Upper semicontinuity of random attractors for stochastic three-component reversible Gray–Scott system

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A B S T R A C T
We consider the upper semicontinuity of the global random attractors for the stochastic three-component reversible Gray–Scott system on unbounded domains when the intensity of the noise converges to zero.

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1. Introduction

The paper is devoted to concerning with the upper-semicontinuity of the random attractors for the stochastic three-component reversible Gray–Scott system on unbounded domains with multiplicative noise. Given a small positive parameter $\sigma$, consider the following stochastically perturbed system:

\begin{align}
\frac{\partial \bar{u}}{\partial t} &= d_1 \Delta \bar{u} - (F + k) \bar{u} + \bar{u}^2 \bar{v} - \bar{G} \bar{u}^3 + N \bar{w} + f_1(x) + \sigma \bar{u} \circ \frac{d\omega}{dt}, \\
\frac{\partial \bar{v}}{\partial t} &= d_2 \Delta \bar{v} - F \bar{v} - \bar{u}^2 \bar{v} + \bar{G} \bar{u}^3 + f_2(x) + \sigma \bar{v} \circ \frac{d\omega}{dt}, \\
\frac{\partial \bar{w}}{\partial t} &= d_3 \Delta \bar{w} + k \bar{u} - (F + N) \bar{w} + f_3(x) + \sigma \bar{w} \circ \frac{d\omega}{dt},
\end{align}

(1.1)

with initial data

$$\bar{u}(0,x) = \bar{u}_0(x), \quad \bar{v}(0,x) = \bar{v}_0(x), \quad \bar{w}(0,x) = \bar{w}_0(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\bar{u} = \bar{u}(x,t), \bar{v} = \bar{v}(x,t), \bar{w} = \bar{w}(x,t)$ are real-valued functions on $\mathbb{R}^n \times [0, \infty)$; $f_i (i = 1, 2, 3)$ are nonlinear functions satisfying certain conditions; all the parameters are arbitrarily given positive constants; $\omega$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\Omega = \{ \omega \in C[\mathbb{R}, \mathbb{R}] : \omega(0) = 0 \}$, the Borel sigma-algebra $\mathcal{F}$ on $\Omega$ is generated by the compact open topology (see [1]), $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$ and $\circ$ denotes the Stratonovich sense of the stochastic term.

Historically, when $\bar{w} = 0, G = 0, f_1 = f_3 = 0, f_2 = F$ and there are not random terms ($\sigma = 0$), system (1.1) reduces to the two-component Gray–Scott system which signified one of the Brussels school led by the renowned physical chemist and...
Nobel Prize laureate (1977), Ilya Prigogine, which originated from describing an isothermal, cubic autocatalytic, continuously fed and diffusive reactions of two chemicals (see e.g. [7–10]). The three-component reversible Gray–Scott model was firstly introduced by Mahara et al., which is based on the scheme of two reversible chemical or biochemical reactions [11]. Then in [2], You took some nondimensional transformations, the three-component reversible system was reduced to the system (1.1) without random forces. In [2], You considered the existence of global attractor for the system (1.1) with Neumann boundary condition on a bounded domain of space dimension $n \leq 3$ by the method of the re-scaling and grouping estimation. In [21], the uniform attractor of a non-autonomous three-component reversible Gray–Scott system was established.

Stochastic differential equations of this type arise from many chemical or biochemical systems when random spatio-temporal forcing is taken into consideration. These random perturbations are intrinsic effects in a variety of settings and spatial scales. They may be most obviously influential at the microscopic and smaller scales but indirectly they play an important role in macroscopic phenomena (see e.g. [12–14]). In [15–17], the influence of additive noise on Gray–Scott systems was discussed. Recently, Gu [18] gave the existence of a compact random attractor for stochastic three-component reversible Gray–Scott system with multiplicative white noise in a bounded domain of $\mathbb{R}^n$ ($n \leq 3$) when $f_1 = f_2 = 0, f_3 = F$ in system (1.1). The existence of random attractor for system 1.1 and 1.2 on unbounded domains was obtained in [19].

In this paper, we will study the limiting behavior of random attractors for the stochastically perturbed reversible three-component reaction–diffusion system (1.1) and (1.2) defined on $\mathbb{R}^n$ when $\sigma \to 0$, and prove the upper semicontinuity of these perturbed random attractors. The main result reveals the robustness of the global asymptotic dynamics for such a class of perturbed stochastic reversible reaction–diffusion systems. The global dynamics considered here is novel and meaningful and it haven’t been reported yet to the best of our knowledge. It is worth mentioning that when there is not random forcing ($\sigma = 0$), the upper semicontinuity of global attractors as the inverse reaction rates $G, N$ tend to $(0, 0^+)$ for the deterministic system was considered in [2] by an approach of transformative decomposition to avoid the singularity factors. The result is different from ours. Here, the main difficulty is the non-compactness of Sobolev embedding on $\mathbb{R}^n$. Based on the methodology of [6], we will overcome the obstacles by using uniform estimate for far-field values of functions lying in the perturbed random attractors. Actually, by a cut-off technique, we will show that the values of all functions in all perturbed random attractors are uniform convergent to zero when spatial variable approach infinity.

The paper is organized as follows. In Section 2, we recall some basic random attractors theory. Section 3 is devoted to the existence of the unique weak solution and define a continuous random dynamical system for system 1.1 and 1.2 in $\mathbb{H}$. In Section 4, we establish some uniform estimates of the solutions. Finally, we obtain the main result of upper- semicontinuity of the global random attractors in the last section.

We denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and inner product in $L^2(\mathbb{R}^n)$ or $\mathbb{H} = [L^2(\mathbb{R}^n)]^3$; let $\forall = [L^4(\mathbb{R}^n)]^3$, $U = [L^6(\mathbb{R}^n)]^3$, $E = [H^1(\mathbb{R}^n)]^3$, $\| \cdot \|_{L^2}$, $\| \cdot \|_{L^4}$ and $\| \cdot \|_{L^6}$, $\| \cdot \|_0$ denote the norm in $L^4(\mathbb{R}^n)$, $L^6(\mathbb{R}^n)$ and $\forall$, $U$.

2. Preliminaries

In this section, we recall some basic concepts related to random attractors for random dynamical systems. We refer the reader to [1.4–6] for more details. Let $(X, \| \cdot \|_X)$ be a separable Hilbert space with Borel sigma-algebra $\mathcal{B}(X)$, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

**Definition 2.1.** $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system, if $\theta : \mathbb{R} \times \Omega \mapsto \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, $\theta_0$ is the identity on $\Omega$, $\theta_{s+t} = \theta_s \theta_t$ for all $s, t \in \mathbb{R}$ and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

**Definition 2.2.** A stochastic process $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ is a continuous random dynamical system (RDS) over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ if $\varphi$ is $(\mathcal{B}(0, \infty) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable, and for all $\omega \in \Omega$,

(i) The mapping $\varphi(t, \omega) : X \mapsto X, x \mapsto \varphi(t, \omega)x$ is continuous for every $t \geq 0$,

(ii) $\varphi(0, \omega)$ is the identity on $X$,

(iii) (cocycle property) $\varphi(s + t, \omega) = \varphi(t, \theta_s \omega) \varphi(s, \omega)$ for all $s, t \geq 0$.

**Definition 2.3.**

(i) A set-valued mapping $\omega \mapsto B(\omega) \subset X$ (we may write it as $B(\omega)$ for short) is said to be a random set if the mapping $\omega \mapsto \text{dist}(x, B(\omega))$ is measurable for any $x \in X$.

(ii) A random set $B(\omega) \subset X$ is said to be bounded if there exist $x_0 \in X$ and a random variable $r(\omega) > 0$ such that $B(\omega) \subset \{x \in X : \|x - x_0\|_X \leq r(\omega), x_0 \in X\}$ for all $\omega \in \Omega$.

(iii) A random set $B(\omega)$ is called a compact random set if $B(\omega)$ is compact for all $\omega \in \Omega$.

(iv) A random bounded set $B(\omega) \subset X$ is called tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{\gamma \to \infty} e^{-\gamma \sup_{t \in \mathbb{R}} |r(t, \omega)|} = 0 \quad \text{for all} \quad \gamma > 0.$$
We consider a continuous RDS \( \{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega} \) over \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) and \(\mathcal{D}\) the set of all tempered random sets of \(X\).

