Portfolio Optimization in a Markov Modulated Market *

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Abstract

We address a portfolio optimization problem in a Markov modulated market. In this paper both of the terminal expected utility optimization on finite time horizon and risk-sensitive portfolio optimization on finite and infinite time horizon are considered. A numerical procedure is also developed to compute the optimal expected terminal utility for finite horizon problem.

Key words. Risk-sensitive control, Markov process, fixed income securities, nonnegative factors.

AMS subject classification. 91B28, 93E20, 49L20, 35K55, 60H30.

1 Introduction

We study a portfolio optimization problem in continuous time for a portfolio consisting of $n + 1$ securities. One of them is a locally risk free money market account with a floating interest rate which is governed by a Markov process and the other $n (\geq 1)$ are risky assets, assumed to follow Markov modulated geometric Brownian motions. The risky assets represent stocks or some security derivatives. There is considerable literature on portfolio optimization problem beginning with the pioneering work by Merton [25], [26]. In the recent literature on portfolio optimization various objectives are treated based on the applicability in cases considered. The levels of difficulty vary from case to case. There are two popular approaches to portfolio optimization. One is to select a planning horizon and a utility function and then maximize the expected utility of the terminal wealth. Another is the classical Markowitz mean variance approach [24] which has been extended to the dynamic situation; see [7] and the references therein. The latter enables an investor to maximize his expected return after specifying his acceptable risk level that is measured by the variance of the return. There are many generalizations and improvement of this concept [9], [19], [28], [30], [31]. In [22] a similar problem is studied with asymptotic criterion. In this approach an investor faces a tradeoff between his portfolio’s growth rate and its asymptotic variance. Similar tradeoff between long run expected growth rate and asymptotic variance is also captured in an implicit way in the risk sensitive optimality criterion. Risk sensitive criterion was studied in [4], [5], [10], [11], [12], [13], [14], [16], [23]. We follow this approach for optimal portfolio selection problem in a Markov modulated market assumption.

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Though option pricing in a regime switching market has been studied extensively (see [6] and the references therein), the corresponding results on portfolio optimization seem to be rather sparse. The portfolio optimization in a Markovian regime switching market along the lines of Merton [25], [26] is studied in [3]. In [33], [34] Yin and Zhou have studied the portfolio optimization problem in Markovian regime switching market using mean-variance approach. In [33] the discrete-time case and its limit to the continuous-time model are studied. The continuous-time case is studied in details in [34]. In this work it is assumed that the state of the market \{X_t\} is a continuous time Markov chain. Using techniques of stochastic linear-quadratic control, mean-variance efficient portfolios and efficient frontiers are derived explicitly.

In this paper we assume that the state process \{X_t\} is a finite state Markov process. We take the floating interest rate as the basis of risk free asset model. We assume the interest rate of the bank account, the volatility matrix and the drift coefficients of risky assets are driven by the process \{X_t\}.

We primarily consider the portfolio optimization problem with risk sensitive criterion for both finite and infinite time horizon. The risk sensitive criterion takes care of the investor’s interest of maximizing the expected growth rate of wealth against his aversion of risk due to deviations of the actually realized rate from the expectation (see (1.1) in [4]). The subjective notion of investor’s risk aversion is parameterized by a single variable $\theta$. Hence the optimal expected utility function depends on $\theta$. We address the optimization problems by studying the corresponding Hamilton-Jacobi-Bellman (HJB) equation.

The analysis of the finite horizon logarithmic and power utility cases are analogous. This, however, is not the case for infinite horizon risk sensitive criterion. In this case we have first shown that the optimal portfolio selection is identical to that of the finite horizon counterpart. We have obtained the optimal growth rate in terms of a maximal eigenvalue of an appropriate matrix using Perron-Frobenius theorem. We also study the asymptotic behaviour of the growth rate with increasing risk aversion. We illustrate the behaviour by considering a numerical example. We use the well known power method to compute the risk sensitive growth rate for Markov modulated market model. The optimal risk sensitive criterion denoted by $\varphi_\theta$ is computed numerically for various values of $\theta$. The dependence of $\varphi_\theta$ on $\theta$ is illustrated graphically.

