Automorphisms of Partially Commutative Groups II: Combinatorial Subgroups

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1 Introduction

Partially commutative groups are a class of groups widely studied on account both of their intrinisically rich structure and their natural appearance in many diverse branches of mathematics and computer science (for properties of these groups see [7] or [20].) It is therefore natural that the pace study of their automorphism groups should be gaining momentum, as it has recently.

A partially commutative group (also known to various authors as a right-angled Artin group, a trace group, a semi-free group or a graph group) is a group given by a finite presentation \( \langle X | R \rangle \), where \( X \) is the vertex set of a simple graph \( \Gamma \) and \( R \) is the set consisting of precisely those commutators \([x, y]\) of elements of \( X \) such that \( x \) and \( y \) are joined by an edge of \( \Gamma \). (A simple graph is one without multiple edges or self-incident vertices. Our convention is that \([x, y] = x^{-1}y^{-1}xy\).

Initial work by Servatius [35] and Laurence [28] established a finite generating set for the automorphism group of a partially commutative group. In a resurgence of interest over the last few years considerably more has been discovered: for example, Bux, Charney, Crisp and Vogtmann [8, 10, 5] have shown that these groups are virtually torsion-free and have finite virtual cohomological dimension and Day has shown how peak reduction techniques may be used on certain subsets of the generators and using this technique given a presentation for the automorphism group [14]. Moreover these groups have a very rich subgroup structure: Guttierrez, Piggot and Ruane [26] have constructed a semidirect product decomposition for the more general case of automorphism groups of graph products of groups. Duncan, Remeslennikov and Kazachkov [18] describe several arithmetic subgroups of the automorphism group of a partially commutative group; different arithmetic subgroups have been found by Noskov [32]. Under certain conditions on \( \Gamma \) Charney and Vogtmann have shown [11] that the Tits alternative holds for the outer automorphism group of \( G(\Gamma) \) and moreover Day [13] has shown that in all cases this group contains either a finite-index nilpotent subgroup or a nonabelian free subgroup. Minasyan has shown [31] that partially commutative groups are conjugacy separable, from which (loc. cit.) it follows that their outer automorphism groups are residually finite. By reduction to the compressed word problem in \( G(\Gamma) \), Lohrey and Schleimer have shown that the word problem in \( \text{Aut}(G(\Gamma)) \) has polynomial time complexity [29]. Charney and Faber [9], and subsequently Day [15], have studied automorphism groups of partially commutative groups associated to random graphs, of Erdős-Rényi type, and found bounds on the edge probabilities so that, with probability tending to one as the number of vertices tends to \( \infty \), such groups have finite outer automorphism groups.
In this paper we shall continue the investigation of [18] into the structure of the automorphism group and its subgroups. We shall introduce several standard automorphisms of a partially commutative group and describe how an arbitrary automorphism may be decomposed as a product of these standard automorphisms. This will reduce the study of the automorphism group to the study of subgroups generated by particular types of standard automorphism. We shall then define subgroups of a geometric character and use these to analyse the group structure. Note that if \( R \) is the ring of integers or a field of characteristic 0 and \( G \) is a partially commutative group in the class of 2-nilpotent \( R \)-groups, the structure of \( \text{Aut}(G) \) has been completely described, by Remeslennikov and Treier [34]; and decomposes as an extension of an Abelian group by a subgroup of \( \text{GL}(n,R) \).

With this program in mind we shall define certain automorphisms, based on the combinatorial properties of the graph \( \Gamma \), and these will form our stock of standard automorphisms. The idea here is to emulate the theory of automorphisms of algebraic and Chevalley groups. There is extensive literature on abstract isomorphisms of the classical linear groups and algebraic groups, over fields and special classes of rings, in which the fundamental results are theorems on splitting of arbitrary automorphisms into special automorphisms (such as algebraic, semialgebraic, simple, central, etc.) [3, 24] and representations of the group of automorphisms as products of the corresponding subgroups. Similarly in the case of Chevalley groups splitting theorems have been established. Steinberg [36] and Humphreys [27] established such results for Chevalley groups over fields and Bunina [4] has defined several special types of automorphism (Central, Ring, Inner and Graph automorphism) and shown that, if \( G \) is a Chevalley group over a commutative local ring (subject to certain restrictions) then an arbitrary automorphism of \( G \) decomposes as a product of such automorphisms. Moreover similar results have been obtained for Kac-Moody groups (see [6] and the references therein).

In [18] we obtained analogous theorems which we extend in this work. We use the orthogonalisation operator \( y^\perp \) and a closure operator \( \text{cl}(Y) \) both defined on subsets \( Y \subseteq X \) in [17]. The closure operator \( \text{cl} \) defines a lattice of “closed” subsets \( L(X) \) of \( X \) and the results of [18] were obtained by considering the action of automorphisms on this lattice. In this paper we consider a similar lattice \( K \) of “admissible” subsets of \( X \) and the action of automorphisms on \( K \). We consider the following subgroups of the automorphism group \( \text{Aut}(G) \) of the partially commutative group \( G \).

- The subgroup \( \text{Aut}(\Gamma) \) of automorphisms induced by isomorphisms of the graph \( \Gamma \).
- The subgroup \( \text{Aut}^{\text{comp}}(\Gamma) \) of \( \text{Aut}(G) \), which is isomorphic to the iso-
morphism group of the graph $\Gamma^{\text{comp}}$, the compressed graph of $\Gamma$.

- The subgroup $\text{Conj}(G)$ of basis-conjugating automorphisms.
- The subgroup $\text{St}(K)$, elements of which stabilise subgroups generated by subsets $A$, where $A$ is an element of the lattice $K$.
- The subgroup $\text{St}^{\text{conj}}(K)$, elements of which map each subgroup $\langle A \rangle$, where $A \in K$, to $\langle A \rangle^{g_A}$, for some $g_A \in G$.
- Subgroups $\text{Conj}_V(G), \text{Conj}_N(G), \text{Conj}_A(G), \text{Conj}_I(G), \text{Conj}_S(G)$ and $\text{Conj}_C(G)$ of $\text{Conj}(G)$.

(Several of these groups are well-known: some are defined for example in [28] and others in [18].)

The first step in our decomposition of $\text{Aut}(G)$ is to separate out the automorphisms induced by isomorphisms of the compressed graph.

**Theorem 4.2.** The group $\text{Aut}(G)$ can be decomposed into the canonical semidirect product of the subgroup $\text{St}^{\text{conj}}(K)$ and the finite subgroup $\text{Aut}^{\text{comp}}(\Gamma)$, i.e.

$$\text{Aut}(G) = \text{St}^{\text{conj}}(K) \rtimes \text{Aut}^{\text{comp}}(\Gamma).$$

This theorem reduces the problem of studying $\text{Aut}(G(\Gamma))$ to the study of the group $\text{St}^{\text{conj}}(K)$.

We may also decompose the automorphism group using the connected components of $\Gamma$. If $\Gamma$ has connected components $\Gamma_1, \ldots, \Gamma_n$ then the partially commutative group determined by $\Gamma$ is the free product of those determined by the $\Gamma_i$. The group of automorphisms of free product of groups has been completely described (from the point of view of generators an defining relations) in papers [21, 22, 23, 12]. We specialise these results to the case under consideration to give generators and relations for the full automorphism group in terms of presentations for the automorphism groups of the factors.

However here we encounter the first of two main obstructions to identifying the structure of $\text{Aut}(G(\Gamma))$. The problem arises when there are isolated vertices in the graph $\Gamma$ (vertices of valency zero). In this case the automorphism group does not have an natural semidirect product decomposition in terms of the automorphism groups of the factors. Nonetheless, in the special case where there are no isolated vertices the quoted results give the following theorem, where $\text{LInn}_{\text{ext}}$ is defined in Definition 3.20, $\Gamma$ has connected components $\Gamma_1, \ldots, \Gamma_n$ and $G_i = G(\Gamma_i)$. 


Theorem 3.26 (cf. [12], Theorem C]). Suppose that no component of $\Gamma$ is an isolated vertex. Define $\tilde{G} = G_1 \times \cdots \times G_n$ and $\text{FR}(G) = \langle \text{LInn}_{\text{ext}} \rangle$. Then $\text{FR}(G)$ is the kernel of the canonical map from $\text{Aut}(G)$ to $\text{Aut}(\tilde{G})$. Moreover $\text{FR}(G)$ has a normal series

$$1 < P_{n-1} < \cdots < P_2 < \text{FR}(G)$$

such that, setting $\text{FR}_i(G) = \text{FR}(G)/P_i$,

(a) $\text{FR}(G) = P_i \rtimes \text{FR}_i(G)$,

(b) $\text{FR}_i(G) = \text{FR}(G_1 \star \cdots \star G_i)$ and

(c) all the $P_i$ are finitely generated.

The last theorem reduces analysis of the structure of $G(\Gamma)$ for the case when $\Gamma = \Gamma_{\text{big}}$ to analysis of $\text{Aut}(G(\Gamma_i))$, $i = 1, \ldots, n$, and of the Fouxe-Rabinovitch core $\text{FR}(G)$.

In the light of these results we may, when expedient, reduce to study of $\text{St}^{\text{conj}}(K)$ where $\Gamma$ is a connected graph. First of all we have

Theorem 4.3. The subgroup $\text{Conj}_N(G)$ of normal conjugating automorphisms is a normal in $\text{St}^{\text{conj}}(G)$ and therefore in $\text{Conj}(G)$.

The next step might appear to be to give an affirmative answer to the following question.

Question 4.6. Let $\Gamma$ be a connected graph. Is $\text{St}^{\text{conj}}(K) = \text{St}(K) \text{Conj}_N(G)$?

However as examples show this is not the case and turns out to be a second major obstruction to the description of the structure of $\text{Aut}(G)$. On the other hand, if there is no occurrence of a vertex $y$ dominating a vertex $x$ in $\Gamma$ (see Definition 3.32), then we obtain a clear description of the structure of $\text{St}^{\text{conj}}(K)$ in terms of $\text{Conj}(G)$ and the stabiliser $\text{St}(L)$ of the lattice of closed sets, studied in detail in [18]. Also in this case $\text{Conj}(G) = \text{Conj}_V(G) = \text{Conj}_N(G)$.

Theorem 4.9. The following are equivalent for a graph $\Gamma$.

(i) $G$ has no dominating vertices.

(ii) $\text{St}^{\text{conj}}(K) = \text{Conj}_N(G) \rtimes \text{St}(L)$.

(iii) $\text{St}^{\text{conj}}(K) = \text{Conj}(G) \rtimes \text{St}(L)$.

(iv) $\text{St}^{\text{conj}}(K) = \text{Conj}(G) \rtimes \text{St}(K)$. 

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Therefore, in the case where there are no dominating vertices the structure of \(\text{Aut}(G)\) is determined by the structure of \(\text{Conj}_N(G)\) and \(\text{St}(L)\) and, as we have shown in [18], \(\text{St}(L)\) is an arithmetic group for which we have a complete structural decomposition.

We conclude by establishing conditions under which \(\text{Aut}(G) = \text{Conj}(G) \text{St}(K)\), even though there are dominated vertices and this product is not semi-direct. In Section 4 we introduce balanced graphs, which include those without dominated vertices, and prove the following theorem.

**Theorem 4.16.** Let \(\Gamma\) be a connected graph and \(G = G(\Gamma)\). Then \(\text{Aut}^*(G) = \text{St}(K) \text{Conj}(G)\) if and only if \(\Gamma\) is a balanced graph.

Therefore in many cases the structure of \(\text{Aut}(G)\) is determined by the structure of \(\text{St}(K)\), \(\text{Conj}(G)\) and \(\text{Conj}_N(G)\) or \(\text{Conj}_V(G)\). In this paper we establish information about structure of the automorphism group in the simplest case, leaving the case where there are dominated vertices to a later paper.
2 Preliminaries

Graph will mean undirected, finite, simple graph throughout this paper. Let \( \Gamma \) be a finite, undirected, simple graph. Let \( X = V(\Gamma) = \{x_1, \ldots, x_n\} \) be the set of vertices of \( \Gamma \) and let \( F(X) \) be the free group on \( X \). Let

\[
R = \{[x_i, x_j] \in F(X) \mid x_i, x_j \in X \text{ and there is an edge of } \Gamma \text{ joining } x_i \text{ to } x_j\}.
\]

We define the partially commutative group with (commutation) graph \( \Gamma \) to be the group \( G(\Gamma) \) with presentation \( \langle X \mid R \rangle \). When the underlying graph is clear from the context we write simply \( G \).

Denote by \( \lg(g) \) the minimum of the lengths of words that represent the element \( g \). If \( w \) is a word representing \( g \) and \( w \) has length \( \lg(g) \) we call \( w \) a minimal form for \( g \). When the meaning is clear we shall say that \( w \) is a minimal element of \( G \) when we mean that \( w \) is a minimal form of an element of \( G \). We say that \( w \in G \) is cyclically minimal if and only if

\[
\lg(g^{-1}wg) \geq \lg(w)
\]

for every \( g \in G \). We write \( u \circ w \) to express the fact that \( \lg(uw) = \lg(u) + \lg(w) \), where \( u, w \in G \). We shall refer to divisors and the greatest divisor of a word \( w \) with respect to a subset \( Y \subseteq X \), as defined in [20]. Let \( u \) and \( w \) be elements of \( G \). We say that \( u \) is a left (right) divisor of \( w \) if there exists \( v \in G \) such that \( w = u \circ v (w = v \circ u) \). We order the set of all left (right) divisors of a word \( w \) as follows. We say that \( u_2 \) is greater than \( u_1 \) if and only if \( u_1 \) left (right) divides \( u_2 \). It is shown in [20] that, for any \( w \in G \) and \( Y \subseteq X \), there exists a unique maximal left divisor of \( w \) which belongs to the subgroup \( \langle Y \rangle < G \) which is called the greatest left divisor \( \text{gd}_L(w) \) of \( w \) in \( Y \). The greatest right divisor of \( w \) in \( Y \) is defined analogously. We omit the indices when no ambiguity occurs. The following lemma will be useful.

**Lemma 2.1.** Let \( x, y \in X \) and \( f, g \in G \) such that \( [x, y] = [x^f, y^g] = 1 \) and

1. \( x^f = f^{-1} \circ x \circ f, \ y^g = g^{-1} \circ y \circ g \) and
2. \( \text{gd}_X(f, g) = 1 \).

Then \( [\alpha(f), \alpha(g)] = [f, y] = [g, x] = 1 \).

**Proof.** From condition (ii) \( fg^{-1} = f \circ g^{-1} \) and we have \( [x^{fg^{-1}}, y] = 1 \). From [DKR2], Lemma 2.3, there exist \( a, b, u \in G \) such that \( fg^{-1} = a \circ b \circ u, \ x^{fg^{-1}} = u^{-1} \circ x \circ u, \ a = x^n \), for some \( n \in \mathbb{Z} \) and \( [b, x] = 1 \). Therefore \( b = x^m \circ c \), for some \( c \in A(x) \), and \( a \circ b = x^k \circ c \), for some \( k \in \mathbb{Z} \). Now \( a \circ b \) is a left divisor of \( f \circ g^{-1} \) and it follows from condition (i) that \( g^{-1} = a \circ b \circ r \),
for some word \( r \). From [DKR2] Corollary 2.6 \([x, y] = [x^u, y] = 1\) implies that \( u \in C(y) \). As \( u \) is a right divisor of \( f \circ g^{-1} \) it follows, from condition (i) again, that \( f = s \circ u \), for some word \( s \). Hence \( f \circ g^{-1} = s \circ u \circ a \circ b \circ r = a \circ b \circ u \). Consideration of word length shows that \( r = s = 1 \) so \( a \circ b \circ u = u \circ a \circ b \). If follows, using condition (ii) once more, that \([\alpha(a \circ b), \alpha(u)] = 1\) and this gives the result.

The subgroup generated by a subset \( Y \subseteq X \) is called a canonical parabolic subgroup of \( G \) and denoted \( G(Y) \). This subgroup is equal to the partially commutative group with commutation graph the full subgraph of \( \Gamma \) generated by \( Y \) [2].

### 2.1 Admissible sets

In this section we give definitions and a summary of the facts we need concerning graphs. For further details the reader is referred to [17].

If \( x \) and \( y \) are vertices of a graph then we define the distance \( d(x, y) \) from \( x \) to \( y \) to be the minimum of the lengths of all paths from \( x \) to \( y \) in \( \Gamma \). Given a subset \( Y \) of \( X \) the orthogonal complement of \( Y \) is defined to be

\[
Y\perp = \{ u \in X | d(u, y) \leq 1, \text{ for all } y \in Y \}.
\]

By convention we set \( \emptyset \perp = X \). It is not hard to see that \( Y \subseteq Y\perp\perp \) and \( Y\perp = Y\perp\perp\perp \) [17, Lemma 2.1]. We define the closure of \( Y \) to be \( \text{cl}(Y) = Y\perp\perp \). The closure operator in \( \Gamma \) satisfies, among others, the properties that \( Y \subseteq \text{cl}(Y) \), \( \text{cl}(Y\perp) = Y\perp \) and \( \text{cl}(\text{cl}(Y)) = \text{cl}(Y) \) [17, Lemma 2.4]. Moreover if \( Y_1 \subseteq Y_2 \subseteq X \) then \( \text{cl}(Y_1) \subseteq \text{cl}(Y_2) \).