**Definition 2.4.** A random set \(K(\omega)\) is called an absorbing set in \(\mathcal{D}\) if for all \(B \in \mathcal{D}\) and a.e. \(\omega \in \Omega\) there exist \(t_B(\omega) > 0\) such that
\[
\varphi(t, \theta_{-t} \omega) B(\theta_{-t} \omega) \subset K(\omega) \quad \text{for all } t \geq t_B(\omega).
\]

**Definition 2.5.** A random set \(A\) is called a global \(\mathcal{D}\)-random attractor (or \(\mathcal{D}\)-pullback attractor) for \(\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}\) if the following hold:

(i) \(A\) is a random compact set, i.e. \(\omega \mapsto d(x, A(\omega))\) is measurable for every \(x \in X\) and \(A(\omega)\) is compact for a.e. \(\omega \in \Omega\);

(ii) \(A\) is strictly invariant, i.e. for \(\omega \in \Omega\) and all \(t \geq 0\) one has \(\varphi(t, \omega) A(\omega) = A(\theta_t \omega)\);

(iii) \(A\) attracts all sets in \(\mathcal{D}\), i.e. for all \(B \in \mathcal{D}\) and a.e. \(\omega \in \Omega\) we have
\[
\lim_{t \to +\infty} \|\varphi(t, \theta_{-t} \omega) B(\theta_{-t} \omega), A(\omega)\| = 0,
\]
where \(\|Y, Z\| = \sup_{y \in Y} \inf_{z \in Z} \|y - z\|_X\) is the Hausdorff semi-metric \((Y \subseteq X, Z \subseteq X)\).

**Proposition 2.6 (Existence of a random attractor).** Let \(K(\omega) \in \mathcal{D}\) be a random absorbing set for the continuous RDS \(\{\varphi(t)\}_{t \in \mathbb{R}}\), which is closed and satisfies for a.e. \(\omega \in \Omega\) the following asymptotic compactness condition: each sequence \(x_n \in \varphi(t_n, \theta_{-t_n} \omega, K(\theta_{-t_n} \omega))\) with \(t_n \to \infty\) has a convergent subsequence in \(X\). Then, the cocycle \(\varphi\) has a unique global random attractor
\[
A(\omega) = \bigcap_{t > t_B(\omega)} \overline{\bigcup_{s \geq t} \varphi(s, \theta_{s-t} \omega, K(\theta_{s-t} \omega)))},
\]

**Proposition 2.7 (Upper semicontinuity of non-compact random attractors).** Suppose the following conditions be satisfied:

(i) Given \(\sigma > 0\), suppose \(\Phi_\sigma\) is a random dynamical system over a metric system \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\). For \(\mathbb{P}\)-a.e. \(\omega \in \Omega\), \(t \geq 0\), \(\sigma_n \to 0\), and \(x_n, x \in X\) with \(x_n \to x\), there holds
\[
\lim_{n \to \infty} \Phi_{\sigma_n}(t, \omega, x_n) = \Phi(t)x.
\]

(ii) Assume that \(\Phi_\sigma\) has a random attractor \(A_\sigma = \{A_\sigma(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\) and a random absorbing set \(E_\sigma = \{E_\sigma(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\) such that for some deterministic positive constant \(c\) and for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\)
\[
\limsup_{\sigma \to 0} \|E_\sigma(\omega)\|_X \leq c,
\]

where \(\|E_\sigma(\omega)\|_X = \sup_{x \in E_\sigma(\omega)} \|x\|_X\).

(iii) There exists \(\sigma_0 > 0\) such that for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\),
\[
\bigcup_{0 < \sigma \leq \sigma_0} A_\sigma(\omega) \quad \text{is precompact in } X.
\]

Then for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\),
\[
\text{dist}(A_\sigma(\omega), A_0) \to 0, \quad \text{as } \sigma \to 0.
\]

**3. Stochastic three-component reversible Gray–Scott system**

In this section, we will give the basic setting of system (1.1) and show that it generates a random dynamical system. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space as in Section 1. Define \((\theta_t)_{t \in \mathbb{R}}\) on \(\Omega\) via \(\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \ t \in \mathbb{R}\), then \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) is an ergodic metric dynamical system (see [1]).

Denote \(\tilde{g} = (\tilde{u}, \tilde{v}, \tilde{w})^T\), system (1.1) with initial data (1.2) can be rewritten as
\[
\frac{\partial \tilde{g}}{\partial t} = A \tilde{g} + \tilde{\lambda}(\tilde{g}) + f(x) + \sigma \tilde{g} \cdot \frac{d\omega}{dt}, \quad t > 0,
\]
\[
\tilde{g}(0, x) = \tilde{g}_0(x), \quad x \in \mathbb{R}^n,
\]
where
\[
A = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \quad \tilde{\lambda}(\tilde{g}) = \begin{pmatrix} -(F + k)\tilde{u} + \tilde{u}^2 \tilde{v} - \tilde{G} \tilde{u}^3 + N \tilde{w} \\ -F \tilde{v} - \tilde{u}^2 \tilde{v} + \tilde{G} \tilde{u}^3 \\ k\tilde{u} - (F + N)\tilde{w} \end{pmatrix},
\]
and \(f(x) = (f_1(x), f_2(x), f_3(x))^T\), here \(T\) denotes the transposition.
For our purpose, it is convenient to transform the problem (3.1) into a deterministic system with a random parameter, and then show that it generates a random dynamical system.

Given \( \omega \in \Omega \), let \( \alpha(t, \omega) = e^{-\sigma(t)\langle \omega \rangle} \). Then \( \alpha \) solves the following stochastic equation in the sense of Stratonovich integration:

\[
\frac{d\alpha}{dt} + \sigma \alpha \frac{d\omega}{dt} = 0. \tag{3.2}
\]

Let

\[
(u(t), \nu(t), w(t)) = \alpha(t, \omega)(\tilde{u}(t), \tilde{v}(t), \tilde{w}(t))
\]

then system (3.1) can be written as

\[
\frac{\partial u}{\partial t} = d_1 u - (F + k)u + \alpha^2(t, \omega)u^3 - G\alpha^2(t, \omega)u^3 + Nw + \alpha(t, \omega)f_1,
\]

\[
\frac{\partial \nu}{\partial t} = d_2 \nu - F\nu - \alpha^2(t, \omega)u^2 \nu + G\alpha^2(t, \omega)u^3 + \alpha(t, \omega)f_2,
\]

\[
\frac{\partial w}{\partial t} = d_3 w - (F + N)w + ku + \alpha(t, \omega)f_3.
\]

that is \( g(t, :) = (u(t, :) , \nu(t, :) , w(t, :) ) \) satisfies

\[
\frac{dg}{dt} = Ag + \Lambda(g, \omega) + \alpha(t, \omega)f(x), \quad t > 0,
\]

\[
g(0, x) = g_0(x) = \bar{g}_0(x), \quad x \in \mathbb{R}^n,
\]

where

\[
\Lambda(g, \omega) = \begin{pmatrix}
-(F + k)u + \alpha^2(t, \omega)u^3 - G\alpha^2(t, \omega)u^3 + Nw \\
-F\nu - \alpha^2(t, \omega)u^2 \nu + G\alpha^2(t, \omega)u^3 \\
-(F + N)w + ku
\end{pmatrix}.
\]

We will consider system (3.7) for \( \mathbb{P} \)-a.e. \( \omega \in \tilde{\Omega} \) and still use \( \Omega \) instead of \( \tilde{\Omega} \) from now on.

As in the case of a bounded domain with Dirichlet boundary conditions which are studied in [21], for \( \Lambda : \mathbb{E} \cap \mathbb{U} \rightarrow \mathbb{H} \) is locally Lipschitz continuous and \( f \in \mathbb{E} \cap \mathbb{U} \), by a Gelerkin method which is similar to the autonomous case studied in [3], one can get that for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( g_0 \in \mathbb{H} \), system (3.7) has a unique solution \( g(\cdot, \omega, g_0) \in C([0, \infty), \mathbb{H}) \cap L^2((0, T); \mathbb{E}) \) with \( g(0, \omega, g_0) = \bar{g}_0 \) for every \( T > 0 \). Because of the continuous nonlinearity \( \Lambda \), one may take the domain to be a sequence of balls with radius approaching \( \infty \) to deduce the existence of a weak solution to (3.7) on \( \mathbb{R}^n \) provided \( f \in \mathbb{E} \cap \mathbb{U} \), and the unique solution generates a continuous random dynamical system \( (\varphi_\sigma(t))_{t \geq 0} \) over \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}) \) according to the conditions (i)–(iii) in Definition 2.2, where

\[
\varphi_\sigma(t, \omega, \bar{g}_0) = (g(t, \omega, \bar{g}_0), \quad \text{for all } (t, \omega, \bar{g}_0) \in \mathbb{R}^+ \times \Omega \times \mathbb{H}.
\]