Our paper is structured as follows. The Markov model description is presented in Section 2. Section 2 also contains the basic definitions, notations and a few fairly standard assumptions on the model. Section 3 contains the portfolio optimization problem on finite time horizon. We obtain explicit or tractable solutions and the corresponding optimal portfolio selections for a risk sensitive criterion. The risk sensitive portfolio optimization problem for infinite horizon case is studied in Section 4. In Section 5 we present a numerical results. We conclude our paper in Section 6 with a few remarks.

## 2 Model Description

Let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space which is assumed to be complete. Let $W = \{W_t\} = [W^1_t, \ldots, W^n_t]'$ (for any vector or matrix $C$, $C'$ denote its transpose) be a standard $n$-dimensional Wiener process. Let $X = \{X_t\}_{t \geq 0}$ be an irreducible Markov process taking values in $\mathcal{X} = \{1, 2, \ldots, k\}$ which denotes the state of the market based on economic indicators. Let matrix $\Lambda := (\lambda_{ij})_{k \times k}$ be the rate matrix of the Markov chain $\{X_t\}$. We assume that $W$ and $X$ are independent. Let $\mathcal{F}_t = \sigma(W_s, X_s, 0 \leq s \leq t)$, assumed to be right continuous and complete without any loss of generality.
We consider a frictionless market consisting of \( n + 1 \) assets \( \{S^0_t, S^1_t, \ldots, S^n_t\} \) which are traded continuously. The first asset is (locally) risk free to be referred as the money market account. Its value at time \( t \) is given by

\[
    dS^0_t = r(t, X_t) S^0_t \, dt, \quad S^0_0 = s_0,
\]

where \( r(t, i) \geq 0, i = 1, 2, \ldots, k; r(t, i) \) is the floating interest rate at time \( t \) corresponding to the market mode \( i \). The other \( n \) assets are risky (may be referred to as stocks). The prices of the \( l \)-th stock is given by the following stochastic differential equation

\[
    dS^l_t = S^l_t [\mu^l(t, X_t) \, dt + \sum_{j=1}^{n} \sigma_{lj}(t, X_t) \, dW^j_t], \quad S^l_0 = s_l,
\]

where for each \( l = 1, 2, \ldots, n; \mu^l(t, i) \) is the growth rate of the \( l \)-th stock in regime (or mode) \( i; \sigma_l(t, i) := [\sigma_{l1}(t, i), \sigma_{l2}(t, i), \ldots, \sigma_{ln}(t, i)] \) is the volatility of the \( l \)-th stock in regime \( i \).

Let

\[
    \sigma(t, i) = [\sigma_{lj}(t, i)]_{n \times n}, \quad i = 1, \ldots, k
\]

denote the volatility matrix in regime \( i \).

Consider a trader (or agent) who employs a self-financing portfolio of the above \( (n + 1) \) assets so as to maximize his terminal expected utility. Let \( u^l(t) \) be the fraction invested in the \( l \)-th asset. Then

\[
    \sum_{l=0}^{n} u^l(t) = 1.
\]

Hence

\[
    u^0(t) = 1 - \sum_{l=1}^{n} u^l(t).
\]

Let \( u(t) = [u^1(t), \ldots, u^n(t)]' \) be the portfolio at time \( t \). Then the value of the portfolio or the wealth process denoted by \( V^u_t \) takes the form

\[
    \frac{dV^u_t}{V^u_t} = \sum_{l=0}^{n} u^l(t) \frac{dS^l_t}{S^l_t}.
\]

i.e.,

\[
    dV^u_t = V^u_t \left[ r(t, X_t) + \sum_{l=1}^{n} [\mu^l(t, X_t) - r(t, X(t))] u^l(t) \right] \, dt
    + V^u_t \sum_{l=1}^{n} \sum_{j=1}^{n} \sigma_{lj}(t, X_t) u^l(t) \, dW^j_t.
\]

Let

\[
    b(t, i) = [\mu^1(t, i) - r(t, i), \ldots, \mu^n(t, i) - r(t, i)].
\]

