A COMBINATORIAL PROPERTY AND POWER GRAPHS OF GROUPS

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Abstract. The *power graph* of a group G is a directed graph with the set G of vertices, and with all edges (u, v) such that $u \neq v$ and v is the power of u. For each directed graph D, we give a complete description of all groups G such that every infinite subset of G contains a power subgraph isomorphic to D. Also, we describe the structure of the power graphs of all finite abelian groups.

Giving an answer to a question of Paul Erdös, B.H. Neumann proved that a group is center-by-finite if and only if every infinite sequence contains a pair of elements that commute. After that several authors have investigated combinatorial properties of groups with all infinite subsets containing certain special elements, and a survey of this direction of research has been given by the first author in [2].

We consider a combinatorial property defined in terms of power graphs. The *power graph* $P(G)$ of a group G has all elements of G as vertices, and it has edges (u, v) for all $u, v \in G$ such that $u \neq v$ and v is a power of u. Let D be a directed graph. We say that a group G is *power D-saturate* if and only if, for every infinite subset T of G, the power graph of G has a subgraph isomorphic to D with all vertices in T.

Our first theorem completely describes all pairs (D, G), where D is a directed graph and G is a group, such that G is power D-saturate (Theorem 1). After that, for each finite abelian group G, we describe the power graph of G (Theorem 2).

For notation and terminology of graph and group theories not mentioned in this paper the reader is referred to [1] and [5], respectively. If p is a prime, then the p-primary component of a group G is denoted by G_p, the cyclic group of order p is denoted by C_p, and C_{p^∞} stands for the quasicyclic p-group. A directed graph is said to be *acyclic* if it has no directed cycles. Obviously, if G is finite or if D is a null graph, then G is power D-saturate. Therefore in our first main theorem we consider only infinite groups G and directed graphs D which have edges. Denote by T_∞ the transitive tournament on the set $\mathbb{N}$ of natural numbers. It has the vertex set $\mathbb{N}$ and the edge set $E(T_\infty) = \{(m, n) \mid m > n\}$.

Theorem 1. *Let $D(V, E)$ be a directed graph with $E \neq \emptyset$, and let G be an infinite group. Then G is power D-saturate if and only if G is a center-by-finite*

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torsion group, the center $C(G)$ has a finite number of primary components, each primary component of $C(G)$ is finite or quasicyclic, the order of $G/C(G)$ is not divisible by p for each quasicyclic p-subgroup of G, and D is isomorphic to a subgraph of T_∞.

Clearly, for each group G and every set of vertices V there exist maximal graphs $D(V,E)$ such that G is D-saturate. Theorem 1 shows that in fact there are only three types of maximal graphs: null graphs, transitive subtournaments of T_∞, and the compete graphs.

For the proof we need the following well-known result due to B.H. Neumann:

Lemma 1. ([3]) *A group is center-by-finite if and only if every infinite sequence contains a pair of elements that commute.*

A directed graph $D(V,E)$ is called a *direct product* of $D_1(V_1,E_1),\dots,D_n(V_n,E_n)$ if $V = V_1\times\cdots\times V_n$ and E is the set of all pairs $((a_1,\dots,a_n),(b_1,\dots,b_n))$ such that $(a_1,\dots,a_n)\neq(b_1,\dots,b_n)$ and $(a_i,b_i)\in E_i\cup(V_i\times V_i)$ for all $1\le i\le n$.

Lemma 2. *If $G=\prod_{i=1}^n G_{p_i}$ is a direct product of p_i-groups, where $p_1,\dots,p_n$ are pairwise distinct primes, then the power graph of G is the direct product of the power graphs of $G_{p_1},\dots,G_{p_n}$.*

Proof. If $b=(b_1,b_2,\dots,b_n)$ is a power of $a=(a_1,a_2,\dots,a_n)$ in G, then obviously every b_i is a power of a_i in G_i, for all $i=1,\dots,n$.

Conversely, suppose that $b_i=a_i^{m_i}$ in G_i, for all $i=1,\dots,n$. Denote the order of a_i by k_i. Given that G_{p_i} is a p_i-group, we see that $k_1,\dots,k_n$ are pairwise coprime. By the Chinese remainder theorem there exists a positive integer k which has remainder m_i upon division by k_i, for all $i=1,\dots,n$. It follows that $a^k=b$, as required. □

Maximal complete subgraphs of a directed graph are called *cliques*. Evidently, the binary relation of every power graph is transitive.