We now define a mapping \( \varphi_\sigma : \mathbb{R}^+ \times \Omega \times \mathbb{H} \rightarrow \mathbb{H} \) by

\[
\varphi_\sigma(t, \omega, \bar{g}_0) = \tilde{g}(t, \omega, \bar{g}_0) = g(t, \omega, \bar{g}_0) \varphi^{-1}(t, \omega)
\]

for all \( (t, \omega, \bar{g}_0) \in \mathbb{R}^+ \times \Omega \times \mathbb{H} \). Then \( \varphi_\sigma \) is a continuous random dynamical system associated with problem (3.1) on \( \mathbb{R}^n \). We remark that the two random dynamical systems are conjugated to each other, thus in the following sections, we only need to consider \( \varphi_\sigma \). In the sequel, we always assume that \( D \) is a collection of random subset of \( \mathbb{H} \) given by

\[
\mathcal{D} = \{ D = \{ D(\omega) \}_{\omega \in \Omega}, D(\omega) \subset \mathbb{H} \text{ and } e^{-\|\cdot\|}; D(\theta^{-1} t) \omega) \rightarrow 0 \text{ as } t \rightarrow \infty \},
\]

where

\[
\|D(\theta^{-1} t)\omega)\| = \sup_{g \in D(\theta^{-1} t) \omega)} \|g\|.
\]

By [19], we have obtained that \( \varphi_\sigma \) has a \( \mathcal{D} \)-pullback random attractor if \( \mathcal{D} \) is the collection of all tempered random subsets of \( \mathbb{H} \). Following the statements of [19], we can also prove that \( \varphi_\sigma \) has a unique \( \mathcal{D} \)-pullback random attractor \( \{ A_\sigma \}_{\omega \in \Omega} \) when \( \mathcal{D} \) is given by (3.9) (the existence will be implied in the next section of the paper). When \( \sigma = 0 \), system (3.1) defines a continuous deterministic dynamical system \( \varphi \) in \( \mathbb{H} \). In this case, the results of [19] imply that \( \varphi \) has a unique global attractor \( \mathcal{A}_0 \) in \( \mathbb{H} \) (see also in [20]). The purpose of the paper is to give the relationship of \( \{ A_\sigma \}_{\omega \in \Omega} \) and \( \mathcal{A}_0 \) when \( \sigma \rightarrow 0 \).
4. Uniform estimates of solutions

In this section, we derive uniform estimates of solutions with respect to the small parameter $\sigma$. Hereafter, we always assume that $\mathcal{D}$ is the collections of random subsets of $\mathbb{H}$ given in (3.9).

**Lemma 4.1.** Let $0 < \sigma \leq 1$ and $f \in H \cap L^1$ hold. Then for every $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there is $T_B(\omega) > 0$, independent of $\sigma$, such that for all $g_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$ and for all $t \geq T_B(\omega)$,

$$
\|g(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))\|^2 \leq c + c\|f\|^2 \chi^2(-t, \omega) \int_{-\infty}^0 e^s \chi^2(s, \omega)ds,
$$

and

$$
\int_0^t e^{(s-t)}||\nabla g(s, \theta_{-t}\omega, g_0(\theta_{-t}\omega))\||^2 ds \leq c + c\|f\|^2 \chi^2(-t, \omega) \int_{-\infty}^0 e^s \chi^2(s, \omega)ds,
$$

where $c$ is a positive deterministic constant dependent of $\sigma$.

**Proof.** Define

$$W(t, x) = \frac{N}{k} W(t, x), \quad \mu = k \frac{N}{k},$$

then the Eqs. (3.4)–(3.6) become

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u - (F + k)u + \chi^2(t, \omega)u^3 + kW + \alpha(t, \omega)f_1, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v - Fv + \chi^2(t, \omega)u^3 + G\chi^2(t, \omega)u^3 + \alpha(t, \omega)f_2, \\
\mu \frac{\partial W}{\partial t} &= \mu d_3 \Delta W - (\mu F + k)W + ku + \alpha(t, \omega)f_3.
\end{align*}
\]

Take the inner products (4.1), (Gu (t)), (4.2), (v (t)) and (4.3), (GW (t)). Then summing up the resulting equalities we get

\[
\begin{align*}
\frac{d}{dt}(G\|u\|^2 + \|v\|^2 + \mu G\|W\|^2) + 2d_1 G\|\nabla u\|^2 + 2d_2 \|\nabla v\|^2 + 2\mu Gd_3 \|\nabla W\|^2 + 2G(F + k)\|u\|^2 + F\|v\|^2 + G(\mu F + k)\|W\|^2
\end{align*}
\]

\[
= 4kG \int_\mathbb{R} uWdx + 2G \int_\mathbb{R} \alpha(t, \omega)uf_1 dx + 2 \int_\mathbb{R} \alpha(t, \omega)vf_2 dx + 2G \int_\mathbb{R} \alpha(t, \omega)Wf_3 dx - 2\chi^2(t, \omega) \int_\mathbb{R} (Gu^2 - u^2) dx,
\]

that is

\[
\frac{d}{dt}(G\|u\|^2 + \|v\|^2 + \mu G\|W\|^2) + F(G\|u\|^2 + \|v\|^2 + \mu G\|W\|^2) + 2d_0(G\|\nabla u\|^2 + \|\nabla v\|^2 + \mu G\|\nabla W\|^2)
\]

\[
\leq \frac{\chi^2(t, \omega)}{F} \left(G\|f_1\|^2 + \|f_2\|^2 + \frac{G}{\mu} \|f_3\|^2\right),
\]

where $d_0 = \min\{d_1, d_2, d_3\}$. Multiplying (4.4) by $e^t$ and integrating the inequality, we get that for all $t \geq 0$,

\[
\|g(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))\|^2 + 2d_0 \int_0^t e^{(s-t)}\|\nabla g(s, \theta_{-t}\omega, g_0(\theta_{-t}\omega))\|^2 ds \leq e^{-t}\|g_0(\theta_{-t}\omega))\|^2 + c_1\|f\|^2 \int_0^t e^{(s-t)} \chi^2(s, \omega)ds,
\]

where $c_1 = \max\{\frac{c_1}{c_2}, \frac{c_2}{c_1}\}$ Since $g_0(\theta_{-t}\omega) \in \mathcal{D}$, there is $T = T_B(\omega)$, independent of $\sigma$, such that for all $t \geq T$,

\[
e^{-T}\|g_0(\theta_{-t}\omega)\|^2 \leq 1.
\]

Now, by replacing $\omega$ by $\theta_{-t}\omega$, we get from (4.5) that

\[
\|g(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))\|^2 + 2d_0 \int_0^t e^{(s-t)}\|\nabla g(s, \theta_{-t}\omega, g_0(\theta_{-t}\omega))\|^2 ds
\]

\[
\leq e^{-t}\|g_0(\theta_{-t}\omega)\|^2 + c_1\|f\|^2 \int_0^t e^{(s-t)} \chi^2(s, \theta_{-t}\omega)ds \leq e^{-t}\|g_0(\theta_{-t}\omega)\|^2 + c_1\|f\|^2 \chi^2(-t, \omega) \int_{-\infty}^0 e^s \chi^2(s, \omega)ds
\]

\[
\leq 1 + c_1\|f\|^2 \chi^2(-t, \omega) \int_{-\infty}^0 e^s \chi^2(s, \omega)ds,
\]

which completes the proof. \(\square\)
We need the propositions to prove the next result.

**Proposition 4.2.** Assume that \( f \in H \cap U \). Then for any \( B(\omega) \in D \) and \( g_0(\omega) \in B(\omega) \), there exists a \( T_B(\omega) > 0 \) such that the solution \( \varphi \) of (3.7) satisfies for \( \mathbb{P}\text{-a.e.} \ \omega \in \Omega \) and \( t \geq T_B(\omega) \),

\[
\int_t^{t+1} \|\varphi(s, \theta_{-1} \omega, g_0(\theta_{-1} \omega))\|_U^6 \, ds \leq c + c\|f\|_U^6 \, z^6(-t - 1, \omega) \int_{-\infty}^0 e^{(s+1)z^6(s, \omega)} \, ds,
\]

where \( c \) is a positive deterministic constant dependent of \( \sigma \).