Then (2.3) takes the form

\[
    dV^u_t = V^u_t \left[ (r(t, X_t) + b(t, X_t) u(t)) \, dt + u(t)' \sigma(t, X_t) \, dW_t \right].
\]

Let \( A \subseteq \mathbb{R}^n \) be the set of all possible portfolio selections. In the case of unrestricted short selling \( A = \mathbb{R}^n \). The restrictions on short selling makes \( A = \prod_{i=1}^{n} [c_i, d_i]; \ c_i, d_i \in \mathbb{R} \). In case of no short selling and no bank loan \( A = [0, 1]^n \).
Definition 2.1 A portfolio strategy \(u(\cdot) = [u^1(\cdot), \ldots, u^n(\cdot)]'\) is said to be admissible if

(i) The process \(u(\cdot)\) is \(A\)-valued and is predictable with respect to \(\{F_t\}\).

(ii) \(E \int_0^T \|u(t)\|^2 \, dt < \infty\), for all \(T > 0\).

We first study the portfolio optimization on a finite planning horizon \(T > 0\). We carry out our study under the following assumptions.

(A1)

(i) For each \(i, l, j\) the functions \(r(t, i), \mu^l(t, i), \sigma^l_j(t, i)\) are continuous on \([0, T]\).

(ii) Let \(a(t, i) := \sigma^l(t, i) \sigma^l(t, i)'\) denote the diffusion matrix. Assume that \(a(t, i)\) is continuous on \([0, T]\) and there exists a \(\delta > 0\) such that for each \(i\)

\[a(t, i) \geq \delta I\] \hspace{1cm} (2.5)

where \(I\) is the \(n \times n\) identity matrix.

Remark 2.1 The non-degeneracy assumption (2.5) ensures that the filtration \(F_t = \sigma(W_s, X_s, 0 \leq s \leq t)\) is as same as \(\sigma(S^1_s, \ldots, S^n_s, X_s, 0 \leq s \leq t)\).

Under assumptions (A1), the equation (2.4) governing the wealth process has a unique strong solution \(\{V^u_t, 0 \leq t \leq T\}\) under a given admissible (portfolio) strategy \(u(\cdot)\). It is also easy to see that \(V^u_T > 0\) a.s. for all \(t\).

3 Finite Horizon Problem

Let \(U\) be a utility function satisfying the usual condition, i.e., \(U\) is nonnegative, strictly increasing and strictly concave. The objective of the trader is to maximize his terminal expected utility

\[J^u(x, i) = E[U(V^u_T) \mid V_0 = x, X_0 = i].\] \hspace{1cm} (3.6)

A portfolio strategy \(u(\cdot)\) is said to be Markov if \(u(t)\) is of the form \(u(t) = \bar{u}(t, V_t, X_t, Y_t)\), where \(\bar{u} : [0, T] \times \mathbb{R} \times X \times [0, T] \to A\) is a measurable function. If \(\bar{u}\) does not have explicit \(t\) dependence, the strategy \(u(t)\) is called stationary Markov. It is clear from (2.4) that under a Markov strategy \(u, (V^u_t, X_t, Y_t)\) is jointly Markov.

For a fixed \(u \in A\) and \(t \in [0, T]\), let the operator \(A^u_t : C^{2,1}(\mathbb{R} \times X) \to C(\mathbb{R} \times X)\) be defined by

\[A^u_t \varphi(x, i) := (r(t, i) + b(t, i)u)x \frac{\partial}{\partial x} \varphi(x, i) + \frac{1}{2}(u'a(t, i)u)x^2 \frac{\partial^2}{\partial x^2} \varphi(x, i) + \sum_j \lambda_{ij} \varphi(x, j).\] \hspace{1cm} (3.7)