**Definition 2.2.** A subset \( Y \) of \( X \) is called closed (with respect to \( \Gamma \)) if \( Y = \text{cl}(Y) \). Denote by \( L = L(\Gamma) \) the set of all closed subsets of \( X \).

For non-empty \( Y \subseteq X \) define \( a(Y) = \cap_{y \in Y} (y\perp)\perp \). Also define \( a(\emptyset) = X \). Subsets of the form \( a(Y) \), where \( Y \subseteq X \) are called admissible sets. Let \( K \) denote the set of admissible subsets of \( X \).

For sets \( U, V \) we sometimes write \( U < V \) when \( U \subseteq V \) and \( U \neq V \). For \( x \neq z \in X \) and subsets \( U \) and \( V \) of \( X \) the following hold.

A subgraph \( S \) of a graph \( \Gamma \) is called a full subgraph if vertices \( a \) and \( b \) of \( S \) are joined by an edge of \( S \) whenever they are joined by an edge of \( \Gamma \). A subset \( Y \) of \( X \) is called a simplex if the full subgraph of \( \Gamma \) with vertices \( Y \) is isomorphic to a complete graph.

**Lemma 2.3.**
(i) If $U \subseteq V$ then $a(V) \subseteq a(U)$.

(ii) $a(U) \cap a(V) = a(U \cup V)$.

(iii) $\text{cl}(x) = a(x) \cap x^\perp$ so $a(x) = \text{cl}(x)$ if and only if $a(x) \subseteq x^\perp$.

(iv) $x^\perp \subseteq a(x)$ if and only if $x^\perp$ generates a complete subgraph.

(v) If $x^\perp \setminus x = z^\perp \setminus z$ then $a(x) = a(z)$.

(vi) If $x^\perp = z^\perp$ then $a(x) = a(z)$.

(vii) If $a(x) = a(z)$ then either $x^\perp = z^\perp$ or $x^\perp \setminus x = z^\perp \setminus z$.

(viii) If $y \in a(x)$ then $a(y) \subseteq a(x)$.

(ix) $a(U) = \cup_{y \in a(U)} a(y)$.

(x) If $\text{cl}(x) = a(x)$ then $\text{cl}(y) = a(y)$, for all $y \in a(x)$.

(xi) If $[x,y] = 1$ then $[G(a(x)), G(a(y))] = 1$.

(xii) $a(y) \subseteq a(x)$ if and only if $x^\perp \setminus x \subseteq y^\perp$.

Proof. Statements (i) to (v) follow directly from the definitions. For (vi) note that in this case $a(x) = (x^\perp \setminus x) = ((x^\perp \setminus \{x,z\}) \cup \{z\})^\perp = (x^\perp \setminus \{x,z\})^\perp \cap z^\perp = (z^\perp \setminus \{x,z\})^\perp \cap x^\perp = a(z)$.

To see (vii) suppose first that $[x,z] = 1$. Then $x \in a(x) = a(z)$ so $x \in a(z) \cap (z^\perp \setminus z)$. Thus $u \in a(x) = a(z)$ implies $u \in x^\perp$, so $a(x) \subseteq x^\perp$. Therefore $\text{cl}(x) = a(x) = a(z) = \text{cl}(z)$; so $x^\perp = z^\perp$. If, on the other hand, $[x,z] \neq 1$ then $z \in a(x)$ and $x \in a(z)$ so $z^\perp \setminus z \subseteq x^\perp$. As $z \neq x$ we have $z^\perp \setminus z \subseteq x^\perp \setminus x$ and by symmetry the reverse inclusion also holds.

For (viii), if $y \in a(x)$ then $[y,u] = 1$ for all $u \in x^\perp \setminus x$. If $y \not\in x^\perp$ then this implies $x^\perp \setminus x \subseteq y^\perp \setminus y$ and so $a(y) \subseteq a(x)$. If $y \in x^\perp$ then $x \in y^\perp$ so $x^\perp \subseteq y^\perp$. Then $z \in a(y)$ implies $[z,v] = 1$, for all $v \in x^\perp \setminus y$ and in particular $[z,x] = 1$; so $z \in x^\perp$. Thus also $[y,z] = 1$ and so $z^\perp \supseteq x^\perp$ and therefore $z \in a(x)$, as required.

Statement (ix) follows from (viii) as if $y \in a(U)$ then $a(y) \subseteq a(U)$.

To see statement (x) observe that $\text{cl}(x)$ is a simplex so if $\text{cl}(x) = a(x)$ and $y \in a(x)$ then $a(y) \subseteq a(x)$ implies that $a(y)$ is a simplex. Therefore $a(y) \subseteq y^\perp$ and the result follows from (iii).

For (xi) suppose that $u \in a(x)$ and $v \in a(y)$. Since $y \in x^\perp \setminus x$ we have $u \in y^\perp$ and similarly $v \in x^\perp$. Since $[u,z] = 1$ for all $z \in x^\perp$, except possibly $x$, it follows that $u$ commutes with $v$ unless $v = x$. However if $v = x$ then, since $v \in (y^\perp \setminus y)^\perp$, $v$ commutes with all elements of $y^\perp$, including $u$. 

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The right to left implication of (xii) is a direct consequence of the definition; since \( x^\perp \setminus x \subseteq y^\perp \) implies \( y \in (x^\perp \setminus x)^\perp \). To see the opposite implication: if \( a(y) \subseteq a(x) \) then \( y \in a(x) \) and if \( u \in x^\perp \setminus x \) then \([u, v] = 1\) for all \( v \in a(x)\), so in particular \([u, y] = 1\).

Let \( \sim_\perp \) be the relation on \( X \) given by \( x \sim_\perp y \) if and only if \( x^\perp = y^\perp \) and \( \sim_\bigcirc \) be the relation given by \( x \sim_\bigcirc y \) if and only if \( x^\perp = y^\perp \setminus y \). These are equivalence relations and the equivalence classes of \( x \) under \( \sim_\perp \) and \( \sim_\bigcirc \) are denoted by \([x]_\perp \) and \([x]_\bigcirc \), respectively. Note that if \(|[x]_\perp| > 1\) then \([x]_\bigcirc = \{x\}\) and the same is true on interchanging \( \perp \) and \( \bigcirc \). Therefore the relation \( \sim \), given by \( x \sim y \) if and only if \( x \sim_\perp y \) or \( x \sim_\bigcirc y \), is an equivalence relation. Denote the equivalence class of \( x \) under \( \sim \) by \([x]\). Then \( x \sim y \) if and only if \( x \sim_\perp y \) or \( x \sim_\bigcirc y \) and \([x] = [x]_\perp \cup [x]_\bigcirc \); that is \( x \sim y \) if and only if \( x^\perp \{x, y\} = y^\perp \{x, y\} \).

**Lemma 2.4.**

(i) \( a(x) = a(z) \) if and only if \( z \in [x] \),

(ii) \([x] = a(x) \setminus (\bigcup \{a(y) | y \in a(x) \text{ and } a(y) < a(x)\})\),

for all \( x, z \in X \).

**Proof.** It suffices to show that \( a(x) = a(z) \) if and only if \( z \in [x] \). From Lemma 2.3 (v), (vi) and (vii), \( z \in [x] \) if and only if \( x^\perp = z^\perp \) or \( x^\perp \setminus x = z^\perp \setminus z \); if and only if \( a(x) = a(z) \).

**Lemma 2.5.**

\([x] = \left\{ \begin{array}{ll} [x]_\perp, & \text{if } a(x) = \text{cl}(x) \\ [x]_\bigcirc, & \text{if } a(x) > \text{cl}(x) \end{array} \right.\)

**Proof.** First suppose \( a(x) = \text{cl}(x) \). Then from Lemma 2.3 (iii) \( a(x) \subseteq x^\perp \). If \( z \in [x] \) then \( z \in [x] \cap x^\perp \) and from Lemma 2.3 (vii) \( z \in [x]_\perp \).

Now suppose that \( \text{cl}(x) < a(x) \). If \( z \in [x] \), \( z \neq x \), and \( z^\perp = x^\bigcirc \) then \( x \in z^\perp \setminus z \), so \( a(x) = a(z) \subseteq x^\perp \), contradicting Lemma 2.3 (iii). Hence, from Lemma 2.3 (vii), \( z^\perp \setminus z = x^\perp \setminus x \), so \( z \in [x]_\bigcirc \).

In view of Lemma 2.3 (ii) above, for an arbitrary subset \( Y \) of \( X \) the \( a \)-\( \text{closure} \) of \( Y \) may be defined to be the admissible set

\( \text{cl}_a(Y) = \cap \{U \subseteq X | Y \subseteq U \text{ and } U = a(V), \text{ for some } V \subseteq X \} \).

Then \( \text{cl}_a(Y) \) is the smallest admissible set containing \( Y \) and \( Y \) is admissible if and only if \( Y = \text{cl}_a(Y) \).
Lemma 2.6. $[x] \cap \text{cl}(x) = [x]_\perp$ and $[x] \cap \text{cl}_a(x) = [x]$.

Proof. If $x^+ = z^\perp$ then $\text{cl}(x) = \text{cl}(z)$ so $z \in \text{cl}(x)$ and therefore $[x]_\perp \subseteq \text{cl}(x)$. Moreover if $x \in a(u)$ then $(u^\perp \setminus u) \subseteq x^+ = z^\perp$, so also $z \in a(u)$. Therefore $[x]_\triangleleft \subseteq \text{cl}_a(x)$. Now if $x^+ \setminus x = z^\perp \setminus z$ the same argument shows that $[x]_\triangleleft \subseteq \text{cl}_a(x)$ (since $u$ commutes with $x$ if and only if it commutes with $z$). However, if $z \in \text{cl}(x)$ then $z \in x^+$ so $z \notin [x]_\triangleleft$ unless $z = x$. \hfill \Box

We have $K \subseteq L$, where $L$ is the set of closed subsets of $X$. The set $K$ with infimum $U \cap V = U \cap V$ and supremum $U \cup V = \text{cl}_a(U \cup V)$ forms a lattice. The lattice $K$ has maximal element $X = \text{cl}_a(X) = a(\emptyset)$ and minimal element $a(X)$.

2.2 Ordering $X$

The next goal is to define a total ordering on $X$ which reflects the structure of the lattice $K$. First define a partial order $<_K$ on $X$ by $x <_K y$ if and only if $a(x) < a(y)$. If $a(x) = a(y)$ we write $x =_K y$. We say $x$ is $K$-minimal if $y \leq_K x$ implies $y =_K x$. The definition of $K$-maximal is then the obvious one. Recall that in [DKR6] the analogous ordering using $L$ instead of $K$ was defined, and the definitions of $L$-minimal and $L$-maximal were also defined using the closure operator instead of the $a$ operator.

Lemma 2.7. An element $x \in X$ is $K$-minimal if and only if $[x] = a(x)$. If $x$ is $K$-minimal then

(i) $x$ is $L$-minimal and

(ii) $\text{cl}(y) = [y]_\perp$, for all $y \in a(x)$.

Proof. The first statement follows immediately from the definitions and Lemma 2.4. For the second suppose $x$ is $K$-minimal. If $[x] = a(x) = \text{cl}(x) = [x]_\perp$ then $\text{cl}(y) = \text{cl}(x)$, for all $y \in \text{cl}(x)$, so $x$ is $L$-minimal and (i) and (ii) hold. If $[x] = a(x) > \text{cl}(x)$ then $[x]_\triangleleft = a(x)$ so $\text{cl}(x) \cap [x]_\triangleleft = \{x\}$, from Lemma 2.3(iii), so again $x$ is $L$-minimal. To see (ii) in this case note that $y \in a(x)$ implies $y \in [x]_\triangleleft = [y]_\triangleleft$ and $\text{cl}(y) = y^\perp \cap a(y) = y^\perp \cap a(x) = y^\perp \cap [y]_\triangleleft = \{y\} = [y]_\perp$. \hfill \Box

Examples show that there may be elements which are $L$-minimal but are not $K$-minimal.

We now define a total order $<_K$ on $X$, which will have the properties that

1. if $x <_K y$ then $y < x$ and
2. if \( z < y < x \) and \( z \in [x] \), then \( y \in [x] \).

Define \( K_X \) to be the subset of \( K \) consisting of sets \( a(x) \), for \( x \in X \). To begin with let

\[
B_0 = \{ Y \in K_X : Y = a(x), \text{ where } x \text{ is } K\text{-minimal} \}.
\]

Suppose that \( B_0 \) has \( k \) elements and choose an ordering \( Y_1 < \cdots < Y_k \) of these elements. If \( i \neq j \) then it follows from Lemma 2.7 that \( Y_i \cap Y_j = \emptyset \).

Therefore we may define the ordering \( \prec \) on \( \cup_{i=1}^k Y_i \) in such a way that if \( x_i \in Y_i \) and \( x_j \in Y_j \) and \( Y_i < Y_j \) then \( x_j < x_i \): merely by choosing an ordering for elements of each \( Y_i \).

We recursively define sets \( B_i \) of elements of \( K_X \), for \( i \geq 0 \), as follows. Assume that we have defined sets \( B_0, \ldots, B_i \), set \( U_i = \cup_{j=0}^i B_j \) and define \( X_i = \{ u \in X : u \in Y, \text{ for some } Y \in U_i \} \). If \( U_i \neq K_X \) define \( B_{i+1} \) by

\[
B_{i+1} = \{ Y = a(x) \in K_X : Y \notin U_i, \text{ and } y \prec_k x \text{ implies that } a(y) \in U_i \}.
\]

If \( U_i \neq K_X \) then \( X_i \neq X \) and \( B_{i+1} \neq \emptyset \). We assume inductively that we have ordered the set \( X_i \) in such a way that if \( 0 \leq a < b \leq i \) then \( x_a \in Y_a \) where \( Y_a \in B_a \) and \( x_b \in Y_b \) where \( Y_b \in B_b \) implies that \( x_b < x_a \). If \( Y = a(x) \in B_{i+1} \) then

\[
[x] = Y \setminus \{ u \in a(y) : y \prec_k x \}
\]

\[
= Y \setminus \{ u \in X_i \}.
\]

Therefore we have defined \( \prec \) on the set \( Y \setminus [x] \). Moreover, if \( Y_1 \neq Y_2 \) and \( Y_1, Y_2 \in B_{i+1} \) then \( Y_1 \cap Y_2 \in K \) so \( z \in Y_1 \cap Y_2 \) implies \( a(z) \subseteq Y_1 \cap Y_2 \). As \( Y_1 \neq Y_2 \) this implies that \( a(z) \) is strictly contained in \( Y_i, i = 1, 2 \). If \( Y_i = a(x_i) \) then \( z <_k x_i \) and so \( z \notin [x_i], i = 1, 2 \). That is, \( [x_1] \cap [x_2] = \emptyset \). Now choose an ordering on the set of elements of \( B_{i+1} \): \( Z_1 < \cdots < Z_k \) say, where \( Z_j = a(x_j) \).

Then \( Z_j \setminus [x_j] \subseteq X_i, j = 1, \ldots, k \). We can extend the total order \( \prec \) on \( X_i \) to

\[
X_{i+1} = X_i \cup \cup_{j=1}^k Z_j = X_i \cup \cup_{j=1}^k [x_j]
\]

as follows. Assume the order has already been extended to \( X_i \cup \cup_{j=1}^{s-1} [x_j] \). Extend the order further by choosing the ordering \( \prec \) on the elements of \( [x_s] \) and then setting its greatest element less than the least element of \( X_i \cup \cup_{j=1}^{s-1} [x_j] \).

At the final stage \( s = k \) and the order on \( X_i \) is extended to \( X_{i+1} \). We continue until \( U_i = K_X \), at which point \( X = X_i \) and we have the required total order on \( X \). Note that, by construction, if \( x, y \in X \) and \( x \prec_k y \) then \( y \prec x \). Also, if \( x \prec y \prec z \) and \( [z] = [x] \) then \( [y] = [x] \). Thus 1 and 2 above hold. If \( a(x) \) belongs to \( B_i \) we shall say that \( x \), \( a(x) \) and \( [x] \) have height \( i \) and write \( h(x) = h(a(x)) = h([x]) = i \).
3 Generators for Aut(G) and Decomposition over Graph Isomorphisms

For the remainder of the paper G will denote a partially commutative group with commutation graph Γ.

3.1 Graph Isomorphisms

For any graph Ω let Isom(Ω) denote the group of graph isomorphisms of Ω to itself. If Ω is labelled then by an isomorphism of Ω we mean a graph isomorphism which preserves labels. We shall use the equivalence ∼ to define a quotient graph of Γ. In [DKR3] it is shown that if $a \in [u]$ and $b \in [v]$ then there is an edge of Γ joining $a$ to $b$ then there is an edge joining $c$ to $d$, for all $c \in [u]$ and $d \in [v]$. Therefore there is a well-defined graph with vertex set consisting of the equivalence classes of ∼ and an edge joining vertices $[u]$ and $[v]$ if and only if there is an edge of Γ joining $u$ and $v$. The resulting graph has no multiple edges but may have loops.

