of Separable MC and importance sampling.

The failure criterion in structural problems is often defined as the response exceeding the capacity. Separable MC can be used if the response and capacity are independent of each other (Smarslok et al., 2010; Ravishankar et al., 2010; Villanueva et al., 2011). Importance sampling for reliability analysis is based on biasing the sampling towards failing designs. The two approaches operate on different principles to reduce variance. Therefore, combining Separable MC and importance sampling makes sense, and it is the objective of this paper. We dub this combination ImpSMC.

Separable MC is described in Section 2. Section 3 discusses an importance sampling procedure based on sampling around most-probable point. The combination of Separable MC and importance sampling is presented in Section 4. A composite plate example is used to demonstrate the accuracy of different Monte Carlo methods in Section 5. Section 6 compares the accuracy of different Monte Carlo methods for a tuned mass-damper example. Conclusion is presented in Section 7.

2 Separable Monte Carlo

The probability of failure for the limit state $G(X)$ where X is a set of random numbers with a given probability density function of $f_X(\ )$ is given by equation (1). The Monte Carlo estimation of the probability is shown in equation (2). I is the indicator function which equals 1 when the condition is true and equals 0 when it is false.

$$p_f = \int I[G(x) < 0] f_X(x) dx \tag{1}$$

$$\hat{p}_{f_{Standard\ MC}} \approx \frac{1}{N}\sum_{i=1}^{N} I[G(X_i) < 0] \tag{2}$$

The limit state is usually expressed in terms of response and capacity variables. The Separable MC method takes advantage of a common situation where the response R and capacity C are stochastically independent random variables (Smarslok et al., 2010). That is, the uncertainty in capacity depends on a set of random variables, X_C and that in response depends on a different set of random variables, X_R which are mutually independent. The limit state for probability of failure calculation can be represented by equation (3).

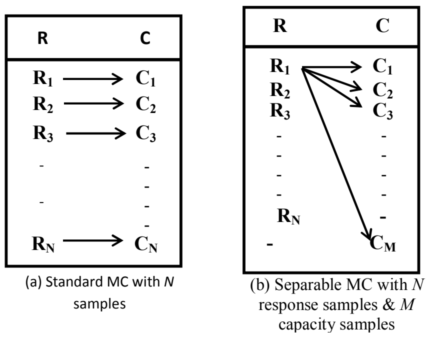
$$G(C, R) = G(C(X_C), R(X_R)) \tag{3}$$

In Separable MC, all possible combinations of M samples of capacity and N samples of response are compared to evaluate failure. This creates a large sample of points with only modest number of samples of either response or capacity. Failure is defined as $G < 0$ and the probability of failure is estimated by equation (4).

$$\hat{p}_{f_{Separable\ MC}} \approx \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N} I[G(X_{C_i}, X_{R_j}) < 0] \tag{4}$$

Separable MC allows different sample sizes for response and capacity, which is very advantageous when working with limited computational budget. Figure 1 shows a comparison between Standard MC and Separable MC. It can be applied any time the capacity and response depend on independent sets of random variables. More details and derivations of the variance reduction properties of Separable MC can be found in the work of Smarslok et al. (2010) and Ravishankar et al. (2010).

Figure 1 Illustration showing a comparison between (a) standard MC and (b) separable MC



Source: Ravishankar et al. (2010)

3 Importance sampling

Importance sampling changes the sampling density function in order to pick *important* values of input random numbers. This improves the accuracy of estimation of statistical response of interest, which, is the probability of failure here. To compensate for the use of different sampling densities, the samples are weighted (Swiler and West, 2010).

In the case of probability of failure, the important regions are the regions of relative high probability where the limit state is near zero, because this is the region most involved with failure.

To ensure that most of the sampled points are in the failure region, a new sampling distribution centred in the failure region $h_V(\ )$ is selected from which the set of random variables V are sampled. To get the probability of failure with this new sampling we use equation (5).

$$p_{f_{imp}} = \int I[G(v) < 0] \frac{f_X(v)}{h_V(v)} h_V(v) dv \tag{5}$$

Thus, the Monte Carlo estimation is now weighted as shown in equation (6).

$$\hat{p}_{f_{imp}} \approx \frac{1}{N} \sum_{i=1}^{N} I[G(V_i) < 0] \frac{f_X(v_i)}{h_V(v_i)} \tag{6}$$

In equation (6), $\frac{f_X(v)}{h_V(v)}$ is the weight function.