3.1 Hamilton-Jacobi-Bellman Equation

Let

\[J^u(t, x, i) = E^u[U(V^u_T) \mid V_t = x, X_t = i]\] \hspace{1cm} (3.8)
be the expected terminal utility and

$$\varphi(t, x, i) = \sup_u J^u(t, x, i)$$  \hspace{1cm} (3.9)$$

where supremum is taken over all admissible strategies. That is \(\varphi(t, x, i)\) is the optimal expected utility. Since \((S^0_t, S^1_t, \ldots, S^n_t, X_t)\) is a Markov process, it follows that for each \(u \in A\), \((V^u_t, X_t)\) is a Markov process with differential generator given by (3.7). Following the general treatment on controlled Markov processes (Theorem 8.1, pp. 141-142, [15]) the HJB equation for \(\varphi\) is given by

$$\frac{\partial}{\partial t} \varphi(t, x, i) + \sup_{u \in A} A^u_t \varphi(t, x, i) = 0$$  \hspace{1cm} (3.10)$$

with the terminal condition

$$\varphi(T, x, i) = U(x) \text{ for all } i = 1, \ldots, m.$$  \hspace{1cm} (3.11)$$

It also follows from Theorem 8.1, pp. 141-142, [15], that if (3.10) has a classical solution and the supremum in (3.10) is attained at \(u(t, x, i)\), then it is an optimal control. Note that (3.10) is a system of equations with weak coupling. The coupling is weak since it occurs through the zeroth order terms.

The equation (3.10) can be rewritten as

$$\frac{\partial}{\partial t} \varphi(t, x, i) + \sup_{u \in A} \left[ (r(t, i) + b(t, i)u)x \frac{\partial}{\partial x} \varphi(t, x, i) + \frac{1}{2}(u'a(t, i)u)x^2 \times \right.$$  

$$\left. \frac{\partial^2}{\partial x^2} \varphi(t, x, i) + \sum_{j=1}^{k} \lambda_{ij} \varphi(t, x, j) \right] = 0.$$  \hspace{1cm} (3.12)$$

### 3.2 Risk Sensitive Criterion

In this subsection we study risk sensitive portfolio optimization with constant risk aversion parameter \(\theta > 0\). We consider finite horizon payoff criteria given below. For an admissible strategy \(u(\cdot)\), finite horizon risk-sensitive criterion is given by

$$J^u_{\theta,T}(x, i) = -\frac{2}{\theta} \log E \left[ e^{-\frac{\theta}{2} \log V^u_T} \mid V^u_0 = x, X_0 = i \right], T > 0.$$  \hspace{1cm} (3.13)$$

For this it suffices to consider the following payoff criterion given by

$$E \left[ (V^u_T)^{-\frac{\theta}{2}} \mid V^u_0 = x, X_0 = i \right]$$  \hspace{1cm} (3.14)$$

since the maximizer of \(J^u_{\theta,T}(x, i)\) is the minimizer of above criterion and vice-versa. Let

$$\tilde{J}^u_{\theta,T}(t, x, i) := E^u \left[ (V^u_T)^{-\frac{\theta}{2}} \mid V^u_t = x, X_t = i \right]$$

and

$$\tilde{\varphi}_\theta(t, x, i) = \inf_u \tilde{J}^u_{\theta,T}(t, x, i).$$
The HJB equation corresponding to the function $\tilde{\varphi}_\theta$ is given by
\[
\frac{\partial}{\partial t} \tilde{\varphi}_\theta(t, x, i) + \inf_{u \in A} \left[ (r(t, i) + b(t, i)u) x \frac{\partial}{\partial x} \tilde{\varphi}_\theta(t, x, i) + \frac{1}{2} (u' a(t, i) u)x^2 \times \right.
\left. \frac{\partial^2}{\partial x^2} \tilde{\varphi}_\theta(t, x, i) \right] + \sum_{j=1}^{k} \lambda_{ij} \tilde{\varphi}_\theta(t, x, j) = 0.
\] (3.15)
with terminal condition
\[
\tilde{\varphi}_\theta(T, x, i) = x^{-\frac{\theta}{2}}.
\] (3.16)

We look for a solution for (3.15)-(3.16) of the form
\[
\tilde{\varphi}_\theta(t, x, i) = x^{-\frac{\theta}{2}} \psi_\theta(t, i).
\] (3.17)

Then substituting (3.17) in (3.15), we obtain
\[
\frac{d}{dt} \psi_\theta(t, i) + \frac{\theta}{2} \inf_{u \in A} \left[ - (r(t, i) + b(t, i) u) + \frac{1}{2} \left( \frac{\theta}{2} + 1 \right) (u' a(t, i) u) \right] \psi_\theta(t, i)
\right.
\left. + \sum_{j=1}^{k} \lambda_{ij} \psi_\theta(t, j) = 0
\] (3.18)
i = 1, 2, \ldots, k, with terminal condition
\[
\psi_\theta(T, i) = 1.
\] (3.19)