Definition 3.1. The compression of the graph Γ is the labelled graph $\Gamma^{\text{comp}}$ with vertices $X^{\text{comp}} = \{ [v] : v \in X \}$ and an edge joining $[u]$ to $[v]$ if and only if $u$ is joined to $v$ by an edge of Γ. Vertices of $\Gamma^{\text{comp}}$ are labelled as follows. Let $u \in X$ and $|[u]| = d$.

1. If $d = 1$ then $[u]$ is labelled $(1, 1)$.
2. If $d > 1$ and $[u] = [u]_\perp$ then $[u]$ is labelled $(\perp, d)$.
3. If $d > 1$ and $[u] = [u]_\diamond$ then $[u]$ is labelled $(a, d)$.

We shall express each isomorphism $\phi$ of Γ as a product $\phi = \alpha \beta$, where $\alpha$ corresponds to a certain isomorphism of $\Gamma^{\text{comp}}$ and $\beta$ is an isomorphism of Γ which maps $[u]$ to itself, for all $u \in X$. First we make some definitions. If $\Omega$ and $\Omega'$ are graphs without multiple edges and $f$ is a map from $V(\Omega)$ to $V(\Omega')$ then we say that $f$ induces a graph homomorphism if, for all $u, v \in V(\Omega)$, $f(u)$ is joined to $f(v)$ whenever $u$ is joined to $v$. It is easy to see (for details see [DKR3]) that the map from $X$ to $X^{\text{comp}}$ sending $u$ to $[u]$ induces a homomorphism $\text{comp} : \Gamma \to \Gamma^{\text{comp}}$. Every isomorphism $\phi \in \text{Isom}(\Gamma)$ induces a label preserving isomorphism $\phi^{\text{comp}} \in \text{Isom}(\Gamma^{\text{comp}})$ (sending $[u]$ to $[u\phi]$). Thus $\phi \to \phi^{\text{comp}}$ maps $\text{Isom}(\Gamma)$ surjectively to $\text{Isom}(\Gamma^{\text{comp}})$.

For $v \in X$ let $S_{[v]}$ denote the group of permutations of $[v]$; so $S_{[v]}$ is a subgroup of $\text{Isom}(\Gamma)$ isomorphic to the symmetric group of degree $|[v]|$. We then have the following lemma, proved in [18].
Lemma 3.2. \( \text{Isom}(\Gamma) = \left( \prod_{[v] \in \comp \Gamma} S_{[v]} \right) \rtimes \text{Isom}(\Gamma^{\text{comp}}) \).

Now we focus attention on isomorphisms which interchange connected components of \( \Gamma \). First of all we fix notation for these connected components. In the following definition we adopt the convention that, if \( m \) is a non-negative integer and \( \Omega \) is a graph then \( \Omega^m \) denotes the disjoint union of \( m \) copies of \( \Omega \) (and is empty if \( m = 0 \)).

**Definition 3.3.** Let \( \Omega_0 \) denote the graph consisting of a single vertex and no edges. Suppose that there exist pairwise non-isomorphic graphs \( \Omega_1, \ldots, \Omega_d \), such that every connected component of \( \Gamma \) with at least two vertices is isomorphic to \( \Omega_i \), for some \( i \geq 1 \), and that \( d \) is minimal with this property. Then

\[ \Gamma \cong \Omega_0^{m_0} \cup \Omega_1^{m_1} \cup \cdots \cup \Omega_d^{m_d}, \]  

(3.1)

for some \( m_i \in \mathbb{Z} \), with \( m_0 \geq 0 \) and \( m_i \geq 1 \), for \( i \geq 1 \). In this case we say that the right hand side of (3.1) is the isomorphism type of \( \Gamma \).

Suppose that \( \Gamma \) has isomorphism type \( \Omega_0^{m_0} \cup \Omega_1^{m_1} \cup \cdots \cup \Omega_d^{m_d} \). Identify each connected component of \( \Gamma \) with a particular copy of \( \Omega_j \) (to which it is isomorphic) in the disjoint union \( \Omega_j^{m_j} \). For \( 0 \leq j \leq d \) and \( 1 \leq a < b \leq m_j \) there is an isomorphism of \( \Omega_0^{m_0} \cup \Omega_1^{m_1} \cup \cdots \cup \Omega_d^{m_d} \) interchanging the \( a \)th and \( b \)th copy of \( \Omega_j \) and fixing all other connected components. This induces, via the fixed identifications of components of \( \Gamma \), an isomorphism \( \omega_{a,b}^j \) of \( \Gamma \), which interchanges two connected components and leaves all others fixed. The subgroup of \( \text{Isom}(\Gamma) \) generated by \( \{ \omega_{a,b}^j : 1 \leq a < b \leq m_j \} \) is then isomorphic to the symmetric group \( S_{m_j} \). Note that \( \Gamma^{\text{comp}} \) has isomorphism type \( \Omega_0^{n_0} \cup \bigcup_{i=1}^d (\Omega_i^{\text{comp}})^{m_i} \), where \( n_0 = 0 \), if \( m_0 = 0 \) and \( n_0 = 1 \) if \( m_0 > 0 \), so we obtain the following decomposition of \( \text{Isom}(\Gamma^{\text{comp}}) \).

**Lemma 3.4.** Let \( \Gamma \) have isomorphism type given by (3.1). Then

\( \text{Isom}(\Gamma^{\text{comp}}) \cong \prod_{i=1}^d \text{Isom}(\Omega_i^{\text{comp}})^{m_i} \rtimes S_{m_i} \).

**Definition 3.5.** Let \( \text{Aut}(G) \) denote the automorphism group of the partially commutative group \( G \) with commutation graph \( \Gamma \).

An element \( \phi \in \text{Aut}(G) \) is a graph automorphism if the restriction \( \phi|_X \) of \( \phi \) to \( X \) is an element of \( \text{Isom}(\Gamma) \). Denote by \( \text{Aut}(\Gamma) \) the subgroup of \( \text{Aut}(G) \) consisting of graph automorphisms.

The following proposition follows from Lemmas 3.2 and 3.4.

**Proposition 3.6.** Let \( \Gamma \) have isomorphism type given by (3.1). Then

(i) \( \text{Aut}(\Gamma) \cong \left( \prod_{[v] \in \comp \Gamma} S_{[v]} \right) \rtimes \text{Aut}(\Gamma^{\text{comp}}) \) and
(ii) \( \text{Aut}(\Gamma_{\text{comp}}) \cong \prod_{i=1}^{d} \text{Aut}(\Omega_{i_{\text{comp}}})^{m_i} \rtimes S_{m_i} \).

Clearly a presentation for \( \text{Aut}(\Gamma) \) may be constructed from the above decomposition, but we leave details of this to the appendix.

### 3.2 Generators for \( \text{Aut}(G) \)

**Definition 3.7.** Given \( x \in X \) the automorphism of \( G \) mapping \( x \) to \( x^{-1} \) and fixing all other generators is called an inversion and denoted \( \iota_x \). The set of all inversions is denoted \( \text{Inv} = \text{Inv}(G) \).

For fixed \( x,y \in X^{\pm 1} \) an automorphism sending \( x \) to \( xy \) and fixing all elements of \( X^{\pm 1} \) other than \( x^{\pm 1} \) is denoted \( \tau_{x,y} \) and called a transvection. The set of all transvections such that \( y \in X \) is denoted \( \text{Tr} = \text{Tr}(G) \).

If \( \tau_{x,y_i} \in \text{Tr} \), for \( y_i \in X^{\pm 1} \) such that \( w = y_1 \cdots y_n \) is a geodesic word in \( G \) then \( \tau_{x,w} = \tau_{x,y_n} \cdots \tau_{x,y_1} \) is called an composite transvection and the set of all composite transvections is denoted \( \text{Tr}_W(G) \).

For distinct \( x,y \in X \), there exists an element \( \tau_{x^\varepsilon,y^\delta} \in \text{Aut}(G) \) if and only if \( x^{\varepsilon} \setminus x \subseteq y^{\delta} \).

**Definition 3.8.** An automorphism \( \phi \in \text{Aut}(G) \) is called a basis-conjugating automorphism if there exists \( g_x \in G \) such that \( x^\phi = x^{g_x} \), for all \( x \in X \). The subgroup of \( \text{Aut}(G) \) consisting of all basis-conjugating automorphisms is denoted \( \text{Conj}(G) \).

The group of inner automorphisms \( \text{Inn}(G) \) is a normal subgroup of \( \text{Conj}(G) \).

**Definition 3.9.** For \( S \subseteq X \) define \( \Gamma_S \) to be \( \Gamma \setminus S \), the graph obtained from \( \Gamma \) by removing all vertices of \( S \) and all their incident edges.

**Definition 3.10.** Let \( x \in X \), let \( C \) be the vertex set of a connected component of \( \Gamma_{x^\perp} \) and let \( \varepsilon = \pm 1 \). The automorphism \( \alpha_{C,x^\varepsilon} \) given by

\[
  y \mapsto \begin{cases} 
    y^{x^\varepsilon}, & \text{if } y \in C \\
    y, & \text{otherwise}
  \end{cases}
\]

is called an elementary conjugating automorphism of \( \Gamma \). If \( L \) consists of a union \( L = \cup_{i=1}^r C_i \) of connected components \( C_i \) of \( \Gamma_{x^\perp} \) then \( \alpha_{L,x^\varepsilon} = \prod_{i=1}^r \alpha_{C_i,x} \) is called an composite elementary conjugating automorphism. The set of all elementary conjugating automorphisms (over all connected components of \( \Gamma_{x^\perp} \) and all \( x \in X \)) is denoted \( \text{LInn} = \text{LInn}(G) \) and the set of all composite elementary conjugating automorphisms is denoted \( \text{LInn}_W = \text{LInn}_W(G) \).
Theorem 3.11 (Laurence [28]). The group of basis-conjugating automorphisms $\text{Conj}(G)$ is generated by the set $L\text{Inn}(G)$.

In [28] it is shown that $\text{Aut}(G)$ is generated by the following automorphisms.

(i) Generators for the graph isomorphisms $\text{Aut}(\Gamma)$.

(ii) The set of inversions $\text{Inv}$.

(iii) The set of transvections $\text{Tr}$.

(iv) The set of elementary conjugating automorphisms $L\text{Inn}$.

It is worth noting the following corollary of this fact.

Corollary 3.12. If $H$ is a canonical parabolic subgroup of $G$ and $\phi$ is an automorphism of $G$ such that $H\phi \subseteq H$ then $H\phi = H$.

Corresponding to various decompositions of $\text{Aut}(G)$ we shall reduce to proper generating subsets of Laurence’s generators. The first such reduction is the following.

Proposition 3.13. Let $\pi_{\text{comp}}$ be the natural homomorphism from $\text{Aut}(\Gamma)$ onto $\text{Aut}(\Gamma_{\text{comp}})$ and let $\iota_{\text{comp}}$ be a homomorphism from $\text{Aut}(\Gamma_{\text{comp}})$ into $\text{Aut}(\Gamma)$ such that $\iota_{\text{comp}}\pi_{\text{comp}} = 1_{\text{Aut}(\Gamma_{\text{comp}})}$ (both of which exist in the light of Proposition 3.6).

Define $\mathcal{P}_{\text{comp}}$ to be a generating set for the image $(\text{Aut}(\Gamma_{\text{comp}}))\iota_{\text{comp}} \subseteq \text{Aut}(\Gamma)$. Then $\text{Aut}(G)$ is generated by $\text{Inv} \cup \text{Tr} \cup L\text{Inn} \cup \mathcal{P}_{\text{comp}}$.

Proof. To see that these automorphisms generate $\text{Aut}(G)$, we show that every automorphism $\phi \in \text{Aut}(\Gamma)$ belongs to the subgroup generated by $\text{Inv}$, $\text{Tr}$ and $\text{Aut}_{\text{comp}}(\Gamma)$. From Proposition 3.6, $\phi$ may be written as $\phi = \alpha\beta$, where $\alpha \in \prod_{[v] \in X_{\text{comp}}} S_{[v]}$ and $\beta \in \langle \mathcal{P}_{\text{comp}} \rangle$. Hence it is enough to show that $S_{[v]} \subseteq \langle \text{Inv}, \text{Tr} \rangle$. As $[v]$ generates a free or free Abelian subgroup of $G$, for all $x, y \in [v]$ and $\varepsilon = \pm 1$ the transvections $\tau_{x,y}$ and inversions $\iota_x$ and $\iota_y$ are automorphisms of $G$ and belong to $\text{Inv} \cup \text{Tr}$. The permutation $\sigma_{x,y}$ sending $x$ to $y$ and $y$ to $x$ and fixing all other generators can be obtained as a word in these generators; $\sigma_{x,y} = \iota_x \tau_{x,y} \tau_{y,x} \iota_{x^{-1},y}$. As $S_{[v]}$ is generated by such elements it follows that $S_{[v]} \subseteq \langle \text{Inv}, \text{Tr} \rangle$ as required.

$\square$
3.3 Decomposition of \( \text{Aut}(G) \) over Graph Isomorphisms

**Definition 3.14.** Let \( \text{Aut}^*(G) \) denote the subgroup of \( \text{Aut}(G) \) generated by the set \( \mathcal{P}_* = \text{Inv} \cup \text{Tr} \cup \text{LInn} \).

Later we shall show that \( \text{Aut}^*(G) \) has a natural description in terms of the stabiliser of the lattice \( K \). Here we establish what we need in order to obtain an initial decomposition of \( \text{Aut}(G) \) in terms of \( \text{Aut}^\text{comp}(G) \). It is useful to establish the following fact first.

**Lemma 3.15.** Let \( x, y \in X \) and let \( C \) be a connected component of \( \Gamma_{y^\perp} \). If \( a(x) \notin C \cup y^\perp \) and \( a(x) \cap C \neq \emptyset \) then \( y \in a(x) \).

**Proof.** Suppose that \( y \notin a(x) \). Then there exists \( u \in x^\perp \setminus x \) such that \( [y, u] \neq 1 \). Thus \( u \notin y^\perp \) and \( u \in x^\perp \setminus x \); so \( [u, v] = 1 \) for all \( v \in a(x) \). This means that \( a(x) \setminus y^\perp \) is contained in some connected component of \( \Gamma_{y^\perp} \). As \( a(x) \cap C \neq \emptyset \) it follows that \( a(x) \subseteq C \cup y^\perp \), and the result follows.

**Proposition 3.16.** Let \( \phi \in \text{Aut}^*(G) \). Then, for all \( x \in X \), there exists \( f_x \in G \) such that \( G(a(x))^\phi = G(a(x))^{f_x} \).

**Proof.** It suffices to prove the statement in the case where \( \phi \) is a generator of \( \text{Aut}^*(G) \). First consider the case where \( \phi \) is a basis-conjugating automorphism. Suppose then that \( y \in X \), \( C \) is a connected component of \( \Gamma_{y^\perp} \) and that \( \phi = a_{C,y} \).

Now let \( x \in X \). If \( C \cap a(x) = \emptyset \) then \( G(a(x))^\phi = G(a(x)) \), so we may assume that \( C \cap a(x) \neq \emptyset \). If \( a(x) \subseteq C \cup y^\perp \) then \( G(a(x))^\phi = G(a(x))^y \), as required. This leaves the case where \( C \cap a(x) \neq \emptyset \) and \( a(x) \notin C \cup y^\perp \). In this case \( y \in a(x) \), from Lemma 3.15, so \( z\phi \in G(a(x)) \), for all \( z \in G(a(x)) \), and, since \( \phi \) is an automorphism, \( G(a(x))^\phi = G(a(x)) \).

The case where \( \phi \in \text{Inv} \) is straightforward (and \( f_x = 1 \), for all \( x \), in this case). Suppose then that \( \phi \) is a transvection; more precisely let \( y, z \in X \) with \( y^\perp \setminus y \subseteq z^\perp \) and \( \phi = \tau_{y^\perp,z} \), where \( \varepsilon \in \{ \pm 1 \} \). Let \( x \in X \). If \( y \notin a(x) \) then \( \phi \) is the identity on \( G(a(x)) \) so we may assume that \( y \in a(x) \). In this case we have \( z \in \text{cl}(z) \subseteq a(y) \subseteq a(x) \). Hence \( G(a(x))^\phi = G(a(x)) \), as required.

**Remark 3.17.** Note that the proof of this proposition shows that if \( \phi \) is in the subgroup of \( \text{Aut}^*(G) \) generated by \( \text{Inv} \) and \( \text{Tr} \) then \( G(a(x))^\phi = G(a(x)) \), for all \( x \in X \).

**Proposition 3.18.** \( \text{Aut}^*(G) \) is a normal subgroup of \( \text{Aut}(G) \) and the latter decomposes as a semidirect product \( \text{Aut}(G) \cong \text{Aut}^*(G) \rtimes \text{Aut}(\Gamma_{\text{comp}}) \).
Proof. To see that $\text{Aut}^*(G)$ is normal in $\text{Aut}(G)$ it is only necessary to check that $\theta^{-1}\phi\theta \in \text{Aut}^*(G)$ where $\theta \in \text{Aut}(\Gamma)$ and $\phi$ is a generator of $\text{Aut}^*(G)$. It is straightforward to check from the definitions that if $\theta \in \text{Aut}(\Gamma)$ then $\theta$ acts by conjugation on the generators of $\text{Aut}^*(G)$ as follows. If $t_z \in \text{Inv}$ then $t_z^\theta = t_{z\theta}$. If $x \in X^{\pm 1}$, $y \in Y$ and $\tau_{x,y} \in \text{Tr}$ then $\tau_{x,y}^\theta = \tau_{x\theta,y\theta}$. If $\alpha_{C,y}$ is an elementary conjugating automorphism then $\alpha_{C,y}^\theta = \alpha_{C\theta,y\theta}$.