For *n mutually independent* random variables, the joint probability density is the product of the single-variable probability densities h_{V_i}.

$$h_V(v_1,\ldots,v_n) = \prod_{i=1}^{n} h_{V_i}(v_i) \tag{7}$$

The selection of $h_V(\ )$ has to be such that maximum information can be extracted out of the samples generated which should be ideally around the maximum likelihood of $f_X(\ )$ and lying on $G(X) = 0$. In this paper, we use h_{V_i} as a normal distribution centred at the Most Probable Point (MPP) of failure on the limit state boundary X_i^* and a coefficient of variation not less than that of the given distribution as suggested by Melchers (1989).

The optimisation problem used for finding the MPP is shown in equation (8). More information about MPP can be found in Appendix A.

$$\begin{aligned} &\min_{X_i} \ \sqrt{\sum_{i=1}^{n} U_i^2} \\ &\text{subject to } G(X_i) = 0 \end{aligned} \tag{8}$$

where, each of the n mutually independent random variables, X_i is converted into standard normal variable, U_i using Rosenblatt transformation (Rosenblatt, 1952). Equation (8) is solved here by MATLAB function *fmincon*.

In this work the main focus is on the combination effectively reducing the error in probability of failure calculation even with a simple formulation of the importance function. Any other superior way of implementing importance sampling (e.g., adaptive importance sampling strategies) could be easily incorporated in this framework.

3.1 Example

We illustrate the procedure for importance sampling with a simple example with two independent random variables with properties defined in Table 1. For this example, the capacity is 18 – X_1 and the response function is X_2. The limit state is given by equation (9), with failure corresponding to $G < 0$.

$$G(X_1, X_2) = (18 - X_1) - X_2 \tag{9}$$

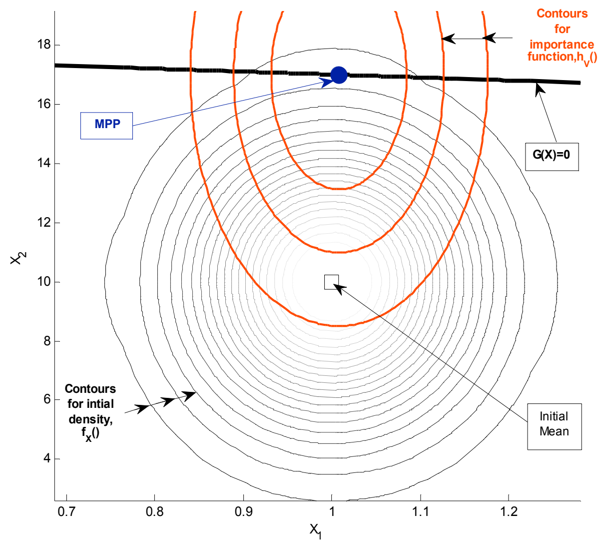
Table 1 Mean, COV and MPP for the random variables, with the limit state of equation (9)

Random variable (X_i)	*Mean*	*COV*	*Actual distribution*	*MPP, X_i^**
X_1	1	0.1	Normal	1.0078
X_2	10	0.3	Normal	16.9922

Notes: Mean and COV describe the true density function and MPP (as mean) and COV describe the importance sampling density function (always taken to be normal distribution in this work).

The MPP obtained by solving equation (8) is also given in Table 1. Figure 2 shows the plot of the original sampling density (f_X) and the importance sampling density function (with original Coefficient of Variation (COV)) from which the sampling is done (h_V) along with the MPP.

Figure 2 Two variables problem showing importance function (see online version for colours)



The Monte Carlo simulation is repeated 1000 times with 10^6 samples in each simulation to evaluate the probability of failure, the standard deviation, associated COV, using Standard MC and importance sampling. The COV for the 1000 replications of each method is a measure of the error in a single run. Importance sampling is done using the new density function with the original COVs of the random variables as well as twice and thrice the given COVs as suggested by Melchers (1989).