It follows that the above Cauchy problem has a classical solution given by
\[
\psi_\theta(t, i) = E[e^{\int_{t}^{T} h_\theta(s, X_s) ds} | X_t = i]
\] (3.20)
where
\[
h_\theta(t, i) = \frac{\theta}{2} \inf_{u \in A} \left[ - r(t, i) - b(t, i) u + \frac{1}{2} (\frac{\theta}{2} + 1) (u' a(t, i) u) \right].
\] (3.21)

Therefore
\[
\tilde{\varphi}_\theta(t, x, i) = x^{-\frac{\theta}{2}} E[e^{\int_{t}^{T} h_\theta(s, X_s) ds} | X_t = i].
\] (3.22)

Thus the risk sensitive optimal expected utility is given by
\[
\varphi_\theta(t, x, i) := \sup_{u} J^T_\theta u(t, x, i)
\]
\[
= - \frac{2}{\theta} \log(\tilde{\varphi}_\theta(t, x, i))
\]
\[
= \log x - \frac{2}{\theta} \log(\psi_\theta(t, i))
\] (3.23)

where $\psi_\theta(t, i)$ is as in (3.20). We now find the optimal portfolio strategy for various cases.

**Case 1:** $A = \mathbb{R}^n$, the minimizing $u^*_\theta(t, i)$ in (3.18) is given by
\[
u^*_\theta(t, i) = \frac{1}{1 + \frac{\theta}{2} a(t, i)^{-1} b(t, i)^T}
\] (3.24)
which is the optimal portfolio strategy for the risk sensitive criterion on the finite horizon.
Case 2: For $A$ an $n$-dimensional rectangle, i.e., $A = \prod_{l \leq n} [c_l, d_l]$, then the $l$ th component of $u_\theta^*(t, i)$ is given by

$$u_\theta^l(t, i) = \begin{cases} 
\frac{1}{1 + \frac{\theta}{2}} (a(t, i)^{-1} b(t, i')) t_l & \text{if } \frac{1}{1 + \frac{\theta}{2}} (a(t, i)^{-1} b(t, i')) t_l < c_l \\
c_l & \text{if } \frac{1}{1 + \frac{\theta}{2}} (a(t, i)^{-1} b(t, i')) t_l = c_l \\
d_l & \text{if } \frac{1}{1 + \frac{\theta}{2}} (a(t, i)^{-1} b(t, i')) t_l > d_l
\end{cases}. \quad (3.25)$$

We now describe the limiting case of risk sensitive criterion as $\theta \downarrow 0$. The optimal portfolio selection for risk-null criterion is obtained by making $\theta = 0$ in (3.24) and (3.25). To obtain the optimal expected utility for risk-null criterion we do the following calculations.

From (3.26), (3.21), (3.20) and (3.23) we have

$$\varphi_\theta(t, x, i) = \log x - \frac{2}{\theta} \log \left( \mathbb{E} \left[ e^{\int_t^T \theta(s, X(s)) ds} \bigg| X_T = i \right] \right)$$

$$= \log x - \frac{2}{\theta} \log \left( \mathbb{E} \left[ 1 + \frac{\theta}{2} \left( \int_t^T \theta(s, X(s)) ds \right)^2 + \cdots \bigg| X_T = i \right] \right)$$

$$= \log x - \frac{2}{\theta} \log \left( 1 + \frac{\theta}{2} \left( \int_t^T \theta(s, X(s)) ds \right)^2 + \cdots \bigg| X_T = i \right)$$

$$= \log x - \frac{2}{\theta} \log \left( 1 + \frac{\theta}{2} \mathbb{E} \left[ \int_t^T \frac{\theta}{2} \theta(s, X(s)) ds \bigg| X_T = i \right] \right)$$

$$= \log x - \frac{2}{\theta} \log \left( 1 + \frac{\theta}{2} \mathbb{E} \left[ \int_t^T \frac{\theta}{2} \theta(s, X(s)) ds \bigg| X_T = i \right] \right)$$

for $\theta \downarrow 0$ and where $h(t, i)$ is given by

$$h(t, i) = \sup_{u \in A} \left[ r(t, i) + b(t, i) u - \frac{1}{2} [u' a(t, i) u] \right]. \quad (3.26)$$