From Remark 3.17 above, if $\phi$ is a transvection or inversion then $a(x)\phi \in G(a(x))$, for all $x \in X$. Moreover, if $\psi$ is a basis-conjugating automorphism then $x\psi$ belongs to a conjugate of $G(a(x))$, by some element of $G$. Therefore, for any $x \in X$ and any $\theta \in \text{Aut}^*(G)$ the image $g = x\theta$ is a product of conjugates of elements of $G(a(x))$. This means that the exponent sum of $y \in X$ in $g$ is zero unless $y \in a(x)$.

We claim that if $\theta \in \text{Aut}(\Gamma)$ then, for all $x \in X$, $x\theta = y$ and $a(x)\theta = a(y)\theta$, for some $y \in X$ with $h(y) = h(x)$. To see this note that $\theta \in \text{Aut}(\Gamma)$ implies $\theta|_X$ is in $\text{Isom}(\Gamma)$ so $a(x)\theta = a(x\theta)$. If $x\theta = y$ then it follows from Lemma 2.4 and induction on the height $h(x)$ of $x$ that $h(y) = h(x)$, and the claim follows.

Now if $\theta$ is a non-trivial element of $\langle \mathcal{P}_{\text{comp}} \rangle$ there is $x \in X$ such that $[x]\phi = [y]$ with $[y] \neq [x]$. As $h(x) = h(y)$ this means that $[y] \cap a(x) = \emptyset$ and the exponent sum of $y$ in $x\phi$ is non-zero; with $y \notin a(x)$. Combining this with the above we have $\text{Aut}^*(G) \cap \langle \mathcal{P}_{\text{comp}} \rangle = 1$. As $\text{Aut}(G)$ is generated by $\text{Inv} \cup \text{Tr} \cup \text{LInn} \cup \mathcal{P}_{\text{comp}}$ and $\iota_{\text{comp}}$ is an isomorphism of $\text{Aut}(\Gamma^{\text{comp}})$ to $\langle \mathcal{P}_{\text{comp}} \rangle$ it follows not only that $\text{Aut}^*(G)$ is normal but that $\text{Aut}(G)$ decomposes as a semidirect product, as claimed. \hfill $\square$

### 3.4 Decomposition over Connected Components

In this section we use the theory of automorphisms of free products developed in [21, 22], [23] and [12] to give a presentation of $\text{Aut}(G)$, and describe its structure, in terms of automorphisms of the groups corresponding to the connected components of $\Gamma$. A presentation for the automorphism group of a free product in terms of presentations for automorphism groups of the factors is given in [21, 22] and reformulated in [23]. Using the latter we construct a presentation for $\text{Aut}(G)$ appropriate to our particular setting.

We use a subset of the set of generators of [28] described in Section 3.2 above. First we make a more explicit choice of the injective homomorphism $\iota_{\text{comp}}$ defined in Proposition 3.13.

**Definition 3.19.** Fix a total order $<$ on $X$. An element $\phi \in \text{Aut}(\Gamma)$ is said to be admissible if the following condition holds, for all $u \in X$. If $v \in [u]$ then $u < v$ if and only if $\phi(u) < \phi(v)$. Let $\text{Aut}^{\text{comp}}(\Gamma)$ denote the
subgroup of admissible isomorphisms of $\text{Aut}(\Gamma)$. Then the restriction of $\pi_{\text{comp}}$ to $\text{Aut}^{\text{comp}}(\Gamma)$ is an isomorphism onto $\text{Aut}(\Gamma^{\text{comp}})$ and we may take $\iota_{\text{comp}}$ to be the inverse of this restriction.

When $\Omega$ is a connected component of $\Gamma$ we shall abuse notation by regarding $\text{Aut}^{\text{comp}}(\Omega)$ as a subgroup of $\text{Aut}(\Gamma)$.

**Definition 3.20.** Let $\Gamma$ have isomorphism type given by (3.1) and let $\Gamma_{i,j}$ be the connected components of $\Gamma$, where $\Gamma_{i,j} \cong \Omega_i$, for $1 \leq j \leq m_i$ and $0 \leq i \leq d$. Let $X_{i,j}$ be the vertex set of $\Gamma_{i,j}$, $S = \{(0,j) : 1 \leq j \leq m_0\}$, $X_S = \{x_s : s \in S\}$, the set of isolated vertices of $\Gamma$, and $J = \{(i,j) : 1 \leq i \leq d, 1 \leq j \leq m_i\}$.

(i) Define the following sets of graph automorphisms.

(a) For $1 \leq i \leq d$, let $P_{\text{comp},i}(G)$ be a generating set for $\text{Aut}^{\text{comp}}(\Gamma_{i,1})$.

(b) For $0 \leq i \leq d$ and $1 \leq a < b \leq m_i$, the group automorphism induced by the graph isomorphism $\omega_{i,a,b}$ (see Section 3.1) is also called $\omega_{i,a,b}$ and we define $P_{\Pi,i}(G) = \{\omega_{i,a,b}^j \mid 1 \leq a < b \leq m_i\} \subseteq \text{Aut}(\Gamma)$.

(c) $P_{\text{comp}}(G) = P_{\Pi,0}(G) \cup \bigcup_{i=1}^d (P_{\text{comp},i}(G) \cup P_{\Pi,i}(G))$.

(ii) Define the following sets of automorphisms which preserve connected components of $\Gamma$ and fix all elements of $G(\Gamma_{i,j})$ when $j \neq 1$: let

(a) $\text{Inv}_{\text{int}}(G) = \{\iota_x \in \text{Inv} \mid x \in X_{i,1}, 0 \leq i \leq d\}$;

(b) $\text{Tr}_{\text{int}}(G) = \{\tau_{x,y} \in \text{Tr} \mid x, y \in X_{i,1}^\pm, 1 \leq i \leq d\}$;

(c) $\text{LInn}_{\text{int}}(G) = \{\alpha_{C,x} \in \text{LInn} \mid x \in X_{i,1}^{\pm 1}, C \subseteq X_{i,1}, 1 \leq i \leq d\}$;

(d) $P_{\text{int}}(G) = \text{Inv}_{\text{int}} \cup \text{Tr}_{\text{int}} \cup \text{LInn}_{\text{int}}$.

(iii) Define the following sets of automorphisms which do not preserve connected components of $\Gamma$: let

(a) $\text{Tr}_{\text{ext}}(G) = \{\tau_{x,y} \in \text{Tr} \mid x \in X_{i,1}^\pm, y \in X\}$;

(b) $\text{LInn}_{\text{ext}}(G) = \{\alpha_{C,y} \in \text{LInn} \mid C = X_j, y \in X_{k,1}^\pm, j \in J, k \in S \cup J, k \neq j\}$;

(c) $P_{\text{ext}}(G) = \text{Tr}_{\text{ext}}(G) \cup \text{LInn}_{\text{ext}}(G)$.

Finally define $P(G) = P_{\text{comp}}(G) \cup P_{\text{int}}(G) \cup P_{\text{ext}}(G)$.
When the group in question is clear from the context we drop the argument \( G \) from these definitions, writing \( P \) for \( \mathcal{P}(G) \), and so on.

**Remark 3.21.**  1. The conditions on \( \text{Tr} \) imply that \( \text{Tr}_{\text{ext}} \) is empty unless there exists an isolated vertex \( x \) of \( \Gamma \), in which case there is an automorphism \( \tau_{x,y} \in \text{Tr}_{\text{ext}} \), for all \( y \in X \setminus \{x\} \).

2. If \( s \in S \) and \( X_s = \{x\} \) then \( \text{Aut}(G) \) contains the automorphism \( \alpha_{X_s,z} \), but also contains \( \tau_{z,\pm 1,z} \), for all \( z \in X_{\pm 1}, z \neq x \pm 1 \); and \( \alpha_{X_s,z} = \tau_{z,\pm 1,z} \), so we make the restriction \( j \in J \); i.e. \( |X_j| \geq 2 \), in Definition 3.20 (iiiib) above.

3. In [23] elements of \( \text{Tr}_{\text{ext}} \cup \text{LInn}_{\text{ext}} \) are called Whitehead automorphisms, and the group generated by \( \bigcup_{i=0}^{d} P_{\Pi,i} \) is denoted \( \Pi \) and called the group of permutation automorphisms.

**Proposition 3.22.** (i) The set \( P_{\text{comp}} \) generates \( \text{Aut}^{\text{comp}}(\Gamma) \).

(ii) The set \( P \) generates \( \text{Aut}(G) \).

**Proof.** (i) follows directly from the definitions and Proposition 3.6.

In the light of Proposition 3.13 and (i), to prove (ii) it suffices to show that every automorphism in \( \text{Inv} \cup \text{Tr} \cup \text{LInn} \) belongs to the subgroup generated by \( P \). For all \( i,j \) with \( 1 \leq i \leq d \) and \( 1 < j \leq m_i \) the automorphism \( \omega_{i,j} \) belongs to \( P_{\Pi,i} \subseteq P_{\text{comp}} \) and for all \( x_j \in X_{i,j} \) there is \( x_1 \in X_{i,1} \) such that \( x_1 = x_j\omega_{i,j} \). Then we have \( t_{x_j} = (\omega_{i,j})^{-1}t_{x_1}\omega_{i,j}^{-1} \), \( \tau_{x_j,y_j} = (\omega_{i,j})^{-1}\tau_{x_1,y_1}\omega_{i,j}^{-1} \), and \( \alpha_{C_j,x_j} = (\omega_{i,j})^{-1}\alpha_{C_1,y_1}\omega_{i,j}^{-1} \), where \( y_j = y_1\omega_{i,j} \) and \( C_j = C_1\omega_{i,j} \). Hence \( \text{Tr}(G_j) \cup \text{LInn}(G_j) \) is contained in the subgroup generated by \( P \), and the claim holds.

**Definition 3.23.** Let \( \text{Aut}^\Pi(\Gamma_i) = \langle P_{\Pi,i} \rangle \) and choose presentations

\[
\langle P_{\Pi,i} \mid R_{\Pi,i} \rangle \text{ for } \text{Aut}^\Pi(\Gamma_i) \cong S_{m_i}, 0 \leq i \leq d, \text{ and } \\
\langle P_{\text{comp},i} \mid R_{\text{comp},i} \rangle \text{ for } \text{Aut}^{\text{comp}}(\Gamma_i), 1 \leq i \leq d.
\]

(For notational convenience set \( P_{\text{comp},0} = R_{\text{comp},0} = \emptyset \).) For \( 0 \leq i \leq d \), let

\[ P_i = P_{\text{int}}(G_i) \cup P_{\text{comp},i}, \]

so \( P_i \) is a set of generators for \( \text{Aut}(G_i) \). Choose a presentation \( \langle P_i \mid R_i \rangle \) for \( \text{Aut}(G_i) \) such that \( R_i \supseteq R_{\Pi,i} \cup R_{\text{comp},i} \). (This may be done using a standard construction and is carried out in detail in the appendix.)

**Proposition 3.24.** \( \text{Aut}(G) \) has a presentation \( \langle P \mid R \rangle \), where \( P \) is given in Definition 3.20 and \( R \) is defined in Definition 3.25 below.
Definition 3.25. Let \( \mathcal{R} \) be the union of the following sets.

(i) \( \mathcal{R}_{\Pi,i} \), for \( i = 0, \ldots, d \).

(ii) \( \mathcal{R}_i \), for \( i = 0, \ldots, d \).

(iii) The sets \( \mathcal{W}_i = \{ [p^\omega, q] | p \in \mathcal{P}_i, \omega \in \mathcal{P}^*_i \} \cup \{ p^{\omega}p^{-1} | p \in \mathcal{P}_i, \omega \in \mathcal{P}^*_i \} \), for \( i = 0, \ldots, d \), where

\[
\mathcal{P}^*_i \cdot \mathcal{P}_i \cdot \mathcal{P}^*_i = \{ \omega^i_{0,a} \in \mathcal{P}_i | 1 < a < m_i \} \quad \text{and} \quad \mathcal{P}^*_i \cdot \mathcal{P}_i \cdot \mathcal{P}^*_i = \{ \omega^i_{1,b} \in \mathcal{P}_i | 1 < b < m_i \}.
\]

(iv) \( \mathcal{D} = \{ [p, q] | p \in \mathcal{P}_i \cup \mathcal{P}_{\Pi,i}, q \in \mathcal{P}_j \cup \mathcal{P}_{\Pi,j}, \text{ with } 0 \leq i < j \leq d \} \).

(v) The set of relations \( \mathcal{R}_i \), for \( i=1, \ldots 11 \), below.

Note that if \( \tau_{x,y} \in \text{Tr}_{\text{ext}} \) then necessarily \( x \in X_i^{\pm 1} \), for some \( i \in S \) and \( y \in X_j \) for some \( j \neq i \). Similarly, if \( \alpha_{C,x} \in \text{LInn}_{\text{ext}} \) then \( x \in X_i^{\pm 1} \), for some \( i \in J \) and \( C = \Gamma_j \), for some \( j \neq i \). In the relations below all the transvections \( \tau \), and elementary conjugating automorphisms \( \alpha \), that are mentioned explicitly, belong to \( \text{Tr}_{\text{ext}} \) or \( \text{LInn}_{\text{ext}} \), respectively. The relations are defined for all \( u, v, x, y, z \in X^{\pm 1} \) and \( i, j, k, l \in \{ 1, \ldots, n \} \), for which the preceding conditions hold. For notational simplicity, for all transvections \( \tau_{x^\varepsilon,y} \in \text{Tr}_{\text{ext}} \), we add the transvection \( \tau_{x^\varepsilon,y}^{-1} \) to the set \( \text{Tr}_{\text{ext}} \). (In \( \text{Aut}(G) \) \( \tau_{x^\varepsilon,y}^{-1} = \tau_{x^{-\varepsilon},y} \) and this equality now becomes a relator.) We write \( \gamma_y \) for the inner automorphism conjugating every element by \( y \); so \( \gamma_y \) is shorthand for the product over all connected components \( C \) of \( \Gamma_y^{-} \) of the automorphisms \( \alpha_{C,y} \).

\( \mathcal{R}_1 \). \( [\tau_{x,y}, \tau_{u,v}] = 1 \) if either

(i) \( u = x^{-1} \) or

(ii) \( x \neq u^{\varepsilon}, x \neq v^{\varepsilon} \) and \( y \neq u^{\varepsilon} \).

\( \mathcal{R}_2 \). \( [\tau_{x,y}^{-1}, \tau_{u,x}] = \tau_{u,y}^{-1} \) (\( x \neq u^{\varepsilon}, x \neq y^{\varepsilon} \) and \( y \neq u^{\varepsilon} \)).

\( \mathcal{R}_3 \). \( \tau_{x,y}^{-1} \tau_{y,x} \tau_{x,y} = \omega_{i,j}^{0}t_{y} \), where \( x \in X_{0,i} \subseteq X_{S} \) and \( y \in X_{0,j} \subseteq X_{S} \), \( i \neq j \).

\( \mathcal{R}_4 \). \( [\alpha_{\Gamma_i,x}, \alpha_{\Gamma_j,y}] = 1 \), if \( x \notin X_j, y \notin X_i, i \neq j, i, j \in J \).

\( \mathcal{R}_5 \). \( [\alpha_{\Gamma_i,x}, \alpha_{\Gamma_j,y} \alpha_{\Gamma_y}] = 1 \), \( i \neq j \) and either \( x \in X_i \) or \( x, y \in X_k, [x,y] = 1, k \neq i, j, i, j, k \in J \).

\( \mathcal{R}_6 \). \( [\tau_{x,y}, \alpha_{\Gamma_i,z}] = 1 \), \( y \notin X_l, x \neq z^{\varepsilon} \).
\[ \tau^{-1}_{x,y} \alpha^{-1}_{\Gamma_i,x} = \alpha_{\Gamma_i,y}, \text{ if } y \notin X_i. \]

\[ \tau_{x,y} \alpha_{\Gamma_i,z} \tau_{x,z} = 1, \text{ if } y \in X_i. \]

\[ \tau_{x,y} \alpha_{\Gamma_i,x} = \gamma_{y^{-1}} \alpha_{\Gamma_i,x} \tau^{-1}_{x^{-1},y}, \text{ if } y \in X_i. \]

\[ \tau_{x,y} \alpha_{\Gamma_i,x} = \gamma_{y^{-1}} \alpha_{\Gamma_i,x} \tau^{-1}_{x^{-1},y}, \text{ if } y \in X_i. \]

\[ \tau_{x,y} \alpha_{\Gamma_i,x} = \gamma_{y^{-1}} \alpha_{\Gamma_i,x} \tau^{-1}_{x^{-1},y}, \text{ if } y \in X_i. \]

\[ \tau_{x,y} \alpha_{\Gamma_i,x} = \gamma_{y^{-1}} \alpha_{\Gamma_i,x} \tau^{-1}_{x^{-1},y}, \text{ if } y \in X_i. \]

R10. Let \( x \in X^\pm_1 \), \( j \in J \) and write \( C = \Gamma_j \). Let \( u \in X \) and \( y, z \in X_i \), \( y \neq z \), such that \([y, z] = 1\). Then

(i) \( \tau_{x,y} \alpha_{\Gamma_i,x} = \gamma_{y^{-1}} \alpha_{\Gamma_i,x} \tau^{-1}_{x^{-1},y}, \text{ if } y \in X_i. \)

(ii) \( \alpha^{-1}_{\Gamma_j,u} = \alpha_{\Gamma_j,u}^{-1} \) and if \( i \neq j \) then \( \alpha_{C,y} \alpha_{C,z} = \alpha_{C,z} \alpha_{C,y}. \)

R11. Let \( y \in X \) and \( \theta \in P_{\text{comp}} \cup P_{\text{int}} \) and let \( y_1^\varepsilon_1 \cdots y_k^\varepsilon_k \) be a geodesic word representing \( y\theta \), with \( y_i \in X \) and \( \varepsilon_i = \pm 1 \).