The results are given in Table 2. As expected, the mean of probability of failure for both Standard MC and importance sampling are the same with the accuracy measured by the coefficient of variation. It is found that for this case the original COVs give the most accurate estimation, with a reduction in error by a factor of four as compared to Standard MC.

Since G is a linear function of normal variables it is also normal and the exact value of probability of failure can also be found analytically as 9.8e-3. The COV of Standard MC can also be found analytically from equation (10) as 0.0101. This value matches the value in Table 2 found from 1000 repetitions.

$$COV(\hat{p}_{f_{Standard\,MC}}) = \sqrt{\frac{(1 - p_{f_{Standard\,MC}})}{p_{f_{Standard\,MC}} N_{Standard\,MC}}} \quad (10)$$

Table 2 Comparing the accuracy (as reflected from 1000 repetitions) of p_f using standard MC and importance sampling

		Number of samples	*Mean p_f*	*Standard deviation[a]*	*COV*
Standard MC		*1e6*	*9.8e-3*	*9.82e-5*	*1%*
Importance sampling	Using original COV	1e6	9.8e-3	2.30e-5	0.23%
	Using twice the original COV	1e6	9.8e-3	4.04e-5	0.41%
	Using thrice the original COV	1e6	9.8e-3	6.25e-5	0.64%

Notes: [a]This represents the unbiased sample standard deviation of 1000 Monte Carlo repetitions for finding p_f.

4 Combining separable MC and importance sampling

Separable MC and importance sampling operate on different principles to reduce variance. Thus it makes sense to combine them in order to obtain better accuracy.

The combination (ImpSMC) aims at utilising the advantages of both methods. The sampling for M samples of capacity and N samples of response is done using the importance sampling strategy of sampling in the regions of failure or around the limit state equal to 0. Then, as in Separable MC, all combinations of M capacity samples and N response samples are compared for evaluating failure. This creates a large sample of points using a modest number of samples for either response or capacity which are mostly around the failure region. The Monte Carlo estimation for probability of failure using ImpSMC is shown in equation (11). V_C and V_R are the samples generated for Capacity (C) and Response (R) from the importance function probability density function h_V. f_X is the actual probability density function of the random variables. All the random variables are assumed to be mutually independent and the joint probability is calculated as shown in equation (7).

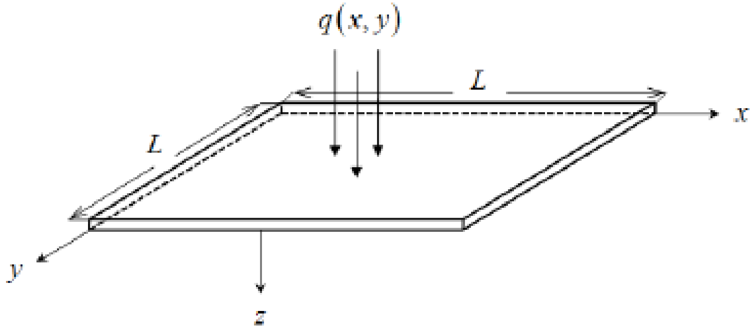
$$\hat{p}_{f_{Combined}} \approx \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} I[G(V_{C_i}, V_{R_j}) < 0] \frac{f_X(V_{C_i}, V_{R_j})}{h_V(V_{C_i}, V_{R_j})} \quad (11)$$

5 Composite plate example

A simply supported square composite plate under transverse loads, shown in Figure 3, is used to demonstrate the accuracy of Separable MC, importance sampling and ImpSMC.

The limit state for this example problem is for the maximum out-of-plane deflection w of the plate. We compare the accuracy of the estimates by repeating the simulations 1000 times.

Figure 3 Composite laminate under transverse loading



Source: Smarslok et al. (2010)

The loading conditions and the stiffness (defining response) are independent of the deflection allowable thus permitting use of Separable MC.