**Proof.** Let \( V(t, x) = \frac{u(t, x)^2}{2} \), then (4.1)–(4.3) can be written as

\[
\frac{\partial u}{\partial t} = d_1 u - (F + k)u + \alpha^2(t, \omega)u^2 \, \nu - Gx^2(t, \omega)u^3 + kW + \alpha(t, \omega)f_1,
\]

\[
\frac{\partial V}{\partial t} = d_2 \Delta V - FV - \alpha^2(t, \omega)u^2 V + \alpha^2(t, \omega)u^3 + \frac{1}{G} \, \alpha(t, \omega)f_2,
\]

\[
\mu \frac{\partial W}{\partial t} = \mu d_3 \Delta W - (\mu F + k)W + ku + \alpha(t, \omega)f_3.
\]

Taking the inner products of (4.7)–(4.9) with \( u^2(t) \), \( GV^2(t) \) and \( W^2(t) \), and summing up the resulting equalities, we get

\[
\frac{1}{6} \frac{d}{dt} \left( \|u\|_{L^6}^6 + G\|V\|_{L^6}^6 + \mu\|W\|_{L^6}^6 \right) + 5 \left( d_1 \|u^2\|_{L^2}^2 + d_2 G\|V\|_{L^2}^2 \right) + \mu d_3 G\|W^2\|_{L^2}^2
\]

\[
= -(F + k)\|u\|_{L^6}^6 - G\|V\|_{L^6}^6 - (\mu F + k)\|W\|_{L^6}^6 + k \int_{\mathbb{R}^n} u^5 \, dx + k \int_{\mathbb{R}^n} uV^5 \, dx - Gx^2 \int_{\mathbb{R}^n} u^3 V \, dx + u^2V^2 + \int_{\mathbb{R}^n} \alpha(t, \omega)f_1 u^5 \, dx + \int_{\mathbb{R}^n} \alpha(t, \omega)f_2 V^5 \, dx + \int_{\mathbb{R}^n} \alpha(t, \omega)f_3 W^5 \, dx,
\]

that is

\[
\frac{d}{dt} \left( \|u\|_{L^6}^6 + G\|V\|_{L^6}^6 + \mu\|W\|_{L^6}^6 \right) + F\left( \|u\|_{L^6}^6 + G\|V\|_{L^6}^6 + \mu\|W\|_{L^6}^6 \right) \leq \frac{1}{F^3} \alpha^6(t, \omega) \left( \|f_1\|_U^6 + \frac{1}{G^2}\|f_2\|_U^6 + \frac{1}{\mu F}\|f_3\|_U^6 \right).
\]

Furthermore, apply Gronwall lemma and let

\[
c_2 = \max \left\{ 1, \frac{1}{c}, \frac{1}{\rho^3} \right\},
\]

then, for \( \nu \geq 0 \), we can deduce from (4.11) that

\[
\|g(v, \omega, g_0(\omega))\|_{L^6}^6 \leq e^{-\nu \tau}\|g_0(\omega)\|_{L^6}^6 + c_2\|f\|_U^6 \int_0^\tau e^{-\psi(t-\nu)} \alpha^6(s, \omega) \, ds.
\]

Letting \( T_B(\omega) \) be a positive variable in Lemma 4.1, \( t \geq T_B(\omega) \), integrating (4.13) for \( \nu \in (t, t + 1) \), we have

\[
\int_t^{t+1} \|g(v, \omega, g_0(\omega))\|_{L^6}^6 \, dv \leq \frac{1}{F} e^{-\nu \tau}\|g_0(\omega)\|_{L^6}^6 + c_2\|f\|_U^6 \int_0^{t+1} e^{-\psi(t-\nu)} \alpha^6(s, \omega) \, ds.
\]

Replacing \( \omega \) by \( \theta_{-1} \omega \) in (4.14), we obtain

\[
\int_t^{t+1} \|g(v, \theta_{-1} \theta_{-1} \omega, g_0(\theta_{-1} \theta_{-1} \omega))\|_{L^6}^6 \, dv \leq \frac{1}{F} e^{-\nu \tau}\|g_0(\theta_{-1} \theta_{-1} \omega)\|_{L^6}^6 + c_2\|f\|_U^6 \int_0^{t+1} e^{-\psi(t-\nu)} \alpha^6(s, \theta_{-1} \theta_{-1} \omega) \, ds
\]

\[
\leq \frac{1}{F} e^{-\nu \tau}\|g_0(\theta_{-1} \theta_{-1} \omega)\|_{L^6}^6 + c_2\|f\|_U^6 \alpha^6(-t - 1, \omega) \int_{-\infty}^0 e^{(s+1)z^6(s, \omega)} \, ds
\]

\[
\leq 1 + c_2\|f\|_U^6 \alpha^6(-t - 1, \omega) \int_{-\infty}^0 e^{(s+1)z^6(s, \omega)} \, ds.
\]

The last but one line in (4.15) due to \( B(\omega) \in D \) is tempered, then for any \( g_0(\theta_{-1} \theta_{-1} \omega) \in B(\theta_{-1} \theta_{-1} \omega) \subset B(\theta_{-1} \omega) \),

\[
\lim_{t \to -\infty} e^{-\nu \tau}\|g_0(\theta_{-1} \theta_{-1} \omega)\|_{L^6}^6 = 0.
\]

The proof is completed. \( \square \)
Also, we have.

**Proposition 4.3.** Assume that $f \in H \cap V$. Then for any $B(\omega) \in D$ and $g_{0}(\omega) \in B(\omega)$, there exists a $T_{B}(\omega) > 0$ such that the solution $\varphi$ of (3.7) satisfies for $\mathcal{P}$-a.e. $\omega \in \Omega$ and $t \geq T_{B}(\omega)$,

$$
\int_{t}^{t+1} \| \varphi(s, \theta_{-t-1} \omega, g_{0}(\theta_{-t-1} \omega)) \|_{1}^{2} ds \leq c + c \| f \|_{1}^{2} \mathcal{A}^{-4}(-t-1, \omega) \int_{-\infty}^{0} e^{F(s+1)} \mathcal{A}^{4}(s, \omega) ds,
$$

where $c$ is a positive deterministic constant dependent of $\sigma$.

**Lemma 4.4.** Assume that $f \in H \cap L$. Then for any $B(\omega) \in D$ and $g_{0}(\omega) \in B(\omega)$, there exists a $T_{B}(\omega) > 0$ such that the solution $\varphi$ of (3.7) satisfies for $\mathcal{P}$-a.e. $\omega \in \Omega$ and $t \geq T_{B}(\omega)$,

$$
\| \nabla \varphi(t, \theta_{-t} \omega, g_{0}(\theta_{-t} \omega)) \|^{2} \leq c + c \| f \|^{2} \mathcal{A}^{-4}(-t-1, \omega) \int_{-\infty}^{0} e^{F(s+1)} \mathcal{A}^{4}(s, \omega) ds + \int_{-\infty}^{0} \mathcal{A}^{2}(s, \omega) ds.
$$

(4.17)

where $c$ is a positive deterministic constant dependent of $\sigma$.

**Proof.** Taking the inner products of (4.1)-(4.3) with $-\Delta u, -\Delta v, -\Delta W$ respectively, and summing up the three resulting equalities, we have

$$
\frac{d}{dt} (\| \nabla u \|^{2} + \| \nabla v \|^{2} + \mu \| \nabla W \|^{2}) + d_{1} \mathcal{A}^{-4}(t, \omega) \int_{\mathbb{R}^{n}} u^{4} \Delta u dx + d_{2} \mathcal{A}^{-4}(t, \omega) \int_{\mathbb{R}^{n}} v^{4} \Delta v dx + \left( \frac{2}{d_{1}} + \frac{2}{d_{2}} + \mu \right) \mathcal{A}^{-4}(t, \omega) \int_{\mathbb{R}^{n}} f_{3} dx
$$

$$
= \mathcal{A}^{-4}(t, \omega) \int_{\mathbb{R}^{n}} u^{4} \Delta u dx + \mathcal{A}^{-4}(t, \omega) \int_{\mathbb{R}^{n}} v^{4} \Delta v dx + k \int_{\mathbb{R}^{n}} u \Delta W dx + f_{3} \mathcal{A}^{-4}(t, \omega) \int_{\mathbb{R}^{n}} f_{3} dx.
$$

By Hölder inequality, it yields

$$
\frac{d}{dt} (\| \nabla u \|^{2} + \| \nabla v \|^{2} + \mu \| \nabla W \|^{2}) + F(\| \nabla u \|^{2} + \| \nabla v \|^{2} + \mu \| \nabla W \|^{2})
$$

$$
\leq \left( \frac{2}{d_{1}} + \frac{2}{d_{2}} \right) \mathcal{A}^{-4}(t, \omega) \int_{\mathbb{R}^{n}} u^{4} \Delta u dx + \left( \frac{2}{d_{1}} + \frac{2}{d_{2}} \right) \mathcal{A}^{-4}(t, \omega) \int_{\mathbb{R}^{n}} v^{4} \Delta v dx + \frac{1}{d_{1}} \mathcal{A}^{-4}(t, \omega) \| f_{3} \|^{2}
$$

\[ + \frac{1}{d_{2}} \mathcal{A}^{-4}(t, \omega) \| f_{3} \|^{2} \leq \frac{1}{d_{0}} \left( \frac{8}{3} + 4 \mathcal{A}^{2}(t, \omega) \right) \mathcal{A}^{-4}(t, \omega) (\| u \|_{[\frac{3}{2}]}^{2} + \| v \|_{[\frac{3}{2}]}^{2}) + \max \left\{ \frac{1}{d_{0}}, \frac{1}{d_{0}}, \mathcal{A}^{-4}(t, \omega) \| f \|^{2} \right\}.
$$

(4.18)