Thus

$$\varphi_0(t, x, i) = \lim_{\theta \to 0} \varphi_\theta(t, x, i) = \log x + \mathbb{E} \left[ \int_t^T h(s, X(s)) ds \bigg| X_T = i \right]. \quad (3.27)$$

Remarks 3.1 (i) The logarithmic utility is given by

$$U(x) = \log x, \quad x > 0. \quad (3.28)$$

The HJB equation (3.10) with terminal condition (3.11), (3.28) has the unique classical solution given by

$$\varphi(t, x, i) = \log x + \mathbb{E} \left[ \int_t^T h(s, X(s)) ds \bigg| X_T = i \right]$$

which is the optimal expected terminal logarithmic utility. The right side of (3.27) shows that the limiting case of risk sensitive criterion as $\theta \downarrow 0$ the optimal expected utility coincides with the
optimal expected logarithmic utility.

(ii) It is interesting to note that all the results obtained in this section holds for a negative value of \( \theta \) as well. A risk taker investor may have a negative risk aversion parameter.

(iii) Set \( \gamma := -\frac{\theta}{2} > 0 \). Thus the criterion in (3.14) is analogous to the terminal power utility, given by

\[
U(x) = \frac{1}{\gamma} x^\gamma, \quad 0 < \gamma < 1.
\]

Therefore the results obtained in this section includes the case of optimizing expected terminal power utility. Moreover the equation (3.27) shows (by comparing with (3.29)) the assymptotic relation between power utility and logarithmic utility.

4 Infinite Horizon Problem

We first note from (3.24) and (3.25) that the optimal portfolio choice in the risk-sensitive criterion with finite horizon is independent of the length of the horizon. We study the infinite horizon problem via the limiting case of the finite horizon problem as the length of the horizon \( T \to \infty \).

The infinite horizon risk sensitive payoff criterion is given by

\[
J^\theta(x, i) := \liminf_{T \to \infty} \frac{1}{T} J^u_{\theta,T}(x, i).
\]

Note that

\[
\frac{1}{T} J^u_{\theta,T}(x, i) \geq \frac{1}{T} J^\theta_{\theta,T}(x, i)
\]

for any admissible strategy \( u \), where \( u^*_\theta \) is as in (3.24) or (3.25). Hence

\[
\liminf_{T \to \infty} \frac{1}{T} J^u_{\theta,T}(x, i) \geq \liminf_{T \to \infty} \frac{1}{T} J^\theta_{\theta,T}(x, i)
\]

for all admissible strategy \( u \). Thus \( u^*_\theta \) given by (3.24) or (3.25) is optimal for the respective action spaces for the infinite horizon case as well.

Some comments are in order.

Remarks 4.1 (i) Note that, the optimal strategy given by (3.24) or (3.25) is nonstationary. If the coefficients in (2.3) are independent of \( t \), then the corresponding optimal strategy would be stationary.

(ii) For the finite horizon case the optimal expected utility for risk-sensitive criterion is given by (3.23). For the infinite horizon case the optimal expected growth rate of the portfolio would be given by

\[
\lim_{T \to \infty} \frac{1}{T} J^u_{\theta,T}(x, i)
\]

provided the above limit exists. We have obtained the optimal strategy for infinite horizon risk sensitive criterion without the existence of the limit in (4.32). Thus the result is obtained without any ergodicity condition. This is in contrast with [4] where a certain ergodicity condition is assumed to derive the optimal portfolio selection.
For the infinite horizon case we restrict ourselves to the autonomous model, i.e., the coefficients in (2.3) are independent of $t$. From the equations (3.20) and (3.23) it is clear that if the limit in (4.32) exists, then

$$\lim_{T \to \infty} \frac{1}{T} j^T \theta^T (x, i) = -\frac{2}{\theta} \lim_{T \to \infty} \frac{1}{T} \log \left( E[e^{h^T \theta (X_s) ds} | X_0 = i] \right)$$

where

$$h_{\theta}(i) = \frac{\theta}{2} \inf_{u \in A} \left[ -r(i) - b(i) u + \frac{1}{2} (\theta + 1) \right]$$

The limit on the right side of (4.33) is the large deviation limit for Markov process. The existence of this limit is proved in [8] when $\{X_t\}$ is a compact state irreducible continuous time Markov chain.