The limit state for this problem is shown in equation (12), where the capacity is the allowable deflection w_{all} and the response is the deflection w from loading.

$$G\left(w_{all}, w\right) = w_{all} - w \tag{12}$$

The loading condition $q(x, y)$ is sinusoidal pressure given in equation (13).

$$q(x, y) = q_0 \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right) \tag{13}$$

The maximum out-of-plane displacement is at the centre of the plate for a symmetric balanced laminate and is given as:

$$w = \frac{q_0}{D^*} \tag{14}$$

where, D^* is composed of terms of the laminate bending stiffness matrix.

$$D^* = \frac{\pi^4}{L^4}\left[D_{11} + 2(D_{12} + 2D_{66}) + D_{22}\right] \tag{15}$$

D^* is a function of plate length L, lamina thickness t, stacking sequence and lamina material properties. The calculations involved in the evaluation of the terms of the laminate bending stiffness are taken from the Haftka and Walsh paper (1992). Thus, D^* is a function of the random variables E_1, E_2, v_{12}, G_{12} and L. A $[90°, 45°, –45°]_S$ IM7/977-2 carbon fibre/epoxy symmetric laminate with ply thickness of 125 μm is used. Randomness in angle and ply thickness is not considered. The distributions of random variables and their coefficients of variation are given in Table 3.

Table 3 Mean, COV and MPP coordinate for the random variables of composite plate

Random variable (X_i)	*Mean*	*COV/range*	*Actual distribution*	*MPP, X_i^**
w_{all} (mm)	8	3%	Lognormal	7.9
q_0 (kPa)	130	15%	Normal	166.31
E_1 (GPa)	150	±9% (*Range% = 18%*)	Uniform	142.48
E_2 (Gpa)	9	5%	Normal	8.97
v_{12}	0.34	5%	Normal	0.34
G_{12} (Gpa)	4.6	5%	Normal	4.59
L (mm)	75	2%	Normal	76.9

Notes: Mean and COV describe the true density function and MPP (as mean) and COV describe the importance sampling density function (always taken to be normal distribution in this work). For uniform distribution the COV is given by *Range%/(2√3) = 5.2%.*

When either of the response or capacity is more expensive than the other, the limit state can be reformulated to reduce uncertainty in the more expensive term and thus improve the overall accuracy. For this problem Smarslok et al. (2010) has shown that it pays to reformulate the limit state as

$$G(w_{all}, q_0, D^*) = \frac{w_{all}}{q_0} - \frac{1}{D^*} \tag{16}$$

In this case w_{all}/q_0 denotes the capacity and $1/D^*$ denotes the response.

5.1 Comparing various Monte Carlo methods

The probability of failure for the composite laminate is found using the limit state function given in equation (16). The Monte Carlo methods used for estimating this probability of failure are Standard MC, Separable MC, importance sampling and ImpSMC.

In importance sampling, the random variables are sampled from a normal density function centred at the MPP. Finding the MPP required 93 function evaluations. The MPP is given in Table 3. The original coefficient of variation for the random variables is used.

First Order Reliability Method (FORM) is also used to compute the probability of failure and it is found to be 7.9e-3. As compared to the value of probability of failure found by Standard MC (6.3e-3) the estimate of p_f is around 25% off.

We first compare the accuracy of Standard MC, Separable MC, importance sampling and ImpSMC for large sample sizes of 1e6 and 1e5 in Table 4, and find that importance sampling improves the accuracy by approximately a factor of five, and Separable MC improves the accuracy by a factor of 2 and ImpSMC improves the accuracy by a factor of 10 when compared to Standard MC for the same sample size.

The comparison of all three methods and the combination is then carried out for a smaller sample size of 1000 as we often do not have the resources to have a larger sample size, and the results are shown in Table 4. For the Standard MC method, equation (10) predicts that the errors are inversely proportional to the square root of sample size. From the results in Table 4, this holds for the other methods as well.