(i) Let \( x, z \in X^\pm_1 \) such that \( z = x\theta \). Then

\[ \tau_{x,y} \theta = \theta \tau_{z,y} \tau_{x,z}^\varepsilon_1 \cdots \tau_{z,y}^\varepsilon_k \]

for all \( \tau_{x,y} \in \text{Tr}_{\text{ext}} \).

(ii) Let \( j \in J \) and \( \Gamma_j \theta = \Gamma_i \). Then, with \( C = \Gamma_j \) and \( D = \Gamma_i \),

\[ \alpha_{C,y} \theta = \theta \alpha_{D,y}^\varepsilon_1 \cdots \alpha_{D,y}^\varepsilon_k \]

for all \( \alpha_{C,y} \in \text{LInn}_{\text{ext}} \).

The proof of this theorem is left to the appendix.

In the case where \( m_0 = 0 \), that is, no component of \( \Gamma \) is an isolated vertex, the set \( \text{Tr}_{\text{ext}} \) is empty and the the relations of this presentation reduce to the union of \( \cup_{i=1}^d \mathcal{R}_{\mathcal{H}_i}, \cup_{i=1}^d \mathcal{R}_i, \mathcal{W}, \mathcal{D}, \mathcal{R}_4, \mathcal{R}_5, \mathcal{R}_{10}(ii) \) and \( \mathcal{R}_{11}(ii) \). In this case \( \text{Aut}(G) \) decomposes as a semidirect product \( \text{Aut}(G) = \langle \text{LInn}_{\text{ext}} \rangle \rtimes (\mathcal{P}_{\text{comp}} \cup \mathcal{P}_{\text{int}}) \); and \( \langle \text{LInn}_{\text{ext}} \rangle \) is called the Fouxe-Rabinowitsch kernel and denoted \( \text{FR}(G) \) (see [12] for more details). The structure of \( \text{Aut}(G) \) is then given by the following (special case of a) theorem from [12].

**Theorem 3.26** (cf. [12], Theorem C). Suppose that no component of \( \Gamma \) is an isolated vertex. Define \( \tilde{G} = G_1 \times \cdots \times G_n \) and \( \text{FR}(G) = \langle \text{LInn}_{\text{ext}} \rangle \). Then \( \text{FR}(G) \) is the kernel of the canonical map from \( \text{Aut}(G) \) to \( \text{Aut}(\tilde{G}) \). Moreover \( \text{FR}(G) \) has a normal series

\[ 1 < P_{n-1} < \cdots < P_2 < \text{FR}(G) \]

such that, setting \( \text{FR}_i(G) = \text{FR}(G)/P_i \),

\[ 22 \]
(a) $\text{FR}(G) = P_i \times \text{FR}_i(G)$,
(b) $\text{FR}_i(G) = \text{FR}(G_1 \ast \cdots \ast G_i)$ and
(c) all the $P_i$ are finitely generated.

In the light of the results of this section we may when necessary reduce to the study of $\text{Aut}(\Gamma)$ where $\Gamma$ is a connected graph. In particular, to give an explicit presentation of $\text{Aut}(G)$ it remains to determine the sets $R_i$ of Definition 3.23.

3.5 Conjugating Automorphisms

The subgroup of basis-conjugating automorphisms, which we consider here, plays an important role in the structure of $\text{Aut}(G)$ and has a rich and complex structure, even in the case of free groups: see for example [30, 25, 33, 1]. We shall consider several subgroups of the basis-conjugating automorphisms $\text{Conj}(G) = \langle \text{LInn}(G) \rangle$ which we now define.

Let $x \in X$ and, as usual, denote by $\Gamma_x$ the full subgraph of $\Gamma$ generated by $X \setminus \{x\}$ and note that if $y \in X$ lies in a connected component $C$ of $\Gamma_x$ then $y^x \subseteq C \cup \{x\}$.

Definition 3.27. Let $x \in X$, let $\Omega$ be the connected component of $\Gamma$ containing $x$ and let $C \subseteq \Omega$ be a connected component of $\Gamma_x$. Then the automorphism $\beta_{C,x}$ given by

$$y \beta_{C,x} = \begin{cases} y^x, & \text{if } y \in C \text{ or } y \notin \Omega \\ y, & \text{otherwise} \end{cases}$$

is called an aggregate conjugating automorphism. The subgroup of all aggregate conjugating automorphisms is denoted $\text{Conj}_A(G)$.

Definition 3.28. An element $\phi \in \text{Conj}(G)$ is said to be a normal conjugating automorphism if, for every element $x \in X$ there exists $f_x \in G$ such that $y \phi = y^{f_x}$, for all $y \in a(x)$. The subgroup of all normal conjugating automorphisms is denoted $\text{Conj}_N(G)$.

Definition 3.29. An element $\phi \in \text{Conj}(G)$ is said to be a vertex conjugating automorphism if, for every element $x \in X$ there exists $f_x \in G$ such that $y \phi = y^{f_x}$, for all $y \in [x]$. The subgroup of all vertex conjugating automorphisms is denoted $\text{Conj}_V(G)$.

If $\Gamma$ is compressed ($\Gamma = \Gamma^{\text{comp}}$) then $\text{Conj}_V(G) = \text{Conj}(G)$. 

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Definition 3.30. An elementary conjugating automorphism $\alpha_{C,u}$, where $u = x^{\pm 1}$, for some $x \in X$ is called an elementary singular conjugating automorphism if $C = \{y\}$, for some $y \in X$, and the set of all such elementary conjugating automorphisms is denoted $\text{LInn}_S = \text{LInn}_S(G)$. The subgroup of $\text{Conj}(G)$ generated by $\text{LInn}_S(G)$ is called singular and denoted $\text{Conj}_S(G)$.

Definition 3.31. Let $\text{Tr}_\perp = \{\tau_{x^\pm,y} \in \text{Tr} | x \in y^\perp, \varepsilon = \pm 1\}$ and $\text{Tr}_\perp = \{\tau_{x^\pm,y} \in \text{Tr} | x \notin y^\perp, \varepsilon = \pm 1\}$. For $\tau_{x^\pm,y} \in \text{Tr}$ write $\hat{\tau}_{x^\pm,y} = \tau_{x^{-\varepsilon},y}$.

Definition 3.32. If $x$ and $y$ are vertices of $X$ such that $x^\perp \cap y^\perp = y^\perp \setminus y$ then we say that $x$ dominates $y$. The set of all vertices dominated by $x$ is denoted $\text{Dom}(\text{Dom}(\Gamma)) = \{u \in X | x \text{ dominates } u\}$. The set of all dominated vertices is denoted $\text{Dom}(\text{Dom}(\Gamma)) = \bigcup_{x \in X} \text{Dom}(x)$. For fixed $y \in X$ the set of all $x$ such that $y \in \text{Dom}(x)$ and $[y] \neq [x]$ is the outer admissible set of $y$, denoted $\text{out}(y)$.

From the definition and Lemma 2.3 (xii) it follows that $x$ dominates $y$ if and only if $[x,y] \neq 1$ and $a(x) \subseteq a(y)$. Thus $\text{out}(y) = \{x \in a(y) : x \notin [y] \cup y^\perp\}$.

If $\alpha_{C,x} \in \text{LInn}_S(G)$ then $C = \{y\}$ is a connected component of $\Gamma_x$ so $y^\perp \setminus y \subseteq x^\perp$ and $x \notin x^\perp$. Therefore $\tau_{y,x} \in \text{Tr}_\perp$ and $\alpha_{C,x} = \tau_{y,x} \hat{\tau}_{y,x}$. Hence $\text{Conj}_S$ is the subgroup of $\text{Aut}(G)$ generated by the set $\{\tau_{y,x} \hat{\tau}_{y,x} | x \text{ dominates } y\}$.

Definition 3.33. Let $x,u \in X$ such that $x$ dominates $u$ and let $[u] \setminus \{x\} = \{v_1, \ldots, v_n\}$. The conjugating automorphism

$$\alpha_{[u],x} = \prod_{i=1}^n \alpha_{\{v_i\},x}$$

is called a basic collected conjugating automorphism and the set of all basic collected conjugating automorphisms is denoted $\text{LInn}_C = \text{LInn}_C(G)$. The subgroup of $\text{Conj}(G)$ generated by $\text{LInn}_c(G)$ is denoted $\text{Conj}_C = \text{Conj}_C(G)$.

Definition 3.34. The set of regular elementary conjugating automorphisms is $\text{LInn}_R = \text{LInn}_R(G) = (\text{LInn}(G) \cap \text{Conj}_V(G)) \setminus \text{LInn}_S(G)$. The set of all basic vertex conjugating automorphisms is $\text{LInn}_V = \text{LInn}_V(G) = \text{LInn}_R(G) \cup \text{LInn}_c(G)$.

We record some straightforward properties of these definitions in the following lemma.

Lemma 3.35. Let $\Gamma$ be a connected graph.

(i) $\text{Inn} \leq \text{Conj}_A \leq \text{Conj}_N \leq \text{Conj}_V \leq \text{Conj}$.  

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(ii) $\text{LInn}_V \subseteq \text{Conj}_V$.

(iii) If $\phi \in \text{Conj}_S$ then $x\phi = x^{f_x}$, where $\alpha(f_x) \subseteq a(x)$, for all $x \in X$.

Proof. To see that $\text{Conj}_A \subseteq \text{Conj}_S$ suppose that $u \in X$ and consider the aggregate conjugating automorphism $\beta = \beta_{C,x}$, where $x \in X$, $\Omega$ is the connected component of $\Gamma$ containing $x$ and $C \subseteq \Omega$ is a connected component of $\Gamma_x$. If $x \in u^{-1}u$ then $v\beta = v$, for all $v \in a(u)$, so assume that this is not the case. If $x \in a(u)$ then $x \notin u^{-1}u$ implies that $a(u) \subseteq C' \cup \{x\}$, for some component $C'$ of $\Gamma_x \cap \Omega$, so we may also assume that $x \not\in a(u)$. If $u \notin \Omega$ then, for all $v \in a(u)$, we have $v\beta = v^2$, so we may further assume that $u \in \Omega$.

Now let $v$ and $w$ be distinct elements of $a(u)$ and $r$ be any element of $u^{-1}u$. Then the path $v, r, w$ does not contain $x$; so $v$ and $w$ are either both in $C$ or both outside $C$. Hence $\beta_{C,x}$ either fixes every element of $a(u)$, or acts as conjugation by $x$ on every element of $a(u)$. Thus all elements of $\text{Conj}_A$ are normal. The remainder of the first statement follows immediately from the

definitions.

Statement (ii) follows directly from the definitions and the fact that the
sets $[x]$ partition $X$.

An induction on the length of $\phi$ as a word in the generators $\text{LInn}_S$ establishes (iii). If $\phi$ is trivial there is nothing to be proved, so assume inductively that the result holds for words of length at most $n - 1$ and that $\phi = \phi_0\phi_1$, where $\phi_0$ has length $n - 1$ as a word in $\text{LInn}_S^{\pm1}$ and $\phi_1 \in \text{LInn}_S^{\pm1}$, say $\phi_1 = \alpha_{C,z}$, for some $z \in X^{\pm1}$ and $C = \{y\}$. Then

$x\phi_0 = x^{f_x}$, where $\alpha(f_x) \subseteq a(x)$, for all $x \in X$. Let $x \in X$ and $f_x = y_1 \cdots y_r$, with $y_i \in X^{\pm1}$. Then $f_x\alpha_1 = y_1^{\gamma_1} \cdots y_r^{\gamma_r}$, where $\gamma_i = z$ or $1$. If $\gamma_i = 1$, for all $i$, then $\alpha(f_x) \subseteq a(x)$. If $\gamma_i = z$, for some $i$, then $y_i = y^{\pm1}$; in which case $z^{\pm1} \in a(y)$ and, as $\alpha(f_x) \subseteq a(x)$, it follows that $z^{\pm1} \in a(y) \subseteq a(x)$. Thus in all cases $\alpha(f_x) \subseteq a(x)$. Now $x\phi = (x\phi_1)^{f_x}\phi_1$ and since $x\phi_1 = x^z$ if and only if $x = y^{\pm1}$ it follows that $\alpha(x\phi) \subseteq a(x)$, as required. \qed

We shall use the following definition of Laurence [28].

Definition 3.36. Let $\phi$ be a conjugating automorphism and for each $x \in X$ let $g_x \in G$ be such that $x\phi = g_x^{-1}x \circ g_x$. The length $|\phi|$ of $\phi$ is $\sum_{x \in X} \log(g_x)$.

We shall prove, in Propositions 3.39 and in a subsequent paper, versions of Theorem 3.11 (i.e. Theorem 2.2 of [28]) appropriate to $\text{Conj}_V$ and $\text{Conj}_N$ and to do so make use of Lemma 2.5 and Lemma 2.8 (loc. cit.) which we state here for reference.

Lemma 3.37 ([28][Lemma 2.5 & Lemma 2.8]). Let $\phi$ be a non-trivial element of Conjug and, for each $x \in X$, let $g_x \in G$ such that $x\phi = g_x^{-1}x \circ g_x$. Then

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Proposition 3.39. Let \( \phi \in \text{Conj}_V \) and for each \( y \in X \) let \( g_y \in G \) be such that \( y\phi = g_y^{-1} \circ y \circ g_y \).

(i) If \( [y] = [y]_\perp \) then \( g_u = g_y \), for all \( u \in [y] \).

(ii) If \( [y] = [y]_\perp \) and \( |[y]| \geq 2 \) then there exists \( v \in [y] \) and \( m_y \in \mathbb{Z} \) such that \( g_u = v^m \circ g_v \), for all \( u \in [y]\backslash \{v\} \). Moreover if \( m_y \neq 0 \) then \( v \) is the unique element of \( [y] \) with this property and, setting \( \epsilon = -m_y/|m_y| \), \( S = [y]\backslash \{v\} \) and \( \alpha = \prod_{u \in S} \alpha_{(u),v^\epsilon} \) we have \( \alpha \in \text{Linn}_{C}^{\pm 1} \) and \( |\alpha\phi| < |\phi| \).

Proof. Since \( \phi \in \text{Conj}_V \), for all \( y \in X \), there exists \( f_y \in G \) such that \( u\phi = u^{f_y} \), for all \( u \in [y] \), and we may choose an \( f_y \) of minimal length with this property.

Fix \( y \in X \). Then \( u^{f_y} = u\phi = u^{g_u} \) so \( g_u f^{-1}_y \in C_G(u) \), for all \( u \in [y] \). Therefore there are \( a, b, c \in G \) such that \( g_u = acb, f_y = cob \) and \( g_u f^{-1}_y = a\circ c^{-1} \in C_G(u) \).

As \( g_u \) has no left divisor in \( C_G(u) \) this means that \( a = 1 \) and so \( f_y = c_u \circ g_u \), for \( c = c_u \in C_G(u) \). If \( [y] = [y]_\perp \) then \( C_G(u) = C_G(y) \), for all \( u \in [y] \), so in this case \( g_y = f_y = g_u \), for all \( u \in [y] \).

Assume then that \( [y] = [y]_\perp \), with \( |[y]| \geq 2 \), and let \( u, v \in [y] \), \( v \neq u \), so \( [u, v] \neq 1 \). Suppose \( v \in \alpha(f_y) \). Then \( f_y = c_v \circ g_v = c'_v \circ v^m \circ g_v \), where \( c'_v \in G(v^\perp \backslash v) \) and \( m \in \mathbb{Z} \). Then \( u^{f_y} = u^{v^m \circ g_v} \), since \( v^\perp \backslash v = u^\perp \backslash u \). As \( g_v \) has no left divisor in \( C_G(v) \) and \( [v, u] \neq 1 \) we have \( u^{v^m \circ g_v} = g_v^{-1} \circ v^{-m} \circ u \circ v^m \circ g_v \), so \( g_u = u^m \circ g_v \). By choice of \( f_y \) we have \( c'_v = 1 \), and if \( m \neq 0 \) then no element \( u \in [y] \), \( u \neq v \), can be a left divisor of \( v^m \circ g_v \), so the first statement of (ii) also holds.

Proposition 3.39. \( \text{Conj}_V \) is generated by \( \text{Linn}_V = \text{Linn}_R \cup \text{Linn}_C \) and \( \text{Conj}_V \cap \text{Conj}_S = \text{Conj}_C \).