Let $c_{3} = \frac{1+4c^{2}}{d_{0} \min \{ \frac{1}{2}, \frac{1}{3} \}}, c_{4} = \frac{\max \{ \frac{1}{2}, \frac{1}{3} \}}{d_{0} \min \{ \frac{1}{2}, \frac{1}{3} \}}$ and $T_{B}(\omega)$ be a positive variable in Lemma 4.1, $t \geq T_{B}(\omega)$ and $s \in (t, t+1)$. Then integrate (4.18) over $(s, t+1)$, we get

$$
\| \nabla g(t+1, \omega, g_{0}(\omega)) \|^{2} \leq F \int_{s}^{t+1} \| \nabla g(s, \omega, g_{0}(\omega)) \|^{2} ds + c_{3} \int_{s}^{t+1} \mathcal{A}^{-4}(s, \omega) \| g(s, \omega, g_{0}(\omega)) \|^{6} ds + c_{4} \| f \|^{2}
$$

(4.19)

Now integrating (4.19) with respect to $s$ over $(t, t+1)$, we find that

$$
\| \nabla g(t+1, \omega, g_{0}(\omega)) \|^{2} \leq \int_{t}^{t+1} \| \nabla g(s, \omega, g_{0}(\omega)) \|^{2} ds + c_{3} \int_{t}^{t+1} \mathcal{A}^{-4}(s, \omega) \| g(s, \omega, g_{0}(\omega)) \|^{6} ds + c_{4} \| f \|^{2}
$$

(4.20)

Then, replacing $\omega$ with $\theta_{-t-1} \omega$, it follows from (4.20) that
\[ \| \nabla g(t + 1, \theta_{t-1}, g_0(\theta_{t-1}\omega)) \|^2 \leq \int_t^{t+1} \| \nabla g(s, \theta_{t-1}, g_0(\theta_{t-1}\omega)) \|^2 ds + c_3 \int_t^{t+1} \chi^4(s, \theta_{t-1}, \omega) \| g(s, \theta_{t-1}, g_0(\theta_{t-1}\omega)) \|_1^6 ds + c_4 \| f \|^2 \int_t^{t+1} \chi^2(s, \theta_{t-1}, \omega) ds \]
\[ \leq \int_t^{t+1} \| \nabla g(s, \theta_{t-1}, g_0(\theta_{t-1}\omega)) \|^2 ds + c_3 \chi^4(-t, \omega) \sup_{0 < \tau < 1} \chi^4(-t, \omega) \times \int_t^{t+1} \| g(s, \theta_{t-1}, g_0(\theta_{t-1}\omega)) \|_1^6 ds \]
\[ + c_4 \| f \|^2 \chi^2(-t-1, \omega) \int_1^0 \chi^2(s, \omega) ds. \] (4.21)

Given \( t \geq 0 \), replacing \( t \) by \( t + 1 \) in Lemma 4.1 we find that
\[ \int_t^{t+1} e^{\rho(s-t-1)} \| \nabla g(s, \theta_{t-1}, g_0(\theta_{t-1}\omega)) \|^2 ds \leq 1 + c \| f \|^2 \chi^2(-t-1, \omega) \int_{-\infty}^0 e^{\rho(-1)} \chi^2(s, \omega) ds. \]
Since \( e^{\rho(s-t-1)} \geq e^{-\rho} \) for \( s \in (t, t+1) \), the above implies that, for all \( t \geq T_\rho(\omega) \),
\[ \int_t^{t+1} \| \nabla g(s, \theta_{t-1}, g_0(\theta_{t-1}\omega)) \|^2 ds \leq e^\rho (1 + c \| f \|^2 \chi^2(-t-1, \omega) \int_{-\infty}^0 e^{\rho(s-t-1)} \chi^2(s, \omega) ds). \]
Then by Proposition 4.2, it yields from (4.21) that
\[ \| \nabla g(t + 1, \theta_{t-1}, g_0(\theta_{t-1}\omega)) \|^2 \leq e^\rho \left( c + c \| f \|^2 \chi^2(-t-1, \omega) \int_{-\infty}^0 e^{\rho(s-t-1)} \chi^2(s, \omega) ds \right) \]
\[ + c_3 \chi^4(-t-1, \omega) \sup_{0 < \tau < 1} \chi^4(-t, \omega) \left( c + c \| f \|^2 \chi^2(-t-1, \omega) \int_{-\infty}^0 e^{\rho(s-t-1)} \chi^2(s, \omega) ds \right) \]
\[ + c_4 \| f \|^2 \chi^2(-t-1, \omega) \int_{-\infty}^0 \chi^2(s, \omega) ds \]
\[ \leq c + c \| f \|^2 \chi^2(-t-1, \omega) \left( \int_{-\infty}^0 e^{\rho(s-t-1)} \chi^2(s, \omega) ds + \int_{-\infty}^0 \chi^2(s, \omega) ds \right) \]
\[ + c \sup_{0 < \tau < 1} \chi^4(-t, \omega) \| f \|^2 \chi^2(-t-1, \omega) \int_{-\infty}^0 e^{\rho(s-t-1)} \chi^2(s, \omega) ds. \]

Note that \( f \in \mathcal{H} \cap U \). This completes the proof. \( \square \)

**Lemma 4.5.** Assume that \( f \in \mathcal{H} \). Let \( \{ B(\omega) \} \in D \) and \( g_0(\omega) \in B(\omega) \). Then, for every \( \epsilon > 0 \), there exist \( T' = T(\epsilon, \omega, B) > 0 \) and \( K' = K(\epsilon, \omega) > 0 \), such that the solution \( \phi \) of problem (3.7) satisfies for \( \mathcal{P} \)-a.e. \( \omega \in \Omega \), \( \forall t \geq T' \),
\[ \int_{|x| > K'} |\phi(t, \theta_{t}, \omega, g_0(\theta_{t}\omega))| dx \leq \epsilon. \] (4.22)

**Proof.** Choose a smooth cut-off function satisfying \( 0 \leq \rho(s) \leq 1 \) for \( s \in \mathbb{R}^+ \) and \( \rho(s) = 0 \) for \( 0 \leq s \leq 1 \), \( \rho(s) = 1 \) for \( s \geq 2 \). Suppose there exists a constant \( c \) such that \( |\rho(s)| \leq c \) for \( s \in \mathbb{R}^+ \).

Taking the inner product of 4.1, 4.2 and 4.3 with \( G\rho(\|\cdot\|^2) v \) and \( G\rho(\|\cdot\|^2) W \) in \( L^2(\mathbb{R}^n) \), respectively, we get
\[ \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) |u|^2 dx - d_1 \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) u \Delta u dx + G(F + k) \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) |u|^2 dx = G \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) \chi^{-2}(t, \omega) u^2 \nu dx \]
\[ - G^2 \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) \chi^{-2}(t, \omega) u^4 dx + kG \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) u \nu dx + G \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) \phi(t, \omega) f_1 \nu dx. \] (4.23)
\[ \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) |v|^2 dx - d_2 \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) v \Delta v dx + F \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) |v|^2 dx \]
\[ = - \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) \chi^{-2}(t, \omega) u^2 v^2 dx + G \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) \chi^{-2}(t, \omega) \nu^2 dx + \int_{\mathbb{R}^n} \rho \left( \frac{|\chi|^2}{K^2} \right) \phi(t, \omega) f_2 v dx, \] (4.24)
\[
\frac{\mu G}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |W|^2 \, dx - \mu G d_3 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) W \Delta W \, dx + G(\mu F + k) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |W|^2 \, dx \\
= kG \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) W_t \, dx + G \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) \alpha(t, \omega) f_3 \, dx.
\]

Adding up the three equalities, we have

\[
\frac{d}{dt} \left( G \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |u|^2 \, dx + \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |v|^2 \, dx + \mu G \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |W|^2 \, dx \right) \\
- 2 \left( d_1 G \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) u \Delta u \, dx + d_2 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) v \Delta v \, dx + \mu G d_3 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) W \Delta W \, dx \right) \\
+ F \left( G \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |u|^2 \, dx + \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |v|^2 \, dx \right) + \mu G \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |W|^2 \, dx \\
\leq \frac{G}{2F^2} \alpha^2(t, \omega) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) f_1(x, t)^2 \, dx + \frac{G}{2F^2} \alpha^2(t, \omega) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) f_2(x, t)^2 \, dx \\
+ \frac{1}{2F} \int_{\mathbb{R}^n} \alpha^2(t, \omega) \rho \left( \frac{|x|^2}{K^2} \right) f_3(x, t)^2 \, dx.
\]