Table 4 Comparing p_f using standard MC, importance sampling, separable MC and combination for indicated sample sizes with 1000 repetitions for composite plate (FORM estimate is 7.9e-3)

	Number of samples	*Mean* p_f	*Standard deviation*[a]	*COV*
Standard MC	N = 1e6	6.3e-3	7.7e-5	1.2%
	N = 1e5	6.3e-3	2.5e-4	4.0%
	N = 1000	6.4e-3	2.5e-3	39%
Importance sampling	N = 1e6	6.3e-3	1.5e-5	0.24%
	N = 1e5	6.3e-3	4.9e-5	0.8%
	N = 1000	6.3e-3	4.4e-4	8.0%
Separable MC	N = 1e5, M = 1e5	6.3e-3	1.2e-4	2.0%
	N = 1000, M = 1000	6.3e-3	1.2e-3	19%
Separable MC with importance sampling (ImpSMC)	N = 1e5, M = 1e5	6.3e-3	2.3e-5	0.4%
	N = 1000, M = 1000	6.3e-3	2.4e-4	3.8%
	N = 500, M = 1000	6.3e-3	2.8e-4	4.4%

Notes: [a]This represents the unbiased sample standard deviation of 1000 Monte Carlo repetitions for finding p_f.

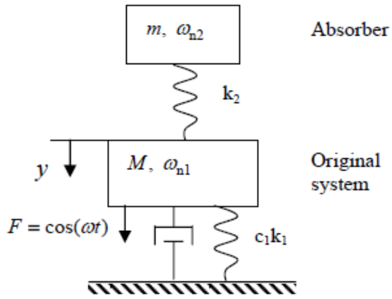
For a sample size of 1000 for both capacity and response, Separable MC at 19% COV is more accurate than Standard MC of same sample size but not as accurate as importance sampling. The accuracy of the combination is 3.8%, which is twice as accurate as importance sampling alone for the same sample size, as was the case for higher sample sizes. Even if the sample size for response is decreased (N = 500) the accuracy of ImpSMC is still really good at 4.4% coefficient of variation for the probability of failure.

6 Tuned mass-damper example

A tuned mass-damper example (Ramu et al., 2006) is presented here to show the effectiveness of the proposed method when FORM is highly inaccurate at predicting failure probability. This problem consists of a single degree of freedom system with a dynamic vibration absorber as shown in Figure 4. The original system is externally excited by a harmonic force and the absorber works to reduce the vibrations.

The amplitude of vibration depends on the mass ratio, $R = m/M$, the mass ratio of the absorber to the original system which is assumed to be 0.01 and the damping ratio of the original system, ζ which is assumed to be 0.02 in this case. It also depends on $\beta_1 = \omega_{n1}/\omega$, ratio of the natural frequency of the original system to the excitation frequency and $\beta_2 = \omega_{n2}/\omega$, ratio of the natural frequency of the absorber to the excitation frequency.

Figure 4 Tuned mass-damper



Source: Ramu et al. (2006)

The amplitude of the original system, normalised by the quasi static response, is denoted as y and is given by equation (17).

$$y=\frac{\left|1-\left(\frac{1}{\beta_2}\right)^2\right|}{\sqrt{\left[1-R\left(\frac{1}{\beta_1}\right)^2-\left(\frac{1}{\beta_1}\right)^2-\left(\frac{1}{\beta_2}\right)^2+\frac{1}{\beta_1^2\beta_2^2}\right]^2+4\zeta^2\left[\left(\frac{1}{\beta_1}\right)+\frac{1}{\beta_1\beta_2^2}\right]^2}} \tag{17}$$

This example considers β_1, β_2 and the allowable amplitude of vibration, y_{all} as random variables with distributions described in Table 5. The normalised amplitude of vibration is plotted in Figure 5. Failure in this case is considered to be when the normalised amplitude exceeds the allowable amplitude of vibration, y_{all}. The failure region can be seen in Figure 5 as the light grey-white part. This failure region is bounded by safe regions on either side making it difficult for approaches like FORM to predict the probability of failure correctly.

In the commonly used tuned mass-damper problem there are two disjoint failure regions but in this paper the example has been modified by moving the mean and changing the distributions of the random variables β_1 and β_2. This makes the occurrence of the second failure region have negligible probability and thus it doesn't contribute to the probability of failure calculations in this case.