Lemma 3. *Let p be a prime, and let a and b be two distinct elements in a cyclic p-group C_{p^n}, where a has order p^r and b has order p^s. Then*

(i) *a belongs to a clique of order p^r-p^{r-1};*
(ii) *$(a,b)\in E(P(C_{p^n}))$ if and only if $r\ge s$.*

Proof. (i): Denote by g a generator of the C_{p^n}. There are precisely p^r-p^{r-1} elements of order p^r in C_{p^n}, namely all elements g^k such that $(k,p^n)=p^{n-r}$. Each element of order p^r generates the same subgroup and can be expressed as a power of every other element of the same order. Thus all elements of order p^r induce a clique of order p^r-p^{r-1} in $P(C_{p^n})$. When $r=0$, the identity element forms a clique of order 1.

(ii): Suppose that $r \geq s$. The order of $f = a^{p^{r-s}}$ is equal to p^s. Since all elements of the same order belong to the same clique in $P(C_{p^n})$, it follows that $(f, b) \in E(P(C_{p^n}))$. Hence $(a, b) \in E(P(C_{p^n}))$.

Conversely, suppose that $(a, b) \in E(P(C_{p^n}))$. Take any generator g of C_{p^n}. Then $a = g^c$, where $(c, p^n) = p^{n-r}$. Similarly, $b = g^d$, where $(d, p^n) = p^{n-s}$. Given that $a^m = b$, we get $(g^c)^m = g^d$; whence $mc \equiv d \pmod{p^n}$. This congruence is solvable if and only if $(c, p^n)|d$. Hence $(c, p^n)|(d, p^n)$, $p^{n-r}|p^{n-s}$ and we get $r \geq s$, as required. $\square$

Proof of Theorem 1. The 'only if' part. Suppose that G is power D-saturate, i.e., every infinite subset of G contains a power subgraph isomorphic to D. Hence every infinite subset has at least two elements a, b such that b is a power of a, and so a and b commute. Lemma 1 implies that G is center-by-finite.

If G has an element g of infinite order, then the vertices $g^2, g^3, g^5, \dots$ are not adjacent in the power graph of G. Since $E(D)$ contains edges between distinct vertices and G is D-saturate, we see that G has to be torsion.

If G contains elements g_i of orders p_i, for infinitely many primes $p_1, p_2, \dots,$ then the vertices $g_1, g_2, \dots$ are not adjacent in $P(G)$. This contradicts the D-saturateness of G again. Therefore G has a finite number of primary components.

If a p-primary component $C(G)_p$ of the center $C(G)$ has infinite p-rank, then $C(G)_p$ contains independent elements $g_1, g_2, \dots$ (see [5], 4.2). Clearly, these elements are not adjacent in the power graph of $C(G)$. Thus the p-rank of $C(G)_p$ is finite.

It follows that $C(G)_p$ is a direct product of finitely many cyclic or quasicyclic groups (see [5], 4.3.13). Suppose that $C(G)_p$ is infinite, but is not quasicyclic. Then it contains a subgroup isomorphic to $C_p \times C_{p^\infty}$. Let g be a generator of C_p, and let $g_1, g_2, \dots$ be generators of C_{p^∞} such that $g_1^p = e$ and $g_{i+1}^p = g_i$ for all $i = 1, 2, \dots$. Then the set $(g, g_1), (g, g_2), (g, g_3), \dots$ induces a null subgraph in the power graph. Therefore G is not D-saturate. Thus each primary component of $C(G)$ is finite or quasicyclic.

Take any prime number p such that G has a quasicyclic subgroup C_{p^∞}. If $g_1, g_2, \dots$ are the same generators of C_{p^∞} as above, then we see that they induce a subgraph of the power graph of G isomorphic to T_∞, that is $(g_i, g_j) \in E(P(G))$ if and only if $i > j$. Since G is D-saturate, D is a subgraph of T_∞.

Suppose that p divides $|G/C(G)|$ and that G has a quasicyclic subgroup C_{p^∞} with generators $g_1, g_2, \dots$ as above. Pick an element h in G such that its image $hC(G)$ has order p in $G/C(G)$. Then all vertices $(h, g_1), (h, g_2), \dots$ are not adjacent in $P(G)$, and so G is not D-saturate. This contradiction shows that $|G/C(G)|$ is not divisible by p for each quasicyclic p-subgroup of G.

The 'if' part. Assume that D has edges, G is a torsion group with a finite number of primary components, each primary component of G is finite or quasicyclic, and the order of $G/C(G)$ is not divisible by p for each quasicyclic

p-subgroup of G. In particular, $G = G_{p_1} \times \cdots \times G_{p_n}$ for pairwise distinct prime numbers $p_1, \ldots, p_n$. For $1 \le i \le n$, denote by $\pi_i : G \to G_{p_i}$ the projection of G onto G_{p_i}.