We know that

\[
\int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) u \Delta u \, dx \leq - \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) \nabla u^2 \, dx + \frac{C_5}{K} (\|u\|^2 + \|\nabla u\|^2),
\]

where \(C_5\) is a positive constant which depends on \(c\). Then from (4.26) and (4.27), we get

\[
\frac{d}{dt} \left( G \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |u|^2 \, dx + \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |v|^2 \, dx + \mu G \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |W|^2 \, dx \right) \\
+ F \left( G \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |u|^2 \, dx + \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |v|^2 \, dx \right) + \mu G \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |W|^2 \, dx \\
\leq \frac{G}{2F^2} \alpha^2(t, \omega) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) f_1(x)^2 \, dx + \frac{G}{2F^2} \alpha^2(t, \omega) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) f_2(x)^2 \, dx \\
+ \frac{1}{2F} \int_{\mathbb{R}^n} \alpha^2(t, \omega) \rho \left( \frac{|x|^2}{K^2} \right) f_3(x)^2 \, dx + \frac{2d^0 C_5}{K} (\|u\|^2 + \|v\|^2 + \mu G \|W\|^2) + \frac{2d^0 C_5}{K} (\|\nabla u\|^2 + \|\nabla v\|^2 + \mu G \|\nabla W\|^2).
\]

By Gronwall lemma and let

\[
c_6 = \max \left\{ 1, \frac{G}{\mu F} \right\}, \quad c_7 = F d^0 c_6, \quad d^0 = \max \{d_1, d_2, d_3\},
\]

then, for any \(t \geq \bar{T}\), we obtain

\[
\int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |g(t, \omega, g_0(\omega))|^2 \, dx \leq e^{-F(t-\bar{T})} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |g(\bar{T}, \omega, g_0(\omega))|^2 \, dx + c_6 I_1^t e^{-F(s-\bar{T})} \alpha^2(s, \omega) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |f|^2 \, dx \, ds \\
+ \frac{c_7}{K} I_1^t e^{-F(s-\bar{T})} \|g(s, \omega, g_0(\omega))\|^2 \, ds + \frac{c_7}{K} I_1^t e^{-F(s-\bar{T})} \|g(s, \omega, g_0(\omega))\|^2 \, ds.
\]

By replacing \(\omega\) by \(\omega_{\cdot, \omega}\), it then follows from (4.30) that

\[
\int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |g(t, \omega, g_0(\omega_{\cdot, \omega}))|^2 \, dx \leq e^{-F(t-\bar{T})} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |g(\bar{T}, \omega_{\cdot, \omega}, g_0(\omega_{\cdot, \omega}))|^2 \, dx \\
+ c_6 I_1^t e^{-F(s-\bar{T})} \alpha^2(s, \omega_{\cdot, \omega}) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{K^2} \right) |f|^2 \, dx \, ds + \frac{c_7}{K} I_1^t e^{-F(s-\bar{T})} \|g(s, \omega_{\cdot, \omega}, g_0(\omega_{\cdot, \omega}))\|^2 \, ds \\
+ \frac{c_7}{K} I_1^t e^{-F(s-\bar{T})} \|g(s, \omega_{\cdot, \omega}, g_0(\omega_{\cdot, \omega}))\|^2 \, ds.
\]

We now estimate each term in (4.31) on the right-hand side one by one. By substituting \(t\) by \(\bar{T}\) and \(\omega\) by \(\omega_{\cdot, \omega}\) in (4.5) and combine with the first term of (4.31), we get
\[ e^{-F(t-T)} \int_{\Omega} \rho \left( \frac{|x|^2}{K^2} \right) |g(T, \theta, \omega, g_0(\theta, \omega))|^2 dx \leq e^{-F(t-T)} \left( e^{-F(t-T)} \|g_0(\theta, \omega)\|^2 + c_1 \|f\|^2 \int_0^T e^{F(s-t)} \|x^2(s, \omega)\| ds \right) \]
\[ = e^{-R} \|g_0(\theta, \omega)\|^2 + c_1 \|f\|^2 \int_0^T e^{F(s-t)} \|x^2(s, \omega)\| ds. \]  
(4.32)

Obviously, there exists \( T_1 = T_1(\varepsilon, \omega, B) > T \) such that for \( t > T_1 \),
\[ e^{-F(t-T)} \int_{\Omega} \rho \left( \frac{|x|^2}{K^2} \right) |g(T, \theta, \omega, g_0(\theta, \omega))|^2 dx \leq \frac{\varepsilon}{4}. \]  
(4.33)

For the second term on the right-hand side of (4.31), since \( f \in \mathcal{H} \), there exist \( T_2 = T_2(\varepsilon, \omega, B) > T \) and \( K_1 = K_1(\varepsilon, \omega) > 0 \) such that for all \( t > T_2 \) and \( K > K_1 \), then
\[ c_0 \int_t^T e^{F(s-t)} \|x^2(s, \theta, \omega)\| dx \leq c_0 \varepsilon \int_t^T e^{F(s-t)} \|x^2(s, \omega)\| dx \leq \frac{\varepsilon}{4}. \]  
(4.34)

For the third term, by replacing \( t \) by \( s \) and \( \omega \) by \( \theta, \omega \) in (4.5),
\[ \frac{c_2}{K} \int_t^T e^{-F(s-t)} \|g(s, \theta, \omega, g_0(\theta, \omega))\|^2 ds \leq \frac{c_2}{K} \int_t^T e^{-F(s-t)} \left( e^{-F(s-t)} \|g_0(\theta, \omega)\|^2 + c_1 \|f\|^2 \int_0^T e^{F(s-t)} \|x^2(s, \theta, \omega)\| dv \right) ds \]
\[ \leq \frac{c_2}{K} (t - T) e^{-R} \|g_0(\theta, \omega)\|^2 + c_2 c_1 \|f\|^2 \int_t^T e^{F(s-t)} \|x^2(v, \theta, \omega)\| dv ds. \]  
(4.35)

Since \( f \in \mathcal{H} \), there exist \( T_3 = T_3(\varepsilon, \omega, B) > T \) and \( K_2 = K_2(\varepsilon, \omega) > 0 \) such that for all \( t > T_3 \) and \( K > K_2 \), we have
\[ \frac{c_2}{K} \int_t^T e^{-F(s-t)} \|g(s, \theta, \omega, g_0(\theta, \omega))\|^2 ds \leq \frac{\varepsilon}{4}. \]  
(4.36)

Finally, we estimate the last term on the right-hand side of (4.31). Since \( f \in \mathcal{H} \), by using Lemma 4.1, there exist \( T_4 = T_4(\varepsilon, \omega, B) > T \) and \( K_3 = K_3(\varepsilon, \omega) > 0 \) such that for all \( t > T_4 \) and \( K > K_3 \), we obtain
\[ \frac{c_2}{K} \int_t^T e^{-F(s-t)} \|\nabla g(s, \theta, \omega, g_0(\theta, \omega))\|^2 ds \leq \frac{\varepsilon}{4}. \]  
(4.37)

Now, denoting \( T^* = \max\{T_1, T_2, T_3, T_4\}, \ K^* = \max\{K_1, K_2, K_3\} \)
and combining with 4.33, 4.34, 4.36 and 4.37, we get
\[ \int_{|x| > K^*} \rho \left( \frac{|x|^2}{K^2} \right) |g(t, \theta, \omega, g_0(\theta, \omega))(x)|^2 dx \leq \varepsilon, \]  
(4.39)
which completes the proof. \( \square \)

5. Upper semicontinuity of random attractors

In this section, we prove the upper semicontinuity of random attractors for the Gray–Scott system defined on \( \mathbb{R}^3 \) when the stochastic perturbations approach zero. To this end, we first establish the convergence of solutions of problem (3.7) when \( \sigma \to 0 \), and then show that the union of all perturbed random attractors is precompact in \( \mathcal{H} \).

To indicate the dependence of solutions on \( \sigma \), we write the solutions of problem (3.7) as \( g_\sigma = (u_\sigma, v_\sigma, w_\sigma) \), and the corresponding cocycle as \( \phi_\sigma \). By Lemma 4.1, Lemma 4.4 and Lemma 4.5 (see also [18]), we have known that system (3.7) has a \( \mathcal{D} \)-pullback random attractor \( \mathcal{A}_\sigma \subset \mathcal{D} \). Given \( 0 < \sigma \leq 1 \), it follows from Lemma 4.1 that, for every \( B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D} \) and \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), there exists \( T = T(\varepsilon) > 0 \), independent of \( \sigma \), such that for all \( t \geq T \),
\[ ||\phi_\sigma(t, \theta, \omega, B(\theta, \omega))|| \leq \tilde{R}_\sigma(t, \omega), \]  
where \( \tilde{R}_\sigma(t, \omega) \) is given by
\[ \tilde{R}_\sigma^2(t, \omega) = 1 + c_1 \|f\|^2 e^{-2(\sigma_0(t-\tau))} + c_3 \|e^{-4(\sigma_0(t-\tau))} \| e^{4(\sigma_0(t-\tau))} \| e^{-6(\sigma_0(t-\tau))} \| e^{6(\sigma_0(t-\tau))} \| e^{-8(\sigma_0(t-\tau))} \| e^{8(\sigma_0(t-\tau))} \| \]  
(5.1)
Denote by
\[ M_0 = \{ M_0(t, \omega) = \{ g \in H : \|g\| \leq \tilde{R}_0(t, \omega) \}, \quad t \in \mathbb{R}, \quad \omega \in \Omega \} \] (5.3)
and
\[ M_0 = M_0(\omega) = \{ g \in H : \|g\| \leq \tilde{R}_0 \}, \] (5.4)
where \( \tilde{R}_0 \) is the constant
\[ \tilde{R}_0 = \sqrt{1 + \left( \frac{C_1}{F} + c_4 \right) \|f\|^2 + c_4 \left( 1 + \frac{C_2 e^F}{F} \right) \|f\|_4^4}. \] (5.5)