Figure 5 Normalised vibration amplitude with respect to β_1 and β_2

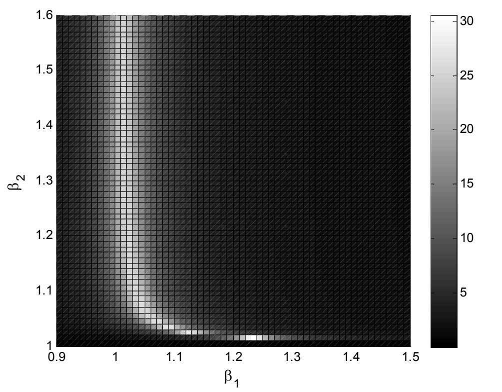


Table 5 Mean, COV and MPP coordinate for the random variables of tuned mass-damper

Random variable (X_i)	*Mean*	*COV/range*	*Actual distribution*	*MPP, X_i^**
y_{all}	30	15%	Normal	25.31
β_1	1.2	±25% (*Range% = 50%*)	Uniform	1.013
β_2	1.3	5%	Normal	1.299

Notes: Mean and COV describe the true density function and MPP (as mean) and COV describe the importance sampling density function (always taken to be normal distribution in this work). For uniform distribution the COV is given by *Range%/(2√3) = 14.4%.*

The limit state for this example is shown in equation (18), where the capacity is the allowable vibration amplitude y_{all} and the response is the normalised vibration amplitude y.

$$G(y_{all}, \beta_1, \beta_2) = y_{all} - y(\beta_1, \beta_2) \tag{18}$$

where, $y(\beta_1, \beta_2)$ is given by equation (17). Failure occurs when the limit state is less than 0.

6.1 Comparing various Monte Carlo methods

The MPP for this case is given in Table 5, and finding it required 266 function evaluations. The simulations are repeated 1000 times to find the accuracy of each method.

FORM is not accurate for this example because of the failure region being bounded by safe regions on either side. It gives a probability of failure of 8.6e-2 compared to the 4.3e-3 found by Standard MC.

Table 6 compares Standard MC and importance sampling for a large sample size of 1e6 where it is seen that importance sampling improves the accuracy by almost two times.

Table 6 Comparing p_f using standard MC, importance sampling, separable MC and combination for indicated sample sizes with 1000 repetitions for tuned mass-damper. FORM's estimate is 8.6e-2

	Number of samples	*Mean p_f*	*Standard deviation*[a]	*COV*
Standard MC	N = 1e6	4.3e-3	6.5e-5	1.5%
	N = 1000	4.4e-3	2.2e-3	49%
Importance sampling	N = 1e6	4.3e-3	2.7e-5	0.6%
	N = 1000	4.4e-3	8.7e-4	20%
Separable MC	N = 1000, M = 1000	4.3e-3	7.5e-4	17%
Separable MC with importance sampling (ImpSMC)	N = 1000, M = 1000	4.3e-3	5.3e-4	12%
	N = 500, M = 1000	4.3e-3	7.5e-4	17%

Notes: [a]This represents the unbiased sample standard deviation of 1000 Monte Carlo repetitions for finding p_f.

Table 6 also shows that for smaller sample size of 1000 the error of Standard MC and importance sampling increase as expected by the square root of the ratio of sample sizes (31.6). Importance sampling at 20% COV and Separable MC at 17% COV are almost 2.5–3 times more accurate than Standard MC. The accuracy of ImpSMC is 12%, which is much better compared to the other methods used alone. Even if the sample size for response is halved (N = 500) the accuracy of ImpSMC at 17% COV for the probability of failure is still as good as Separable MC with double the sample size.

7 Concluding remarks

This paper explored the utility of combining two variance reduction methods, Separable MC and importance sampling. For the examples tested, the combination (ImpSMC) reduced the variance more than each method individually. It also allows reduction of error and sample size simultaneously. For each Monte Carlo method and ImpSMC it was also observed that as the sample size was decreased the error increased by the square root of the ratio of the sample sizes.