The existence of limit (4.32) follows from the results in [8], [17]. Here we obtain a simple expression of this limit in terms of the dominant eigenvalue of certain matrix.

For autonomous model using (4.34), the equation (3.18) with the terminal condition (3.19) is reduced to the following system of ordinary differential equations

$$\frac{d}{dt} \psi_{\theta}(t, i) + h_{\theta}(i) \psi_{\theta}(t, i) + \sum_{j} \lambda_{ij} \psi_{\theta}(t, j) = 0 \tag{4.35}$$

$$\psi_{\theta}(T, i) = 1 \tag{4.36}$$

From (4.35) and (4.36) we have

$$\psi_{\theta}(t, i) = \sum_{j=1}^{k} \exp \left( \left( \Lambda + \text{diag}(h_{\theta}(\cdot)) \right)(T - t) \right)(i, j). \tag{4.37}$$

where $\text{diag}(h_{\theta}(\cdot))$ is a diagonal matrix with $h_{\theta}(i)$ at the $i$th diagonal element and $\exp \left( \left( \Lambda + \text{diag}(h_{\theta}(\cdot)) \right)(T - t) \right)(i, j)$ is the $(i, j)$th element of the matrix $\exp \left( \left( \Lambda + \text{diag}(h_{\theta}(\cdot)) \right)(T - t) \right)$. Again from (3.20) and (4.37) we have

$$E[e^{h^T \theta (X_s) ds} | X_t = i] = \sum_{j=1}^{k} \exp \left( \left( \Lambda + \text{diag}(h_{\theta}(\cdot)) \right)(T - t) \right)(i, j). \tag{4.38}$$

Thus the limit in (4.33) exists if the matrix $\Lambda + \text{diag}(h_{\theta}(\cdot))$ has a real eigenvalue which is greater than the real parts of any other eigenvalues. We prove this in the following theorem.

**Lemma 4.1** The matrix $\Lambda + \text{diag}(h_{\theta}(\cdot))$ has a real eigenvalue which is greater than the real parts of any other eigenvalues.

**Proof:** Let $c := \min_{i} (\lambda_{ii} + h_{\theta}(i))$ and

$$\Lambda^\theta := \Lambda + \text{diag}(h_{\theta}(\cdot)) - c I_{k \times k} \tag{4.39}$$

Thus $\Lambda^\theta$ is a non-negative matrix. Since $\{X_t\}$ is an irreducible Markov chain, $\Lambda^\theta$ is an irreducible matrix. By Perron-Frobenius theorem ([21], Chapter 8) the spectral radius of $\Lambda^\theta$ is an eigenvalue denoted by $\rho$. Then $\rho + c$ is the real eigenvalue of $\Lambda + \text{diag}(h_{\theta}(\cdot))$ such that the real parts of any other eigenvalues are less than $\rho + c$. \[\blacksquare\]
It follows from (4.33), (4.38) and Lemma 4.1 that

$$\lim_{T \to \infty} \frac{1}{T} J^\theta_{u^\theta} T(x, i) = -\frac{2}{\theta}(\rho + c). \quad (4.40)$$

Thus we obtain the following result.

**Theorem 4.2** For the semi-Markov modulated market the infinite horizon risk sensitive optimal portfolio $u^\theta$ is given by (3.24) or (3.25) for $t \geq 0$. If the process $\{X_t\}$ is Markov, the optimal expected growth rate of portfolio is given by (4.40) where $\rho$ is the dominant eigenvalue of the matrix $\Lambda^\theta$, defined in (4.39).