Proof. That \( \langle \text{Linn}_V \rangle \leq \text{Conj}_V \) is Lemma 3.35 (ii). For the opposite inclusion we use induction on the length of an automorphism \( \phi \) in \( \text{Conj}_V \). If \( |\phi| = 0 \) then \( \phi = 1 \) and there is nothing to prove. Assume that \( |\phi| > 1 \) and that for all conjugating automorphisms \( \psi \) of shorter length \( \psi \in \text{Conj}_V \) implies \( \psi \in \langle \text{Linn}_V \rangle \). If there exists \( y \in X \) such that, in the notation of Lemma 3.38,
$m_y \neq 0$, then it follows from that lemma and induction that $\phi \in \langle L\text{Inn}_V \rangle$, as claimed. Hence we assume that $m_y = 0$, for all $y \in X$. From Lemma 3.37 (i) there exist $x, y \in X, \varepsilon \in \{\pm 1\}$ such that $x\phi = g_x^{-1} \circ x \circ g_x$, $y\phi = g_y^{-1} \circ y \circ g_y$ and $x^\varepsilon g_x$ is a right divisor of $g_y$. Then $y \notin x^\perp$, as otherwise $g_y^{-1}yg_y$ is not reduced, and as $m_y = 0$, for all $u \in [y]$, $u\phi = u^{g_y}$ and so $u \notin x^\perp$.

Let $[y] = \{v_1, \ldots, v_r\}$ and let $C_1, \ldots, C_s$ be the components of $\Gamma_x^\perp$ containing $v_1, \ldots, v_r$. Then, from Lemma 3.37 (ii), $x^\varepsilon g_x$ is a right divisor of $g_c$ for all $c \in C_1 \cup \cdots \cup C_s$. Let $\alpha = \prod_{i=1}^s \alpha_{C_i \backslash x^\varepsilon}$. Then $|\alpha\phi| < |\phi|$. We claim that $\alpha \in \text{Conj}_V$. Suppose not, so there is some $z \in X$ and elements $u, v \in [z]$ such that $u \in C_i$, for some $i$, but $v \notin \cup_{i=1}^s C_i$. This implies that $u^+ \backslash u = v^+ \backslash v \subseteq \alpha u^+ \perp \alpha v^+$ and, as $u \in C_i$ implies $x \notin u^+$, so $x$ dominates $u$. Then $C_i = \{u\}$ so $u \in [y]$ and $[z] = [y] \subseteq \cup_{i=1}^s C_i$, a contradiction. Thus no such $z$ exists and $\alpha \in \text{Conj}_V$.

If $r = 1$ and $|C_1| \geq 2$ then $\alpha \in \text{LInn}_{R}$. If $r = 1$ and $|C_1| = 1$ then $x$ dominates $y$ and $\alpha \in \text{LInn}_{C}$. If $r > 1$ then $x^\perp \supseteq y^+ \perp y$ and $x$ dominates every element of $[y]$. In this case $\alpha \in \text{LInn}_{C}$ again. Hence by induction $\phi \in \langle \text{LInn}_R \cup \text{LInn}_C \rangle$.

Suppose then that $\phi \in \text{Conj}_V \cap \text{Conj}_S$. From Lemma 3.35 (iii) it follows that $a(x) \subseteq a(y) = a(v_i)$; so $x$ dominates $v_i$, for $i = 1, \ldots, r$. Therefore $\alpha \in \text{LInn}_{C} \subseteq \text{Conj}_S$ and, by induction on $|\phi|$ again, $\phi \in \langle \text{LInn}_C \rangle = \text{Conj}_{C}$, as claimed.

To describe the structure of $\text{Conj}_A(G)$ it is convenient to work with outer automorphisms. Denote the group $\text{Aut}(G)/\text{Inn}(G)$ of outer automorphisms by $\text{Out}(G)$ as usual and given a subgroup $B$ of $\text{Aut}(G)$ let $\overline{B}$ denote the group $B \text{Inn}(G)/\text{Inn}(G)$. We write $\overline{\beta}$ for the image of $\beta \in \text{Aut}(G)$ in $\text{Out}(G)$ and $\gamma_x$ for the inner automorphism of $G$ mapping $g$ to $g_x$, for all $g \in G$.

**Proposition 3.40.** Let $G = G(\Gamma)$, where $\Gamma$ is a connected graph. Then $\overline{\text{Conj}}_A(G)$ is torsion-free and $\overline{\text{Conj}}_A(G)$ is free Abelian and a normal subgroup of $\overline{\text{Aut}}^\perp(G)$. Moreover, if $c(x)$ is the number of connected components of $\Gamma_x$, for all $x \in X$, then the $\overline{\text{Conj}}_A(G)$ has rank $\sum_{x \in X} (c(x) - 1)$.

**Proof.** First we show that $\overline{\text{Conj}}_A(G)$ is a free Abelian group. Let $x \in X$ and suppose that $\Gamma_x$ has connected components $C_1, \ldots, C_r$. If $y \in X, y \neq x$, then there is some $i$ such that $y^\perp \subseteq C_i \cup \{x\}$. Assume that $\Gamma_y$ has components $D_1, \ldots, D_s$. We claim that there is a $j$ such that

(i) $D_j \supseteq C_k \cup \{x\}$, for all $k \neq i$, and

(ii) $C_i \supseteq D_k \cup \{y\}$, for all $k \neq j$. 

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To see this choose $j$ such that $x \in D_j$, so $x^\perp \subseteq D_j \cup \{y\}$. Let $u \in C_k$, $k \neq i$. Then there exists a path in $\Gamma$ from $u$ to $x$ and, as $y \in C_i$, this path may be chosen so that none of its vertices is $y$. Hence $u$ and $x$ belong to the same component of $\Gamma_y$. Thus, if $D_j$ is the component of $\Gamma_y$ containing $x$ then $D_j \supseteq C_k \cup \{x\}$, for all $k \neq i$. This shows that the first statement of the claim holds and the second follows by symmetry.

The subgroup $\overline{\text{Conj}}_A(G)$ is generated by the images $\overline{\beta}_{C,x}$ in $\text{Out}(G)$ of aggregate conjugating automorphisms $\beta_{C,x}$, where $x$ ranges over $X$ and $C$ ranges over the connected components of $\Gamma_x$. Let $\overline{\beta}_{C,x}$ and $\overline{\beta}_{D,y}$ be generators of $\overline{\text{Conj}}_A(G)$. If $x = y$ then these two generators commute, so we assume $x \neq y$ and that components $C_i$ and $D_j$ of $\Gamma_x$ and $\Gamma_y$, respectively, have been chosen as in the claim above; so $x \in D_j$ and $y \in C_i$. If $C = C_i$ then let $\beta_1 = \gamma_x \beta_{C,x}$, so $\overline{\beta}_{C,x} = \overline{\beta}_1$ and $\beta_1 = \prod_{C_i \neq C} \beta_{C_i}^{-1} \in \text{Conj}_A(G)$. Otherwise let $\beta_1 = \beta_{C,x}$. In either case $u \beta_1 = u$, for all $u \in C_i$. Similarly we may choose a representative $\beta_2$ of $\overline{\beta}_{D,y}$ such that $u \beta_2 = u$, for all $u \in D_j$. Then, from (i) and (ii) above, it follows that $\beta_1 \beta_2 = \beta_2 \beta_1$ and so $\overline{\beta}_{C,x} \overline{\beta}_{D,y} = \overline{\beta}_{D,y} \overline{\beta}_{C,x}$, and $\overline{\text{Conj}}_A(G)$ is Abelian as claimed.

Now, for $i = 1, \ldots, r$, let $\overline{\beta}_i = \overline{\beta}_{C_i,x} \in \overline{\text{Conj}}_A(G)$. As $\prod_{i=1}^r \overline{\beta}_i = 1$, given any element $\phi \in \overline{\text{Conj}}_A(G)$ we may write $\phi = \overline{\gamma}_0 \overline{\gamma}_1$, where $\overline{\gamma}_0 = \prod_{i=1}^{r-1} \beta_i^{m_i}$, for some $m_i \in \mathbb{Z}$, and $\overline{\gamma}_1$ is a product of generators $\overline{\beta}_{D,y}$, with $y \neq x$. Let $y \in C_i$, where $1 \leq i \leq r - 1$. Then $y \gamma_0 = y^{x^{m_i}}$ and $y \gamma_1 = y^h$, for some $h \in G$ such that $x \notin \alpha(h)$. Also $x \gamma_1 = x^g$, for some $g \in G$ such that $x \notin \alpha(g)$. Then

$$y \gamma_0 \gamma_1 = (y^h)(x^{m_i})^g.$$

If $w$ is a geodesic word representing $h(x^{m_i})^g$ then the exponent sum $|w|_x$ of $x$ in $w$ equals $m_i$; so $y \gamma_0 \gamma_1 = v^{-1} \circ y \circ v$, where $|v|_x = m_i$. For $z \in C_r$ we have $z \gamma_0 \gamma_1 = z \gamma_1 = u^{-1} \circ z \circ u$, for some $u \in G$ such that $|u|_x = 0$. If $\phi = 1$ then $\gamma_0 \gamma_1 \in \text{Inn}(G)$ and so it must be that $m_1 = \cdots = m_{r-1} = 0$. It follows by induction, on the minimal number of generators appearing in a word representing $\phi$, that $\text{Conj}_A(G)$ is free Abelian of rank $\sum_{x \in X} (c(x) - 1)$, as claimed.

To see that $\text{Conj}_A(G)$ is torsion free it suffices to note that $\text{Inn}(G)$ is torsion free; since $\text{Inn}(G) \cong G/Z(G)$, which is a partially commutative group. In fact $G/Z(G) \cong G(\Gamma_Z)$, where $Z$ is the subset of $X$ consisting of vertices connected to all vertices of $\Gamma$ and $\Gamma_Z$ is the full subgraph of $\Gamma$ on $X \setminus Z$. As both $\text{Inn}(G)$ and $\overline{\text{Conj}}_A(G) = \overline{\text{Conj}}_A(G)/\text{Inn}(G)$ are torsion-free, so is $\overline{\text{Conj}}_A(G)$.

To show that $\overline{\text{Conj}}_A(G)$ is normal in $\text{Aut}^*(G)$ we shall show that if $\overline{\beta}_{C,x}$ is an arbitrary generator of $\overline{\text{Conj}}_A(G)$ and $\phi$ is a generator of $\text{Aut}^*(G)$, which is not in $\overline{\text{Conj}}_A(G)$, then $\phi^{-1} \overline{\beta}_{C,x} \phi \in \overline{\text{Conj}}_A(G)$. We consider the cases where $\phi$ is the image in $\text{Aut}^*(G)$ of an inversion and a transposition separately.
Let $\phi = \bar{\iota}_z$; an inversion. Straightfoward checking shows that

(a) if $x = z$ then $\iota_z^{-1}\beta_{C,x}\iota_z = \beta_{C,x}^{-1}$ and

(b) if $x \neq z$ then $\iota_z^{-1}\beta_{C,x}\iota_z = \beta_{C,x}$.

Hence the result holds in this case.

Next suppose that $\phi = \hat{\tau}_{v,z}$, where $v = y$ or $y^{-1}$. If $y = x$ then we have $x^\perp \subseteq z^\perp$ which implies that $\Gamma_x$ is connected and so $\beta_{C,x}$ is an inner automorphism; as are all its conjugates. Thus we assume that $y \neq x$ and $\Gamma_x$ is not connected. Let $C_1$ be the component of $\Gamma_x$ containing $y$. Then $y^\perp \subseteq z^\perp$ and $y^\perp \subseteq C_1 \cup \{x\}$. If $z \in C_2$, for some component $C_2$ of $\Gamma_x$ with $C_2 \neq C_1$ then $z^\perp \subseteq C_2 \cup \{x\}$ so $y^\perp \subseteq (C_1 \cup \{x\}) \cap (C_2 \cup \{x\}) = \{x\}$; in which case $y^\perp = \{x,y\}$ and $x \in z^\perp$. These conditions imply that $\tau_{y,z}\beta_{C,x}\tau_{y,z} = \beta_{C,x}$, so we may now assume that $z \in C_1$. Assume in addition that the connected components of $\Gamma_x$ are $C_1, \ldots, C_r$. If $C = C_i$, where $i \neq 1$ then $\phi^{-1}\beta_{C,x}\phi = \beta_{C,x}$. If $C = C_1$ then set $\beta_1 = \prod_{i=2}^{r} \beta_{C_i,x}^{-1}$, so $\beta_{C,x} = \beta_1$ and $\phi^{-1}\beta_1\phi = \beta_1$. Thus $\phi^{-1}\beta_{C,x}\phi = \beta_1 = \beta_{C,x}$, and the result follows.

The previous proposition can not be extended to $\text{Conj}(G)/\text{Inn}(G)$ as the following example shows.

**Example 3.41.** Let $\Gamma$ be the graph of Figure 3.1. Then $\Gamma_x^\perp$ has a component $C = \{c, d, e, y\}$ and $\Gamma_y^\perp$ has a component $D = \{a, b, c, x\}$. Let $\alpha = \alpha_{C,x}$ and $\beta = \alpha_{D,y}$. The images of $c$ and $g$ under $[\alpha, \beta]$ are $c^{[x,y]}$ and $g$, respectively. Therefore $\alpha\beta \neq \beta\alpha$ modulo $\text{Inn}(G)$. In this example $\text{Inn}(G) = \text{Conj}_A(G)$, so in general $\text{Conj}(G)/\text{Conj}_A(G)$ is also non-Abelian.
Figure 3.1: $\text{Conj}(G)/\text{Inn}(G)$ is non-Abelian
4 The stabilisers \( \text{St}(K) \) and \( \text{St}^{\text{conj}}(K) \).

Define

\[
\text{St}(K) = \{ \phi \in \text{Aut}(G) \mid G(Y)\phi = G(Y), \text{ for all } Y \in K \}.
\]

Then, from Lemma 2.3 (ix) it follows that \( \phi \in \text{St}(K) \) if and only if \( G(a(x))\phi = G(a(x)) \), for all \( x \in X \). Also, from Remark 3.17 the subgroup of \( \text{Aut}^*(G) \) generated by \( \text{Inv} \) and \( \text{Tr} \) is contained in \( \text{St}(K) \).

Define

\[
\text{St}^{\text{conj}}(K) = \{ \phi \in \text{Aut}(G) \mid G(Y)\phi = G(Y)^{f_Y}, \text{ for some } f_Y \in G \text{ for all } Y \in K \}.
\]

We shall make use of the following fact in the proof of the next proposition.

**Lemma 4.1.** Let \( H \) and \( K \) be canonical parabolic subgroups of \( G \) and let \( \theta \in \text{Aut}(G) \) be such that \( H\theta = H \) and \( K\theta = K^g \), for some \( g \in G \) which has no left divisor in \( K \). Then \( (H \cap K)\theta = (H \cap K)^g \).

**Proof.** As \( H \cap K \) is a canonical parabolic subgroup it suffices to show that, for all \( x \in X \cap H \cap K \) there is some \( s \in H \cap K \) such that \( x\theta = s^g \). Given such an \( x \) there are elements \( z \in H \) and \( w \in K \) such that \( x\theta = z = w^g \).

From [16, Corollary 2.4] there are elements \( a, b, c, d, e, h, u, v \in G \) such that \( g = a \circ b \circ c \circ h, w = u^{-1} \circ v \circ u, u = d \circ a^{-1}, dh = d \circ h \) and \( w^g = (dh)^{-1} \circ e \circ (dh), \) with \( \alpha(e) = \alpha(v), \) cyclically minimal, \( e = \psi, \alpha(b) \subseteq \alpha(v) \) and \( [\alpha(b \circ c), \alpha(d)] = [\alpha(c), \alpha(v)] = 1 \). As \( g \) has no left divisor in \( K \) we have \( a = b = 1, \) so \( e = v, \) \( g = ch \) and \( x\theta = (d^{-1}vd)^g \). Thus \( (H \cap K)\theta = (H \cap K)^g \), as required. \( \square \)

**Theorem 4.2.** \( \text{St}^{\text{conj}}(K) = \text{Aut}^*(G) \) and the group \( \text{Aut}(G) \) can be decomposed into the canonical semidirect product of \( \text{St}^{\text{conj}}(K) \) and the finite subgroup \( \text{Aut}^{\text{comp}}(G) \):

\[
\text{Aut}(G) = \text{St}^{\text{conj}}(K) \rtimes \text{Aut}^{\text{comp}}(\Gamma).
\]

**Proof.** That \( \text{Aut}(G) = \text{St}^{\text{conj}}(K) \rtimes \text{Aut}^{\text{comp}}(\Gamma) \) follows immediately from Proposition 3.18 once the first statement has been proved. From Lemma 4.1, \( \phi \in \text{St}^{\text{conj}}(K) \) if and only if \( G(a(x))\phi = G(a(x))^{f_x} \), for some \( f_x \in G \), for all \( x \in X \). It follows then from Proposition 3.16 that \( \text{Aut}^*(G) \subseteq \text{St}^{\text{conj}}(K) \).