Using a composite plate example, we found that Separable MC reduced the error in Standard MC by a factor of two. ImpSMC reduced the error in the probability of failure in Standard MC by a factor of 10 and in importance sampling by a factor of two. A tuned mass-damper example, for which FORM is grossly inaccurate, also shows that ImpSMC reduced the error in the probability of failure in Standard MC by a factor of 4 and is much better than importance sampling and Separable MC individually. ImpSMC also allowed simultaneous reduction of error and sample size in both examples, as can be seen by reducing the number of samples for the expensive response samples and still incurring less error than using the MC methods individually.

Acknowledgements

This work was supported by US Air Force Office of Scientific Research grant FA9550-09-1-0153.

References

Au, S.K. and Beck, J.L. (1999) 'A new adaptive importance sampling scheme for reliability calculations', *Structural Safety*, Vol. 21, No. 2, pp.135–158.

Dawson, R.J. and Hall, J.W. (2006) 'Adaptive importance sampling for risk analysis of complex infrastructure systems', *Proceedings of the Royal Society of London: Mathematical, Physical and Engineering Sciences*, Vol. 462, No. 2075, pp.3343–3362.

Du, X. and Chen, W. (2001) 'A most probable point based method for uncertainty analysis', *Journal of Design and Manufacturing Automation*, Vol. 1, Nos. 1/2, pp.47–66.

Geyer, C.J. (1992) 'Practical Markov chain Monte Carlo', *Statistical Science Journal*, Vol. 7, No. 4, pp.473–483.

Gilks, W.R. (2005) 'Markov chain Monte Carlo', *Encyclopedia of Biostatistics*.

Glynn, P.W. and Iglehart, D.L. (1989) 'Importance sampling for stochastic simulations', *Management Science Journal*, Vol. 35, No. 11, pp.1367–1392.

Haftka, R.T. and Walsh, J.L. (1992) 'Stacking-sequence optimization for buckling of laminated plates by integer programming', *AIAA Journal*, Vol. 30, No. 3, pp.814–819.

Hurtado, J.E. and Barbat, A.H. (1998) 'Monte Carlo techniques in computational stochastic mechanics', *Archives of Computational Methods in Engineering*, Vol. 5, No. 1, pp.3–29.

Kalos, M.H. and Whitlock, P.A. (2008) 'Monte Carlo evaluation of finite-dimensional integrals: importance sampling', *Monte Carlo Methods*, Wiley-Blackwell, Weinheim, pp.79–88.

Kuczera, R.C., Mourelatos, Z.P. and Nikolaidis, E. (2010) 'A re-analysis methodology for system RBDO using a trust region approach with local metamodels', *SAE International Journal of Materials and Manufacturing*, Vol. 3, No. 1, pp.356–371.

Melchers, R.E. (1989) 'Importance sampling for structural systems', *Structural Safety Journal*, Vol. 6, No. 1, pp.3–10.

Melchers, R.E. (1999) *Structural Reliability Analysis and Prediction*, 2nd ed., Wiley, New York, USA, pp.73–83.

Nikolaidis, E. and Mourelatos, Z.P. (2011) 'Imprecise reliability assessment when the type of the probability distribution of the random variables is unknown', *International Journal of Reliability and Safety*, Vol. 5, No. 2, pp.140–157.

Oh, M.S. and Berger, J.O. (1992) 'Adaptive importance sampling in Monte Carlo integration', *Journal of Statistical Computation and Simulation*, Vol. 41, Nos. 3/4, pp.143–168.

Ramu, P., Kim, N.H., Haftka, R.T. and Queipo, N.V. (2006) 'System reliability analysis using tail modeling', *11th AIAA/ISMO Multidisciplinary Analysis and Optimization Conference*, 6–8 September, Portsmouth, Virginia, USA.

Ravishankar, B., Smarslok, B.P., Haftka, R.T. and Shankar, B.V. (2010) 'Error estimation and error reduction in separable Monte-Carlo method', *AIAA Journal*, Vol. 48, No. 11, pp.2624–2630.

Rosenblatt, M. (1952) 'Remarks on a multivariate transformation', *The Annals of Mathematical Statistics*, Vol. 23, No. 3, pp.470–472.