Again by Perron-Frobenius theorem

$$\rho \leq \max_i \left( \sum_j \lambda_{ij} + h^\theta(i) - c \right).$$

Hence

$$\rho + c \leq \max_i h^\theta(i) < 0.$$

Therefore the right hand side of (4.40) is positive and has a lower bound

$$\min \sup_i \left[ r(i) + b(i)u - \frac{1}{2}(\frac{\theta}{2} + 1)u'a(i)u \right]$$

$$= \min_i \left[ r(i) + \frac{1}{2 + \theta}b(i)a(i)^{-1}b(i)' \right]$$

$$\geq \min_i r(i) \quad \forall \theta. \quad (4.41)$$

Here we show that the risk sensitive optimal growth rate converges for increasing risk aversion to the lowest interest rate.

**Theorem 4.3**

$$\lim_{\theta \to \infty} \lim_{T \to \infty} \frac{1}{T} J^\theta_{u^\theta} T(x, i) = \min_i r(i) \quad (4.42)$$

**Proof:** From Lemma 4.1 we have, $\frac{2}{\theta}(\rho + c)$ is the dominant eigenvalue of

$$\frac{2}{\theta} [\Lambda + \text{diag}(h^\theta(\cdot))]$$

$$= \frac{2}{\theta} \left[ \Lambda - \frac{\theta}{2} \text{diag} \left( r(\cdot) + \frac{1}{2 + \theta} b(\cdot)a(\cdot)^{-1}b(\cdot)' \right) \right]$$

$$= -\text{diag}(r(\cdot)) + \frac{2}{\theta} \Lambda - \frac{1}{2 + \theta} \text{diag} \left( b(\cdot)a(\cdot)^{-1}b(\cdot)' \right) \quad (4.43)$$

From (4.43) it follows that

$$\lim_{\theta \to \infty} \frac{2}{\theta}(\rho + c) = \max_i (-r(i)). \quad (4.44)$$

The result follows from (4.40) and (4.44).
5 Numerical Methods

In this section we present numerical results for infinite horizon risk sensitive criterion. We investigate the sensitivity of the optimal growth rate due to different values of risk aversion parameter.

To illustrate the results we consider an example of a Markov modulated market with three regimes. The state space $\mathcal{X} = \{1, 2, 3\}$. The drift coefficient, volatility and instantaneous interest rate at each regime are chosen as follows:

\[
\left(\mu(i), \sigma(i), r(i)\right) := \begin{cases} 
(0.3, 0.2, 0.2) & \text{if } i = 1 \\
(0.6, 0.4, 0.5) & \text{if } i = 2 \\
(0.8, 0.3, 0.7) & \text{if } i = 3
\end{cases}
\]

The rate matrix $\Lambda$ is assumed to be given by

\[
\Lambda = \begin{pmatrix}
-1 & 2/3 & 1/3 \\
1 & -2 & 1 \\
1/3 & 2/3 & -1
\end{pmatrix}.
\]

In Figure 2 the infinite horizon risk sensitive optimal portfolio growth rates are plotted along vertical axis against different risk aversion parameters along horizontal axis. There are four line plots. The circled line plot is for the Markov switching case. The other three line plots are for three different fixed regimes. The optimal risk null criteria are computed by using (3.27). All the lines show the strict diminishing of growth rate corresponding to increasing risk aversion. For the fixed regime cases the growth rates are bounded below by the corresponding interest rates 0.2, 0.5 and 0.7 respectively. On the contrary, for the regime switching case the risk-null growth rate appears as an average of those of fixed regimes and it decreases to the lowest interest rate as $\theta$ increases.
6 Conclusion

In this paper the portfolio optimization problem in a Markov modulated market is studied. We find optimal portfolios by optimizing expected utility of terminal wealth with risk sensitive utility functions. We also obtain optimal portfolios for logarithmic as well as power utility function. We show that for the optimal portfolio selection remains the same for both finite and infinite horizon cases. The computation of the optimal growth rate for the infinite horizon case is more involved since this involves the large deviation principle for Markov processes. We obtain a simple expression for the optimal growth rate of the risk sensitive portfolio in terms of a maximal eigenvalue of an appropriate matrix. This enables us to compute the growth rate numerically.

References