Take any infinite subset L of G. By induction on $i = 0, 1, \ldots, n$ we define infinite subsets L_i of L such that every image $\pi_k(L_i)$ forms a chain (i.e., a transitive tournament) in the power graph of G_{p_k} for $k = 1, \ldots, i$. First, put $L_0 = L$. Suppose that the set L_i has already been defined for some $0 \le i < n$.

If $\pi_{i+1}(L_i)$ is finite, then we can find an infinite subset L_{i+1} of L_i such that $\pi_{i+1}(L_{i+1})$ has only one element, and so forms a chain. (Note that in this part of our proof we allow consequtive repetitions of the same vertex in a chain or, equivalently, we attach all loops to the graphs.)

Next consider the case, where $\pi_{i+1}(L_i)$ is infinite. Then $G_{p_{i+1}}$ is infinite too, and so it is quasicyclic. Putting $p = p_{i+1}$,we get $G_p = C_{p^\infty}$. Since $|\pi_{i+1}(L_i)| = \infty$ and C_{p^∞} is the union of an ascending chain of cyclic groups, we can choose an infinite sequence $t_1, t_2, \ldots \in L_i$ such that each element $\pi_{i+1}(t_j)$ has order p^{ℓ_j}, for $j = 1, 2, \ldots$, and $\ell_1 < \ell_2 < \ldots$. Take any positive integers $j < k$. There exists a cyclic subgroup C_{p^ℓ} of C_{p^∞} such that both $\pi_{i+1}(t_j)$ and $\pi_{i+1}(t_k)$ belong to C_{p^ℓ}. Lemma 3 shows that $\pi_{i+1}(t_j)$ is a power of $\pi_{i+1}(t_k)$. It follows that the sequence $\pi_{i+1}(t_1), \pi_{i+1}(t_2), \ldots$ forms an infinite chain in the power graph of G. We can take $L_{i+1} = \{t_1, t_2, \ldots\}$.

Thus we have defined the sets $L_1, \ldots, L_n$. All projections of the infinite set L_n form ascending chains in $G_{p_1}, \ldots, G_{p_n}$. Lemma 2 implies that L_n induces an infinite chain C in the power graph of G.

A vertex u is said to be an *ancestor* of a vertex v, if there is a directed path from v to u. Easy induction shows that the number of ancestors in C of every vertex of C is finite. Hence C is isomorphic to T_∞. Thus D embeds in T_∞, which completes our proof. $\square$

In order to describe the power graphs of all finite abelian groups, we take any finite abelian group G and any elements a, b in G, and introduce the following notation.

Denote the primary components of G by $G_{p_1}, \ldots, G_{p_n}$, and express each G_{p_i} as a direct product of cyclic groups $G_{p_i} = (C_{p_i^{w_{i,1}}})^{q_{i,1}} \times (C_{p_i^{w_{i,2}}})^{q_{i,2}} \times \cdots \times (C_{p_i^{w_{i,m_i}}})^{q_{i,m_i}}$ and $w_{i,1} < w_{i,2} < \cdots < w_{i,m_i}$. For $i = 1, \ldots, n$, denote the projections of a and b on G_{p_i}, by a_i and b_i, respectively. Choose generators $g_{i,j,k}$ in the cyclic groups of G_{p_i} above, where $1 \le j \le m_i$ and $1 \le k \le q_{i,j}$. Write a_i and b_i in the form $a_i = g_{i,1,1}^{c_{i,1,1}} \cdots g_{i,m_i,q_{i,m_i}}^{c_{i,m_i,q_{i,m_i}}}$, and $b_i = g_{i,1,1}^{d_{i,1,1}} \cdots g_{i,m_i,q_{i,m_i}}^{d_{i,m_i,q_{i,m_i}}}$, where $c_{i,j,k} = u_{i,j,k} {p_i}^{w_{i,j} - r_{i,j,k}}$, $d_{i,j,k} = v_{i,j,k} {p_i}^{w_{i,j} - s_{i,j,k}}$ and $(u_{i,j,k}, p_i) = 1$, $(v_{i,j,k}, p_i) = 1$.