Schueller, G.I. and Stix, R. (1987) 'A critical appraisal of methods to determine failure probabilities', *Structural Safety*, Vol. 4, No. 4, pp.293–309.

Smarslok, B.P., Haftka, R.T., Carraro, L. and Ginsbourger, D. (2010) 'Improving accuracy of failure probability estimates with separable Monte Carlo', *International Journal of Reliability and Safety*, Vol. 4, No. 4, pp.393–414.

Smarslok, B.P., Haftka, R.T. and Kim, N.H. (2006) 'Taking advantage of separable limit states in sampling procedures', *47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics & Materials Conference*, 1–4 May, Newport, Rhode Island.

Smith, A.F.M. and Roberts, G.O. (1993) 'Bayesian computation via the Gibbs sampler and related Markov chain Monte Carlo methods', *Journal of the Royal Statistical Society*, Vol. 55, No. 1, pp.3–23.

Swiler, L.P. and West, N.J. (2010) 'Importance sampling: promises and limitations', *51st AIAA Structures, Structural Dynamics & Materials Conference*, 12–15 April, Orlando, Florida, USA.

Villanueva, D., Haftka, R.T. and Sankar, B.V. (2011) 'Including the effect of a future test and redesign in reliability calculations, *AIAA Journal*, Vol. 49, No. 12, pp 2760–2769.

Zhang, G., Nikolaidis, E. and Mourelatos, Z.P. (2009) 'An efficient re-analysis methodology for probabilistic vibration of large-scale structures', *Journal of Mechanical Design*, Vol. 131, No. 5, p.051007

Zheng, Y. and Das, P.K. (2000) 'Improved response surface method and its application to stiffened plate reliability analysis', *Engineering Structures*, Vol. 22, No. 5, pp.544–551.

Appendix A

Most probable point

Most Probable Point (MPP) is the point at which the actual probability density function, $f_X(\ )$ is the maximum and it lies on the limit state equal to 0. This is the region of maximum interest as there are more chances of failure in this region. The optimisation problem used for finding the MPP is shown in equation (A1).

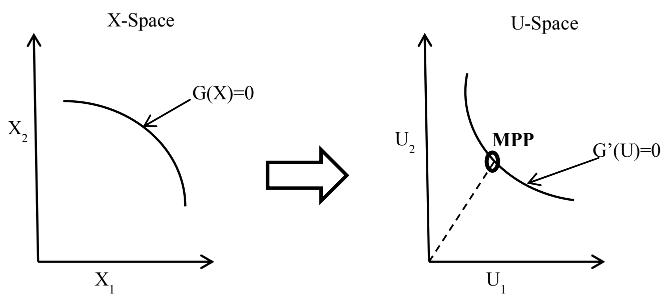
$$\beta = \min_{X_i} \sqrt{\sum_{i=1}^{n} U_i^2} \quad \text{subject to } G(X_i) = 0 \tag{A1}$$

where, each of the n mutually independent random variables, X_i is converted into standard normal variable, U_i using Rosenblatt transformation as shown in equation (A2) (Rosenblatt, 1952). β is the reliability index which is equal to the shortest distance from the origin for standard normal variables (U-space).

$$U_i = \Phi^{-1}[f_i(x_i)] \quad for\, i = 1, \ldots, n \tag{A2}$$

where, Φ^{-1} is the inverse of a normal distribution function. The transformation maintains the CDFs to be identical in both X-space and U-space. Figure A1 shows an illustration for explaining MPP.

Figure A1 Illustration showing MPP with limit state of $G(X) \leq 0$



Source: Du and Chen (2001)

The MPP in the standard normal space has the highest probability of producing the value of the limit state function equal to 0. Since, β is also referred to as safety index or reliability index in reliability analysis, the MPP is also referred to as the critical design point.

Appendix B

Nomenclature

G: Limit state.

C: Capacity.

R: Response.

M, N: Number of capacity and response samples.

p_f: Probability of failure.

I: Indicator function.

f_X: Actual probability density function.

h_V: Probability density function for importance function.

X: Set of Random design variables with probability density function $f_X(\,)$.

V: Set of Random design variables with probability density function $h_V(\,)$.

U: Standard normal variables.