It is evident that Lemma 4.1 implies that \( M_0 \) is a \( D_0 \)-pullback absorbing set of \( \phi_0 \) in \( H \) (see also in [20]). Given \( t \in \mathbb{R} \) and \( \omega \in \Omega \), denote by
\[ B(t, \omega) = \{ g \in H : \|g\| \leq \mathcal{N}(t, \omega) \}, \] (5.6)
where \( \mathcal{N}(t, \omega) \) is given by
\[ \mathcal{N}^2(t, \omega) = 1 + c_1 \|f\|^2 e^{20|\omega|} \int_{-\infty}^{0} e^{\max(0, |s|)} ds + c_2 \|f\|^4 e^{20|\omega|} \int_{-\infty}^{0} e^{s+1} e^{20|\omega|} ds + c_3 \|f\|^4 e^{20|\omega|} \int_{-\infty}^{0} e^{20|\omega|} ds. \] (5.7)

By (5.3),(5.4) and (5.6),(5.7) we obtain \( M_0(t, \omega) \subseteq B(t, \omega) \) for all \( \sigma \in (0, 1) \), \( t \in \mathbb{R} \) and \( \omega \in \Omega \). This implies that for every \( t \in \mathbb{R} \) and \( \omega \in \Omega \),
\[ \bigcup_{0 < \sigma < 1} A_\sigma(t, \omega) \subseteq \bigcup_{0 < \sigma < 1} M_0(t, \omega) \subseteq B(t, \omega). \] (5.8)

By Lemma 4.4 we find that, for every \( \sigma \in (0, 1) \), \( t \in \mathbb{R} \) and \( \omega \in \Omega \), there exists \( T_1 = T_1(\omega) > 0 \) such that for all \( t \geq T_1 
\[ \|\phi_0(t, \theta_{-t} \omega, A_\sigma(t, \theta_{-t} \omega))\|_H \leq \mathcal{N}(t, \omega). \] (5.9)

By (5.9) and the invariant of \( A_\sigma \), we get that, for every \( t \in \mathbb{R} \) and \( \omega \in \Omega \),
\[ \|g\| \leq \mathcal{N}(t, \omega) \quad \text{for all } g \in A_\sigma(t, \omega) \quad \text{with } 0 < \sigma \leq 1. \] (5.10)

We will use (5.10) to prove the precompactness of the union of \( A_\sigma \) in \( H \) for \( 0 < \sigma \leq 1 \).

**Lemma 5.1.** Let \( f \in H \cap \mathbb{U} \) holds. Then for every \( t \in \mathbb{R} \) and \( \omega \in \Omega \), the union \( \bigcup_{0 < \sigma < 1} A_\sigma(t, \omega) \) is precompact in \( H \).

**Proof.** Given \( \eta > 0 \), we need to show that the set \( \bigcup_{0 < \sigma < 1} A_\sigma(t, \omega) \) has a finite covering of balls of radii less than \( \eta \). Let \( C \) be a positive number and denote by
\[ Q_\sigma = \{ x \in \mathbb{R}^n : |x| < C \} \quad \text{and} \quad Q_\sigma^c = \mathbb{R}^n \setminus Q_\sigma. \]

Let \( B(t, \omega) \) be the random set given in (5.6). By Lemma 4.5. we get that, given \( \eta > 0 \) and \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), there exist \( T = T(\omega, \eta) > 0 \) and \( C = C(\omega, \eta) > 0 \) (independent of \( \sigma \)) such that for all \( t \geq T \) and \( g_{\sigma,\sigma}(\theta_{-t} \omega, \theta_{-t} \omega) \in M_\sigma(t, \theta_{-t} \omega) \),
\[ \int_{|x| > C} |g_{\sigma,\sigma}(\theta_{-t} \omega, \theta_{-t} \omega)(x)|^2 dx \leq \frac{\eta^2}{16}. \] (5.11)

By (5.8), \( g_{\sigma,\sigma}(\theta_{-t} \omega, \theta_{-t} \omega) \in A_\sigma(t, \theta_{-t} \omega) \) implies that \( g_{\sigma,\sigma}(\theta_{-t} \omega) \in B(t, \theta_{-t} \omega) \). Therefore it follows from (5.11) that, for every \( 0 < \sigma \leq 1 \), \( \mathbb{P} \)-a.e. \( \omega \in \Omega, t \geq T \) and \( g_{\sigma,\sigma}(\theta_{-t} \omega) \in A_\sigma(t, \theta_{-t} \omega) \),
\[ \int_{|x| \geq C} |g(x)|^2 dx \leq \frac{\eta^2}{16}, \quad \forall g \in \bigcup_{0 < \sigma < 1} A_\sigma(t, \omega), \] (5.12)

which along with the invariance of \( A_\sigma \) shows that, for \( \mathbb{P} \)-a.e. \( \omega \in \Omega, t \in \mathbb{R}, \)
\[ \|g\|^{2} \leq \frac{\eta^2}{16}, \quad \forall g \in \bigcup_{0 < \sigma < 1} A_\sigma(t, \omega). \] (5.13)

On the other hand, (5.10) implies that the set \( \bigcup_{0 < \sigma < 1} A_\sigma(t, \omega) \) is bounded in \( H^1(Q_\sigma) \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega, t \in \mathbb{R} \). By the compactness of embedding \( H^1(Q_\sigma) \subseteq [L^2(Q_\sigma)] \) we find that, for given \( \eta \), the set \( \bigcup_{0 < \sigma < 1} A_\sigma(t, \omega) \) has a finite covering of balls of radii
less than \( \frac{1}{2} \) in \([L^2(Q_0)]^3\). This along with (5.13) shows that \( \bigcup_{0 < \eta < 1} A_{\sigma}(t, \omega) \) has a finite covering of balls of radii less than \( \eta \) in \( \mathbb{H} \). \( \square \)

Next, we will show that, as \( \sigma \to 0 \), the solutions of the perturbed system (3.7) converges to the limiting deterministic system.

\[
\begin{align*}
\frac{dg}{dt} &= Ag + \Lambda(g, \omega) + f(x), \quad t > 0, \\
g(0, x) &= g_0(x) = \bar{g}_0(x), \quad x \in \mathbb{R}^n, \\
\Lambda(g, \omega) &= \left( -(F + k)u + u^2 v - Gu^3 + Nw \right, \\
&\quad \left. -Fv - u^2 v + Gu^3 \right) \\
&\quad \left. -(F + N)v + ku \right),
\end{align*}
\]

(5.14)

where

Lemma 5.2. Suppose \( f \in \mathcal{H} \cap \mathcal{V} \cap \mathcal{U} \) holds. Let \( g_\sigma \) and \( g \) be the solutions of (3.7) and (5.14) with initial conditions \( g_{0, \sigma} \) and \( g_0 \), respectively. Then, for \( \omega \in \mathcal{H}, T > 0 \) and \( \epsilon \in [0, 1] \), there exists a positive constant \( \sigma_0 = \sigma_0(\omega, T, \epsilon) \) such that for all \( \sigma \leq \sigma_0 \) and \( t \in [0, T] \),

\[
\|g_\sigma(t, \omega, g_{0, \sigma}) - g(t, g_0)\|^2 \leq M\|g_{0, \sigma} - g_0\|^2 + M\epsilon((\|f\|^2 + \|g_{0, \sigma}\|^2 + \|g_{0, \sigma}\|_4^6),
\]

where \( M \) is a positive constant independent of \( \omega, \epsilon \) and \( \sigma \).