For the reverse inclusion note that from Proposition 3.18 every element \( \phi \) of \( \text{Aut} \) can be expressed as \( \phi = \alpha\beta \), with \( \alpha \in \text{Aut}^* \) and \( \beta \in \text{Aut}^{\text{comp}}(\Gamma) \). If \( \phi \in \text{St}^{\text{conj}}(K) \) then, since \( \text{Aut}^*(G) \subseteq \text{St}^{\text{conj}}(K) \) we have \( \alpha^{-1}\phi \in \text{St}^{\text{conj}}(K) \cap \text{Aut}^{\text{comp}}(\Gamma) \). However, from the definitions of \( \text{St}^{\text{conj}}(K) \) and \( \text{Aut}^{\text{comp}}(\Gamma) \) this means that \( \alpha^{-1}\phi = \beta = 1 \), so \( \phi = \alpha \in \text{Aut}^*(G) \), as required. \( \square \)
Theorem 4.3. $\text{Conj}_N(G)$ is a normal subgroup of $\text{St}^{\text{conj}}(K)$.

Proof. Let $\phi \in \text{Conj}_N(G)$ and $\psi \in \text{St}^{\text{conj}}(K)$. Let $x \in X$ and let $g_x$ and $h_x$ be elements of $G$ such that $u\phi = u^x$, for all $u \in a(x)$ and $G(a(x))\psi^{-1} = G(a(x))h_x$. For each $u \in a(x)$ let $w_u$ be an element of $G$ (regarded as a geodesic word) such that $u^x = u^x$, so $\alpha(w_u) \subseteq a(x)$ and $w_u\psi = u^x\psi$. Then $u\psi^{-1}\phi\psi = u^x$, where $f_x = (h_x^{-1}g_x(h_x\phi))\psi$, so $f_x$ is dependent only on $x$ and $\psi^{-1}\phi\psi \in \text{Conj}_N(G)$.

Lemma 4.4. $\text{St}(K) \cap \text{Conj}(G) = \text{Conj}_S(G)$, $\text{St}(K) \cap \text{Conj}_V(G) = \text{Conj}_C(G)$.

If $\phi \in \text{Aut}(G)$ then $\phi \in \text{Conj}_S(G)$ if and only if $x\phi = x^f$ where $\alpha(f_x) \subseteq a(x)$, for all $x \in X$.

Proof. Clearly $\text{Conj}_S \subseteq \text{St}(K) \cap \text{Conj}(G)$. For the converse use induction on $|\phi|$, where $\phi \in \text{St}(K) \cap \text{Conj}(G)$. If $|\phi| = 0$ then $\phi = 1$ and so belongs to $\text{Conj}_S$. Assume then that $|\phi| > 0$. In this case, from Lemma 3.37 (i), there exist $u_1, u_2 \in X$ such that $u_i\phi = u_i^{x_i}$, reduced as written, for some $w_1, w_2 \in G$, and $u_1w_1$ is a right divisor of $w_2$. It follows that $u_1 \notin u_2\perp$. As $\phi \in \text{St}(K)$ we have $w_2 \in G(a(u_2))$ so $u_1w_1 \in G(a(u_2))$. In particular $u_1 \in a(u_2)$ so $u_2^{x_i} \subseteq u_1^x$. Therefore $\tau_{u_2,u_1} \in \text{Tr}_\phi$ and $\beta = \tau_{u_2,u_1}\tau_{u_2,u_1} \in \text{Conj}_S \subseteq \text{St}(K) \cap \text{Conj}(G)$. Therefore $\beta\phi \in \text{St}(K) \cap \text{Conj}(G)$ and $|\beta\phi| < |\phi|$ so, by induction, $\beta\phi \in \text{Conj}_S(G)$. This gives $\phi \in \text{Conj}_S(G)$, as required.

From this and Lemma 3.35 (iii) the last statement of the theorem follows immediately.

That $\text{St}(K) \cap \text{Conj}_V(G) = \text{Conj}_C(G)$ follows immediately from Propositions 3.39.

Lemma 4.5. Let $x$ and $y$ be elements of $X$ such that $x$ dominates $y$ and let $C$ be a component of $\Gamma_{y^\perp}$.

(i) If $x \notin C$ then $C$ is a component of $\Gamma_{y^\perp}$.

(ii) If $x \in C$ and the components of $\Gamma_{x^\perp}$ which meet $C$ are $C_1, \ldots, C_r$ then $C = [(C_1 \cup \cdots \cup C_r) \cup x^\perp] \setminus y^\perp$.

Proof. (i). If $C = \{u\}$, for some $u \in X$, then $y$ dominates $u$ so $u^\perp \setminus u \subseteq y^\perp$. In this case, if $x \in u^\perp$ then $x = u$, since $x \notin y^\perp$, but this contradicts $x \notin C$. Thus $x \notin u^\perp$ and $u^\perp \cap x^\perp = y^\perp \cap u^\perp \cap x^\perp = u^\perp \setminus u$; so $x$ also dominates $u$ and $C$ is a component of $\Gamma_{x^\perp}$. If $C$ contains two elements $u$ and $v$ then there is a path $p$ from $u$ to $v$ which does not meet $y^\perp$. If $u$ and $v$ belong to different components of $\Gamma_{x^\perp}$ then $p$ meets $x^\perp$, and as $x \notin y^\perp$ this means that $x \in C$, a contradiction. Hence $C \subseteq C'$, for some component $C'$ of $\Gamma_{x^\perp}$. As $y^\perp \setminus y \subseteq x^\perp$, every component of $\Gamma_{x^\perp}$ containing at least 2 elements is contained in some component of $\Gamma_{y^\perp}$, so $C = C'$.
(ii). Suppose that \( u \in C_i \), for some \( i \). Either \( u \) belongs to \( y^\perp \) or to some component of \( \Gamma_{y^\perp} \). However \( y^\perp \setminus y \subseteq x^\perp \) and \( u \notin x^\perp \), so \( u \notin y^\perp \setminus y \). Clearly \( u \neq y \) so \( u \) belongs to some component \( C' \) of \( \Gamma_{y^\perp} \). If \( x \notin C' \) then, from (i), \( C' = C_i \), in which case \( C' = C \) and \( x \in C' \), a contradiction. Hence \( x \in C' \) and \( C = C' \); so \( C_i \subseteq C \), for all \( i \). By definition \( C \subseteq \bigcup_{i=1}^r C_i \cup x^\perp \), and the result follows.

The following question now arises naturally.

**Question 4.6.** Let \( \Gamma \) be a (connected) graph. Is \( \text{St}^{\text{conj}}(K) = \text{St}(K) \text{Conj}_N(G) \)?

It seems on first sight very plausible that the answer is “yes”, but, as the subsequent example shows, it turns out to be “no”.

**Example 4.7.** Take \( G \) to be the group \( G(\Gamma) \) where \( \Gamma \) is the graph of Figure 4.1. Denote the components of \( \Gamma_{v^\perp} \) by \( C = \{a, r, s\} \) and \( D = \{b, t\} \) and let \( \alpha = \alpha_{C,v}, \tau = \tau_{v,a}\tau_{v,b}\tau_{v,a}^{-1}, \) and set \( \phi = \alpha\tau \).

\[
\begin{align*}
z\phi = \begin{cases} z, & \text{if } z = b, c, t \\
v^a, & \text{if } z = v \\
v^{ab} & \text{if } z \in C
\end{cases}
\end{align*}
\]

The answer above is no, as \( \phi \) cannot be written as \( \gamma\delta \), where \( \delta \in \text{Conj} \) and \( \gamma \in \text{St}(K) \), as we shall demonstrate. The set \( K \) consists of \( a(v) = \{a, b, c, v\} \), \( a(s) = \{a, r, s\} \), \( a(t) = \{b, c, t\} \) and four more sets each of which is a singleton. As \( \gamma \) maps the subgroup generated by \( a(a) = \{a\} \) to itself we have \( a\gamma = a^{\pm 1} \).
As $a\delta = a^g$, for some $g \in G$, it must be that $a\gamma = a$. Similarly $b\gamma = b$, $c\gamma = c$ and $r\gamma = r$. Combined with the expression for $\phi$ above we obtain $a\delta = a^{vb^a}$, $b\delta = b$ and $c\delta = c$. As $c\delta = c$ it must be that $G(c^\perp)\delta = G(c^\perp)$ and as $\delta$ acts on generators by conjugation, moreover $\delta$ must map $G(a, b, v)$ to itself; so $\nu\delta = \nu^g$, for some $g \in G(a, b, v)$. Applying Lemma 3.37 to $\delta$ restricted to $G(a, b, v)$ we see that either $b^\xi$ or $a^\varepsilon vb^a$ is a right divisor of $g$, or that $\nu g$ is a right divisor of $vb^a$, in which case $g = b^a$. In the latter case consider the automorphism $\alpha_{C,v^{-1}}$. This maps $a$ to $a^{vb^a}$, $b$ to itself, $c$ to itself and $v$ to $vb^a$. Restricting to $G(a, b, v)$ again gives a contradiction to Lemma 3.37. Thus we may assume that either $a^\varepsilon vb^a$ or $b^\xi$ is a right divisor of $g$. Suppose that $m$ is maximal such that $a^\varepsilon v^m vb^a$ is a right divisor of $g$; say $g = g_0 \circ a^\varepsilon v^m vb^a$. If $m \geq 1$ then $\delta_0 = \alpha_{v,ab}^{-m} \delta$ maps $a$ to $a^{vb^a}$, fixes $b$ and $c$ and maps $v$ to $v^{g_0 v^m b^a}$. As this contradicts Lemma 3.37 we have $m = 0$. Similarly, if $b^\xi$ is a right divisor of $g$ then we obtain a contradiction. Hence no such conjugating automorphism $\delta$ exists.

It is also possible to show that $\tau \alpha \notin \Conj(G) \St(K)$. Moreover the example shows that replacing $\St(K)$, $\St(K)$ and $\Conj(G)$ with their canonical images in $\Out(G)$ the equality of Question 4.6 still fails.

If there are no dominated vertices in $\Gamma$, that is $\Dom(\Gamma) = \emptyset$, then following holds. Here the subgroup

$$\St(L) = \{ \phi \in \Aut(G) | G(Y)\phi = G(Y), \text{ for all } Y \in L \}$$

defined originally in [18], where it is shown to be an arithmetic group.

**Lemma 4.8.** Let $\Gamma$ be a graph such that $\Dom(\Gamma) = \emptyset$. Then

(i) $\Conj(G) \cap \St(K) = \Conj_S(G) = \{ 1 \}$ and $\Conj(G) = \Conj_V(G) = \Conj_N(G)$ is normal in $\St(K)$ and

(ii) $\St(L) = \St(K)$.

**Proof.** In this case $\Conj_C(G) = \Conj_S(G) = \{ 1 \}$ so $\Conj_V \cap \St(K) = 1$. To see that $\Conj_N = \Conj$, note first that, from Lemma 2.3 (iii), it follows that $a(x) = \cl(x)$, for all $x \in X$. Let $x, y \in X$ and let $C$ be a component of $\Gamma_{y \perp}$. If $a(x) \cap C \neq \emptyset$ then, from Lemma 3.15, either $a(x) \subseteq C \cup y \perp$ or $y \in a(x)$. If $y \in a(x) = \cl(x)$ then $\cl(x) \subseteq y \perp$; so either $a(x) \cap C = \emptyset$ or $a(x) \subseteq C \cup y \perp$. Therefore, either $u a_C \gamma = u^y$, for all $u \in a(x)$, or $u a_C \gamma = u$, for all $u \in a(x)$; and it follows that $\Conj_N = \Conj$.

\[\square\]

**Theorem 4.9.** The following are equivalent.
(i) $\text{Dom}(\Gamma) = \emptyset$.

(ii) $\text{St}^\text{conj}(K) = \text{Conj}_N(G) \rtimes \text{St}(L)$.

(iii) $\text{St}^\text{conj}(K) = \text{Conj}(G) \rtimes \text{St}(L)$.

(iv) $\text{St}^\text{conj}(K) = \text{Conj}(G) \rtimes \text{St}(K)$.

Proof. In view of Lemma 4.8 it suffices to show that each of the last three statements implies the first. To see that the second or third statement implies the first, suppose $\text{St}^\text{conj}(K)$ decomposes as the given semi-direct product. If $y \in \text{Dom}(x)$, for some $x, y \in X$, then $\tau = \tau_{yx} \in \text{St}^\text{conj}$, so $\tau = \alpha \lambda$, for some $\alpha \in \text{Conj}$ and $\lambda \in \text{St}(L)$. Then, for $z \in X \setminus y$ we have $z = z\tau = z\alpha \lambda = z^g \lambda = z \lambda^g \lambda$. As $z \lambda \in G(\text{cl}(z))$ it follows that $z \lambda = z \circ w$, for some $w \in G(\text{cl}(z))$, so $(zw)^g \lambda = z$, from which, counting exponents of letters, we infer that $w = 1$. Hence $g \lambda \in G(z^\perp)$, so $g \in G(z^\perp)$, which implies that $z\alpha = z$, and consequently $z \lambda = z$. Now $y\alpha = y^h$ and $z \alpha = z \lambda = z$, for some $h \in G$ and all $z \in X \setminus y$. As $\lambda \in \text{St}(L)$ we have $y \lambda \in G(\text{cl}(y))$ and, since $z \lambda = z$ for all $z \neq y$, we have $y \lambda = yw$, for some $w \in G(\text{cl}(y))$. However this means that $y\alpha = y\tau \lambda^{-1} = (yx)\lambda^{-1} = yw^{-1} x$, and the exponent sum of $x$ on the left hand side of this expression is zero, while on the right it is one. Hence no such $x, y$ exist and $\text{Dom}(\Gamma) = \emptyset$.

To see that the fourth statement implies the first; from the fourth statement it follows that $\text{Conj}(G) \cap \text{St}(K) = \{1\}$, so $\text{LInn}_S = \emptyset$ and this implies that $\text{Dom}(\Gamma) = \emptyset$. \qed

4.1 Balanced graphs

Although $\text{Dom}(\Gamma) = \emptyset$ is a necessary condition for the intersection of $\text{Conj}(G)$ and $\text{St}(K)$ to be trivial, the class of graphs for which $\text{St}^\text{conj}(K) = \text{Conj}(G) \rtimes \text{St}(K)$ is much wider than those without dominated vertices: it can, as we shall show, be characterised using the following definition.

Definition 4.10. A graph $\Gamma$ is called balanced if the following condition holds for all $v \in \text{Dom}(\Gamma)$. Either

1. $\text{out}(v) = \emptyset$, or

2. there exists a connected component $C_v$ of $\Gamma_{v^\perp}$ such that $\text{out}(v) \subseteq C_v$.

The following subset of $\text{LInn}$ is important, particularly in the case of balanced graphs.
Definition 4.11. Let \( y \in X \) and \( \alpha_{L,y} \in \text{LInn}_W(G) \). If \( a(y) \cap L = \emptyset \) then \( \alpha_{L,y} \) is called a tame elementary conjugating automorphism of \( G \). The set of all tame elementary conjugating automorphism is denoted \( \text{LInn}_T(G) \).

Lemma 4.12. Let \( y \in X, v, x \in X^{\pm 1}, \alpha = \alpha_{L,y} \in \text{LInn}_T \) and \( \tau = \tau_{v,x} \in \text{Tr} \).

(i) If either \( v \in L \) and \( x \in L \cup y^{\perp} \) or \( v \notin L, v \neq y^{\pm 1} \) and \( x \notin L \) then

\[
\alpha \tau = \tau \alpha.
\]

(ii) If \( v \in L \) and \( x \notin L \cup y^{\perp} \) then \( v \in \text{Dom}(y) \) and

\[
\alpha \tau = \tau_{v,y} \tau_{v,y}^{-1} \alpha.
\]

(iii) If \( v \notin L, v \neq y^{\pm 1} \) and \( x \in L \) then \( v \in \text{Dom}(y) \) and

\[
\alpha \tau = \tau_{v,y} \tau_{v,y}^{-1} \alpha.
\]

(iv) If \( y = v^{\pm 1} \) and \( x \notin L \) then \( L \) is a union of connected components of \( \Gamma_{x^{\perp}} \) and, setting \( \beta = \alpha_{L,x} \),

\[
\alpha \tau = \tau \beta \alpha.
\]

Proof. (i) In \( v \in L \) and \( x \in L \cup y^{\perp} \) then

\[
z \alpha \tau = z \tau \alpha = \begin{cases} 
  z, & \text{if } z \notin L \\
  (vx)^{y}, & \text{if } z = v \\
  z^{y}, & \text{if } z \in L \text{ and } z \neq v
\end{cases}.
\]  \hspace{1cm} (4.1)

If \( v \notin L, v \neq y \) and \( x \notin L \) then

\[
z \alpha \tau = z \tau \alpha = \begin{cases} 
  z, & \text{if } z \notin L, z \neq v \\
  vx, & \text{if } z = v \\
  z^{y}, & \text{if } z \in L
\end{cases}.
\]  \hspace{1cm} (4.2)

(ii) In this case \( [x,v] \neq 1 \) as \( v \in L \) and \( x \notin y^{\perp} \); so \( x \in \text{out}(v) \). As \( v \in L \) and \( x \notin L \cup y^{\perp} \), all paths from \( v \) to \( x \) must intersect \( y^{\perp} \), so \( v^{\perp} \backslash v \subseteq y^{\perp} \); and \( v \notin y^{\perp} \), so \( v \in \text{Dom}(y) \). Then \( z \alpha \tau \) is as given in (4.1), and is equal to \( z \tau_{v,y} \tau_{v,y}^{-1} \alpha \), for all \( z \in X \).

(iii) If \( y \in v^{\perp} \) then, as \( v \neq y \), \( x \in y^{\perp} \), a contradiction. Thus, as in the previous case, \( y \) dominates \( v \). Then \( z \alpha \tau \) is as given in (4.2), and is equal to \( z \tau_{v,y} \tau_{v,y}^{-1} \alpha \), for all \( z \in X \).
(iv) In this case \( v \) is dominated by \( x \) so, from Lemma 4.5, \( L \) is a union of connected components of \( \Gamma_{x^\perp} \).

\[
z\alpha\tau = \begin{cases} 
z, & \text{if } z \notin L, z \neq v \\
vx, & \text{if } z = v \\
zvx, & \text{if } z \in L 
\end{cases}
\]

and this is equal to \( z\tau\beta\alpha \), for all \( z \in X \).

\[\square\]

**Corollary 4.13.** Let \( y \in X \) and \( v \in X^{\pm 1}, v \neq y^{\pm 1} \). Let \( \alpha = \alpha_{L,y} \in L\text{Inn}_T \) and let \( \tau_{v,a} \in \text{Tr}_W \). Then

\[
\alpha\tau_{v,a} = \tau_{v,b}\alpha,
\]

for some \( \tau_{v,b} \in \text{Tr}_W \).

**Proof.** Let \( a = a_1 \cdots a_n \), where \( a_i \in X^{\pm 1} \), be a geodesic word representing \( a \). Then \( \tau_{v,a} = \tau_{v,a_n} \cdots \tau_{v,a_1} \). As \( v \neq y^{\pm 1} \), \( \alpha\tau_{v,a_i} = \tau_{v,y}^{-\varepsilon_i}\tau_{v,a_i} \tau_{v,y}^{\varepsilon_i}\alpha \), with \( \varepsilon_i = 0 \) or \( \pm 1 \), for all \( i \). The corollary follows on setting \( b \) equal to the word obtained by freely reducing \( \prod_{i=1}^n y^{\varepsilon_i}a_i y^{-\varepsilon_i} \).

\[\square\]

**Corollary 4.14.** Let \( v \in X \), \( \alpha = \alpha_{L,v} \in L\text{Inn}_T \) and let \( \tau = \tau_{v,a} \in \text{Tr}_W \). Then

\[
\alpha\tau = \tau\beta,
\]

for some \( \beta \in \langle L\text{Inn}_T \rangle \).

**Proof.** Let \( a = a_1 \cdots a_n \), where \( a_i \in X^{\pm 1} \), be a geodesic word representing \( a \). By definition of \( \text{Tr}_W \) we have \( a_i \in (a(v)\setminus\{v\})^{\pm 1} \), for all \( i \). Hence, by definition of \( L\text{Inn}_T \), \( a_i \notin L \), for all \( i \). Thus

\[
\alpha\tau_{v,a_i} = \tau_{v,a_i}\alpha_{L,a_i}\alpha.
\]

Since \( v \neq a_i \), \( v \notin L \) and \( a_j \notin L \), also

\[
\alpha_{L,a_i}\tau_{v,a_j} = \tau_{v,a_j}\alpha_{L,a_i}
\]

when \( i \neq j \). Therefore

\[
\alpha\tau = \alpha\tau_{v,a_n} \cdots \tau_{v,a_1} \\
= \tau_{v,a_n} \cdots \tau_{v,a_1} \alpha_{L,a_n} \cdots \alpha_{L,a_1}\alpha \\
= \tau\alpha_{L,a_n} \cdots \alpha_{L,a_1}\alpha.
\]

As \( \alpha \in L\text{Inn}_T \), we have \( a(v) \cap L = \emptyset \) and, as \( \tau_{v,a_i} \in \text{Tr} \), we have \( a(a_i) \subseteq a(v) \) so \( a(a_i) \cap L = \emptyset \). Hence \( \alpha_{L,a_i} \in L\text{Inn}_T \); and the result follows. \[\square\]
Proposition 4.15. Let $\Gamma$ be a connected graph. Then $\langle \text{Tr} \cup \text{LInn}_T \rangle = \langle \text{Tr} \rangle \langle \text{LInn}_T \rangle$.

Proof. It suffices to prove the proposition holds with $\text{Tr}_W$ in place of $\text{Tr}$. First suppose that $u$ is a word on the generators $\text{LInn}_T$ and their inverses and that $\tau \in \text{Tr}_W^\pm$. It follows by a straightforward induction on $|u|$ and Corollary 4.14 that $\tau u = \sigma' u'$ in $\text{Aut}^*(G)$, for some $\tau' \in \text{Tr}_W^\pm$ and word $u'$ over $\text{LInn}_T^\pm$.

Now let $w$ be a word in the generators of $\langle \text{Tr}_W \cup \text{LInn}_T \rangle$ and their inverses. If $|w| \leq 1$ then $w \in \langle \text{Tr}_W \rangle \langle \text{LInn}_T \rangle$. Assume inductively that for some $k \geq 1$ all words of length at most $k$ define elements of $\langle \text{Tr}_W \rangle \langle \text{LInn}_T \rangle$.

Let $w$ be a word of length $k + 1$ (in the given generators). Then $w = w_0 \xi$, for some word $w_0$ of length $k$ and generator $\xi \in \langle \text{Tr}_W \cup \text{LInn}_T \rangle^\pm$. By induction there exists words $w_1 \in \langle \text{Tr}_W \rangle$ and $w_2 \in \langle \text{LInn}_T \rangle$ such that $w_0 = w_1 w_2$, in $\text{Aut}^*(G)$. If $\xi \in \text{LInn}_T^\pm$ the proof is complete. Otherwise $\xi \in \text{Tr}_W^\pm$ and, from the first part of the proof we may rewrite $w_2 \xi$ to a word $\xi' w_2'$, with $\xi' \in \text{Tr}_W^\pm$ and $w_2' \in \langle \text{LInn}_T \rangle$, such that $w_2 \xi = \xi' w_2'$ in $\text{Aut}^*(G)$. Then $w = w_1 \xi' w_2' \in \langle \text{Tr}_W \rangle \langle \text{LInn}_T \rangle$, as required. \qed

Theorem 4.16. Let $\Gamma$ be a connected graph and $G = G(\Gamma)$. Then $\text{Aut}^*(G) = \text{St}(K) \text{Conj}(G)$ if and only if $\Gamma$ is a balanced graph.

Proof. First suppose that $\Gamma$ is a balanced graph. Let $\iota = \iota_y \in \text{Inv}$, let $\alpha_{L,x} \in \text{LInn}$ and $\tau = \tau_{v,x} \in \text{Tr}$, where $x, y \in X$ and $v \in X^\pm$. Then $\alpha \iota = \iota \alpha$ unless $y = x$, in which case $\alpha \iota = \iota \alpha^{-1}$. Also $\tau \iota = \iota \tau$ unless $y = x$, in which case $\tau \iota = \iota \tau_{v^{-1},x}$. It therefore suffices to show that elements of $\langle \text{Tr} \cup \text{LInn} \rangle$ belong to $\text{St}(K) \text{Conj}(G)$.

Next we show that, as $\Gamma$ is balanced, $\langle \text{Tr} \cup \text{LInn} \rangle$ is generated by $\text{Tr} \cup \text{LInn} \cup \text{LInn}_T$. To see this suppose that $y \in \text{Dom}(\Gamma)$, $\text{out}(y) \neq \emptyset$ and $C_y$ is the component of $\Gamma_{y^{-1}}$ meeting out$(y)$. Let $L = X \setminus (C_y \cup y^\perp \cup \{y\})$, so $L$ is a union of connected components of $\Gamma_{y^{-1}}$ and $\alpha_{L,y} \in \text{LInn}_T$. Moreover, setting $D_y = \{y\} \setminus y$, we have $\alpha_{C_y,y} \alpha_{D_y,y} \alpha_{L,y} = \gamma_y \in \text{Inn}(G)$, so the generators $\alpha_{C_y,y}$ of this form are contained in the subgroup generated by $\text{Inn}$ and $\text{LInn}_T$ and the set $\{\alpha_{D_y,y} \in \text{LInn}_W : y \in \text{Dom}(\Gamma), \{y\} > 1 \text{ and } D_y = \{y\} \}$.

Now suppose that $y \in \text{Dom}(\Gamma)$ and $\{y\} \neq \{y\}$. Then for all $v \in \{y\}$, $v \neq y$, we have $\alpha_{v,y} = \tau_{v^{-1},y} \tau_{v,y}$, so $\alpha_{D_y,y} = \prod_{v \in D_y} \tau_{v^{-1},y} \tau_{v,y}$. Thus all generators $\alpha_{D_y,y}$ are contained in the subgroup generated by $\text{Tr}$. It follows that every word on generators $\text{Tr} \cup \text{LInn}$ and their inverses may be replaced by a word on $\text{Tr} \cup \text{LInn} \cup \text{LInn}_T$ and their inverses. Thus $\langle \text{Tr} \cup \text{LInn} \rangle = \langle \text{Tr} \cup \text{LInn} \cup \text{LInn}_T \rangle$. As $\langle \text{LInn} \rangle$ is normal in $\text{Aut}(G)$ it suffices to show that elements of $\langle \text{Tr} \cup \text{LInn}_T \rangle$ belong to $\text{St}(K) \text{Conj}(G)$: and this follows from Proposition 4.15.

For the converse suppose that $\Gamma$ is not a balanced graph. We shall show that the obstruction of Example 4.7 is also manifested in $\text{Aut}^*(G)$. Indeed
the argument is a generalised version of the example. If $\Gamma$ is not balanced then there exists a vertex $v \in \text{Dom}(\Gamma)$ such that \( \text{out}(v) \neq \emptyset \) and there is no component $C$ of $\Gamma_{v,\perp}$ such that $\text{out}(v) \subseteq C$. Let $v$ be such a vertex and let $a, b \in \text{out}(v)$ such that $a$ and $b$ are in different components $C$ and $B$, respectively, of $\Gamma_{v,\perp}$.

Suppose that $a_0 \in \text{out}(v)$ and $a_0 <_K a$. Then $a_0 <_K a$ implies that $a_1 = a_0 \setminus a \subseteq a_0$ and $a \in \text{out}(v)$ implies that there exists $u \in a_1 \setminus a$ such that $u \notin v_1$. Thus $a_0 \in u_1$, so $a_0 \in C$. We may therefore assume that $a$ is $K$-minimal among elements of out($v$). Similarly we may assume $b$ is $K$-minimal among elements of out($v$).

Define $\phi = \alpha_{C,v} v_{e,a} v_{v,b}$ so

$$z_\phi = \begin{cases} 
  z, & \text{if } z \notin C \cup \{v\} \\
vba, & \text{if } z = v \\
vba^z & \text{if } z \in C
\end{cases}.$$

Assume that there exist $\gamma \in \text{St}(K)$ and $\delta \in \text{Conj}$ such that $\phi = \gamma \delta$.

Note that $Z(G_v)$ is generated by $a(v) \cap (v_1 \setminus v)$ (which may be empty). Let $c \in a(v) \cap (v_1 \setminus v)$. Then $a(c) \subseteq a(v) \cap (v_1 \setminus v)$ and $a(c) = w_c \in Z(G_v)$. There exists $g \in G$ such that $G_c \delta = G_c^g$, where $g$ has no left divisor in $G_c$ or $G(a(c)_\perp)$. Let $z \in a(c)$. Then $z = z \phi = w_z \delta = (w_z^g)^g$, where $w_z$, $w_z' \in G_c$. Since $a(c)$ generates a complete subgraph this implies that $w_z' = z$ and $[g, z] = 1$. This holds for all $z \in a(c)$: so the hypothesis on $g$ implies $g = 1$. Therefore $c \gamma = c \delta = c$, for all such $c$.

As $a$ is $K$-minimal among elements of out($v$) we have $a(a) \setminus [a] \subseteq a(v) \cap (v_1 \setminus v)$. There exists $g \in G$ such that $G_c \delta = G_c^g_a$ and we may assume that $g$ has no left divisor in $G_a$ or $G(a(a)_\perp)$. Let $z \in [a]$, so $z \phi = z vba$. We have $z \gamma \delta = u_g^z$, for some $u_g \in G_a$. Therefore $u_g^z = z vba$ and so $z vba^{-1} \in G_a$. As neither $v$ nor $b$ commute with $a$ or $z$ it follows that $g = g_1 \circ vba$, and then $z g_1^{-1} \in G_a$. This holds for all $z \in [a]$, and for any $u \in a(a) \setminus [a]$ we have $u \delta = u$, from the paragraph above, so $[u, g] = 1$. Since $g$ has no left divisor in $G(a(a) \cup a(a)_\perp)$, [19, Corollary 2.5] implies that $g_1 = 1$ and $g = vba$. Now $z \delta = z g_z^z$, for some $g_z \in G$, so we have $z g_z^z = w_z vba$, for some $w_z \in G_a$. Again $z = w_z vba^{-1}$, so $z \in a(w_z)$ and $v, b \notin a(a)$, so $g_z = h_z \circ vba$, for some $h_z \in G_a$, and $w_z = z h_z$. As elements of $a(a) \setminus [a]$ belong to the centre of $G_a$, moreover $h_z \in G[a]$. Therefore, for all $z \in [a]$, $z \delta = z h_z vba$, for some $h_z \in G[a]$.

Again we have $a(b) \setminus [b] \subseteq a(v) \cap (v_1 \setminus v)$ and there exists $g \in G$ such that $G_b \delta = G_b^g$ and $g$ has no left divisor in $G_b$ or $G(a(b)_\perp)$. Let $z \in [b]$, so $z \phi = z$. We have $z \gamma \in G_b$ and so $z \gamma \delta = u_g^z$, for some $u_g \in G_b$. Therefore $u_g^z = z$ and $z g_z^{-1} \in G_b$. Thus $[z, g] = 1$, which implies $[[b], g] = 1$. For any
Let us call an element of Conj \( v \)-unlikely if it satisfies all of these four properties. Amongst all \( v \)-unlikely basis conjugating automorphisms choose one, which we shall now also call \( \delta \), of minimal length. As usual, for each \( x \in X \) let \( g_x \in G \) be such that \( x \delta = x^{g_x} \).

From condition 4 the automorphism \( \delta \) restricts to an automorphism of \( G_v \) and, applying Lemma 3.37 to this restriction, there exist elements \( x, y \in \mathfrak{a}(v) \) such that \( x^\varepsilon g_x \) is a right divisor of \( g_y \). Moreover, \( x, y \in \text{out}(v) \cup \{v\} \), as the centre of \( G_v \) is generated by \( \mathfrak{a}(v) \cap (v^+ \setminus v) \). Suppose that \( x, y \in C \) and let \( D \) be the component of \( \Gamma_{x \perp} \) containing \( y \). As \( x, y \in \mathfrak{a}(v) \) we have \( D \subseteq C \).

Define \( \delta_0 = \alpha_{D,x} \varepsilon \delta \). For all \( z \in X \setminus D \) we have \( z \delta_0 = z \delta \) and (applying Lemma 3.37 again) \( |\delta_0| < |\delta| \). If \( a \notin D \) then clearly \( \delta_0 \) is \( v \)-unlikely, contrary to the choice of \( \delta \). If \( a \in D \) then \( x^\varepsilon g_x \) is a right divisor of \( h_x h_{v} \), so \( x \in [a] \), which implies \( h_{z} = h_{z}^{x} h_{x}^{y} h_{v} a \), for all \( z \in [a] \setminus D \). Therefore \( z \delta_0 = z h_{z}^{x} h_{x}^{y} h_{v} a \), for all \( z \in [a] \setminus D \), and again \( \delta_0 \) is \( v \)-unlikely, a contradiction. We may therefore assume that \( \{x, y\} \notin C \).

Assume that \( y \notin C \) and that \( D \) is the component of \( \Gamma_{x \perp} \) containing \( y \). Then \( D \cap C = \emptyset \) and \( g_z = g_z^{x} g_x \), for all \( z \in D \). Again set \( \delta_0 = \alpha_{D,x} \varepsilon \delta \) and \( \delta_0 \) is \( v \)-unlikely with \( |\delta_0| < |\delta| \). This contradiction shows that we may assume \( y \in C \) and \( x \notin C \). Then \( C \) is a component of \( \Gamma_{x \perp} \) and \( x^\varepsilon g_x \) is a right divisor of \( h_x h_{v} \), for all \( z \in [a] \), as \( a, y \in C \). Hence \( \alpha(h_z) \subseteq [a] \) and \( x \notin C \) implies \( x = v \) or \( b \). If \( x = b \) then \( \delta_0 = a \), a contradiction, so we have \( x = v, w = vb \) and \( g_v = ba \).
Let $\delta_0 = \alpha_{C,v}^{-1}\delta$, so $z\delta_0 = z^{h_{2}ba}$, for $z \in [a]$, and $z\delta_0 = z\delta$, for $z \notin C$. Again $\delta_0$ is $v$-unlikely, contrary to minimality of the length of $\delta$. In all cases we obtain a contradiction, so there exists no $v$-unlikely automorphism $\delta$, completing the proof that $\phi \notin St(K)\ Conj$. □
References


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