Proof. Let \( (k_1, k_2, k_3) = (u_\sigma - u, v_\sigma - v, W_\sigma - W) \), here \( W_\sigma - W = \frac{1}{2}(w_\sigma - w) \) and \( \kappa = g_\sigma - g = (u_\sigma - u, v_\sigma - v, W_\sigma - W) \). We use (4.1)–(4.3) to replace system (3.7) and the related deterministic equations to replace system (5.14). Then we have

\[
\begin{align*}
\frac{\partial k_1}{\partial t} &= d_1 \Delta k_1 - (F + k)k_1 + e^{2\sigma \omega(t)}u_\sigma^2 v_\sigma - Ge^{2\sigma \omega(t)}u_\sigma^3 + kK_3 + (e^{-\sigma \omega(t)} - 1)f_1 - u^2 v + Gu^3, \\
\frac{\partial k_2}{\partial t} &= d_2 \Delta k_2 - Fk_2 - e^{2\sigma \omega(t)}u_\sigma^2 v_\sigma + Ge^{2\sigma \omega(t)}u_\sigma^3 + (e^{-\sigma \omega(t)} - 1)f_2 + u^2 v - Gu^3, \\
\mu \frac{\partial k_3}{\partial t} &= d_3 \Delta k_3 - (\mu F + k)k_3 + kK_4 + (e^{-\sigma \omega(t)} - 1)f_3.
\end{align*}
\]

(5.15)–(5.17)

Given \( \tau \in \mathbb{R}, \omega \in \mathcal{H}, T > 0 \) and \( \epsilon \in [0, 1] \), since \( \omega \) is continuous on \( \mathbb{R} \), we obtain that there exists \( \sigma_0 = \sigma_0(\tau, \omega, T, \epsilon) > 0 \) such that for all \( \sigma \leq \sigma_0(\tau, \omega, T, \epsilon) \) and \( t \in [\tau, \tau + T] \),

\[
|e^{\sigma \omega(t)} - 1| + |e^{-\sigma \omega(t)} - 1| + |e^{2\sigma \omega(t)} - 1| + |e^{-2\sigma \omega(t)} - 1| < \epsilon.
\]

(5.18)

Multiplying (5.15)–(5.17) with \( k_1, k_2 \) and \( k_3 \) and integrating over \( \mathbb{R} \), respectively, we get

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|k_1\|^2 + d_1 \|
abla k_1\|^2 + (F + k)\|k_1\|^2 &= \int_{\mathbb{R}^n} \left( e^{2\sigma \omega(t)}u_\sigma^2 v_\sigma - Ge^{2\sigma \omega(t)}u_\sigma^3 + kK_3 + (e^{-\sigma \omega(t)} - 1)f_1 - u^2 v + Gu^3 \right) k_1 dx, \\
\frac{1}{2} \frac{d}{dt} \|k_2\|^2 + d_2 \|
abla k_2\|^2 + F\|k_2\|^2 &= \int_{\mathbb{R}^n} \left( -e^{2\sigma \omega(t)}u_\sigma^2 v_\sigma + Ge^{2\sigma \omega(t)}u_\sigma^3 + (e^{-\sigma \omega(t)} - 1)f_2 + u^2 v - Gu^3 \right) k_2 dx, \\
\mu \frac{d}{dt} \|k_3\|^2 + d_3 \|
abla k_3\|^2 + (\mu F + k)\|k_3\|^2 &= \int_{\mathbb{R}^n} \left( kK_4 + (e^{-\sigma \omega(t)} - 1)f_3 \right) k_3 dx.
\end{align*}
\]

(5.19)

Summing up the three equalities together, it yields that

\[
\begin{align*}
\frac{d}{dt} \left( \|k_1\|^2 + \|k_2\|^2 + \mu \|k_3\|^2 + 2F\|k_1\|^2 + \|k_2\|^2 + \mu \|k_3\|^2 \right) + 2d_0 \left( \|
abla k_1\|^2 + \|
abla k_2\|^2 + \mu \|
abla k_3\|^2 \right) \\
&\leq 2\epsilon \int_{\mathbb{R}^n} \left( |u_\sigma^2 v_\sigma| + |G_\sigma| \right) (\|k_1\| + \|k_2\|) dx + \int_{\mathbb{R}^n} \left( |u_\sigma^2 v_\sigma - u^2 v| + |Gu_\sigma^3 - Gu^3| \right) (\|k_1\| + \|k_2\|) dx \\
&\quad + 2\epsilon \int_{\mathbb{R}^n} \left( |f_1 k_1| + |f_2 k_2| + |f_3 k_3| \right) dx.
\end{align*}
\]

(5.20)
where $c_s$ is a constant depends on $d_0$, $G$, $\mu$ and the Gagliardo-Nirenberg inequality constant. By (5.19) and (5.20), we obtain

$$
\frac{d}{dt} \left( \|K_1\|^2 + \|K_2\|^2 + \mu \|K_3\|^2 \right) \leq c_b \left( \|g\|^2 + \|g_0\|^2 + \|g_{0,\sigma}\|^2 + \|g_{0,\sigma}^2\|^2 \right) \left( \|K_1\|^2 + \|K_2\|^2 + \mu \|K_3\|^2 \right)
+ \frac{3c^2}{f} \left\| g_{0,\sigma}^2 \right\|^6 + \frac{3c^2}{2f} \left\| f \right\|^2.
$$

(5.21)

For all $\sigma \in [0, \sigma_0)$, due to Gronwall lemma and let $c_b = c_{b_{\max}} \left( \frac{1}{2} \right)$ and $c_b = c_{b_{\min}} \left( \frac{1}{2} \right)$ combine (5.21) with Lemma 4.1, Proposition 4.2 and Proposition 4.3, it yields that for $t \in [0, T]$,

$$
\|K(t)\|^2 \leq \left( \|K_0\|^2 + c_b c_9 \int_0^T \|g(s, \omega, g_{0,\sigma}(\omega))\|^6 ds \right) e^{c_b c_9 T} + c_b c_6 \left\| f \right\|^2.
$$

(5.22)

where $Q_0 = Q(\sigma, \|g_0\|, \|g_0^2\|, \|g_{0,\sigma}\|, \|g_{0,\sigma}^2\|, \|g_{0,\sigma}^3\|, \|g_{0,\sigma}^4\|, \|g_{0,\sigma}^5\|, \|g_{0,\sigma}^6\|, \|g_{0,\sigma}^7\|)$ and $Q_0 = Q(\sigma, \|g_0\|, \|g_0^2\|, \|g_{0,\sigma}\|, \|g_{0,\sigma}^2\|, \|g_{0,\sigma}^3\|, \|g_{0,\sigma}^4\|, \|g_{0,\sigma}^5\|, \|g_{0,\sigma}^6\|, \|g_{0,\sigma}^7\|) \leq Q_1 < +\infty, \quad \sigma \in [0, 1],$

where $Q_1$ is the $Q$ when $\sigma = 1$ and all the $\omega$ or $-\omega$ are replaced with $|\omega|$. Now, from (5.22), we find that

$$
\|g_{0,\sigma}(t, \omega, g_{0,\sigma}(\omega))\| - \|g_{0,\sigma}(t, \omega, 0)\| \leq \left( \|g_{0,\sigma} - g_0\|^2 \right) e^{Q_1 t} + \left( \frac{c_2 c_9}{f} \right) \int_0^T e^{6Q_0(s)} ds + \int_0^T e^{5Q_2(s)} ds + \|c_0 T \left\| f \right\|^2 \right) e^{Q_1 t}.
$$

(5.23)

which finishes the proof. $\Box$

We now in the position to present the upper semicontinuity of pullback attractors for the stochastic system (1.1).

**Theorem 5.3.** Provided that $f \in \mathcal{H} \cap \mathcal{V} \cap \mathcal{U}$. Then for every $t \in \mathbb{R}$ and $\omega \in \Omega$,

$$
\lim_{\sigma \to 0} D_{\sigma}(A_\sigma(t, \omega), A_\sigma(t)) = 0.
$$

(5.24)

**Proof.** Let $M_\sigma$ and $M_0$ be the families of subsets of $\mathcal{H}$ given by (5.3) and (5.4), respectively. Then we know $M_\sigma$ is a $\mathcal{D}$-pullback absorbing set of $\varphi_\sigma$, and $M_0$ is a $\mathcal{D}$-pullback absorbing set of $\varphi_0$ in $\mathcal{H}$. By (5.2)–(5.5) we find that, for every $t \in \mathbb{R}$ and $\omega \in \Omega$,

$$
\lim_{\sigma \to 0} \|M_\sigma(t, \omega)\| = \limsup_{\sigma \to 0} \|\mathcal{R}_\sigma(t, \omega)\| = \mathcal{R}_0 = \|M_0(t)\|.
$$

(5.25)

By taking a sequence $\sigma_n \to 0$ and $g_{0,n} \to 0$ in $\mathcal{H}$. By Lemma 5.2 we get, for every $s \in \mathbb{R}^+$, $t \in \mathbb{R}$ and $\omega \in \Omega$,

$$
\varphi_{\sigma_n}(s, t, \omega, g_{0,n}) \to \varphi(s, t, g_0) \quad \text{in} \quad \mathcal{H}.
$$

(5.26)

From (5.25) and (5.26), we see that $\varphi_\sigma$ and $\varphi_0$ satisfy conditions (2.1) and (2.2). On the other hand, by Lemma 5.1 we have that $A_\sigma$ also satisfies (2.3) and (2.4). Thus (5.24) follows from Proposition 2.7 immediately. $\Box$

**References**