Theorem 2. *Let G be a finite abelian group, and let a, b be any elements of G. Suppose that the prime factorization of the order of a is $|a| = \prod_{i=1}^{n} {p_i}^{t_i}$, where $1 \leq t_i \leq w_{i,m_i}$. Then*

(a) *a belongs to a clique of order*

$$\prod_{i=1}^{n} ({p_i}^{t_i} - {p_i}^{t_i-1}),$$

where we replace $({p_i}^{t_i} - {p_i}^{t_i-1})$ by 1 if $t_i = 0$;

(b) *(a, b) is an edge of the power graph of G if and only if, for every $i = 1, \ldots, n$,*

$${p_i}^{w_{i,j}} | v_{i,j,k} u_{i,j,k}^{\phi({p_i}^{w_{i,j}})-1} {p_i}^{r_{i,j,k}-s_{i,j,k}} - v_{i,j',k'} u_{i,j',k'}^{\phi({p_i}^{w_{i,j'}})-1} {p_i}^{r_{i,j',k'}-s_{i,j',k'}},$$

for all $1 \leq j \leq j' \leq m_i$, and $1 \leq k \leq k' \leq q_{i,j'}$.

(c) *If w_{i,h_i} is the smallest exponent in G_{p_i} such that $t_i \leq w_{i,h_i}$ then $P(G)$ contains*

$$\prod_{i=1}^{n} \frac{(p_i^{w_{i,1}})^{q_{i,1}} (p_i^{w_{i,2}})^{q_{i,2}} \ldots (p_i^{w_{i,h_{i-1}}})^{q_{i,h_{i-1}}} ((p_i^{t_i})^{q_{i,h_i}+\cdots+q_{i,m_i}} - (p_i^{t_i-1})^{q_{i,h_i}+\cdots+q_{i,m_i}})}{(p_i^{t_i} - p_i^{t_i-1})}$$

cliques of order $\prod_{i=1}^{w} ({p_i}^{t_i} - {p_i}^{t_i-1})$, for each t_i. If $t_i = 0$ for any i then we replace $({p_i}^{t_i} - {p_i}^{t_i-1})$ by 1.

Proof of Theorem 2. It is enough to focus on a primary component of G, verify all formulas, and then apply Lemma 2 to obtain complete results. To simplify notation we drop all references involving i throughout the proof. In other words, we fix i and put $p = p_i$, $t = t_i$, $w_s = w_{i,s}$, etc.

Each element of order p^t in G_p belongs to a clique of order $p^t - p^{t-1}$. Since the order of elements in different p-components are mutually co-prime, the formula in (a) follows from Lemma 2.

Consider the primary p-component $G_p = (C_{p^{w_1}})^{q_1} \times (C_{p^{w_2}})^{q_2} \times \cdots \times (C_{p^{w_m}})^{q_m}$, where the k^{th} copy of $C_{p^{w_j}}$ has generator $g_{j,k}$, for $1 \leq j \leq m$ and $1 \leq k \leq q_j$. Assume $w_1 < w_2 < \cdots < w_m$. Suppose that $a = g_{1,1}^{c_{1,1}} \ldots g_{m,q_m}^{c_{m,q_m}}$ and $b = g_{1,1}^{d_{1,1}} \ldots g_{m,q_m}^{d_{m,q_m}}$ are two elements in G_p where $g_{j,k}^{c_{j,k}}$ and $g_{j,k}^{d_{j,k}}$ have orders $p^{r_{j,k}}$ and $p^{s_{j,k}}$, respectively. Solving $a^x = b$ yields the system of congruences:

$$xc_{j,k} \equiv d_{j,k} \bmod (p^{w_j}), \text{ for all } j, k. \tag{1}$$

Each congruence, considered in isolation, is solvable if and only if $(c_{j,k}, p^{w_j})$ divides $d_{j,k}$. As in the proof of Lemma 3, this implies that $r_{j,k} \geq s_{j,k}$. Moreover, since $g_{j,k}^{c_{j,k}}$ has order $p^{r_{j,k}}$, we see that $c_{j,k}$ can be expressed as $u_{j,k} p^{w_j - r_{j,k}}$, where $(u_{j,k}, p) = 1$. Similarly, $d_{j,k} = v_{j,k} p^{w_j - s_{j,k}}$, where $(v_{j,k}, p) = 1$, for a positive integer $v_{j,k}$. Thus (1) gives us

$$\begin{aligned} x u_{j,k} &\equiv v_{j,k} p^{r_{j,k} - s_{j,k}} \bmod (p^{w_j}) \\ x &\equiv v_{j,k} u_{j,k}^{\phi(p^{w_j})-1} p^{r_{j,k}-s_{j,k}} \bmod (p^{w_j}) \end{aligned} \tag{2}$$

where ϕ is the Euler phi-function (see [4, 13.4]).

Thus $(a, b) \in E(P(G_p))$ if and only if there exists a solution to (2). The system of congruences in (2) is solvable if and only if

$$p^{w_j} \quad | \quad v_{j,k} u_{j,k}^{\phi(p^{w_j})-1} p^{r_{j,k}-s_{j,k}} - v_{j',k'} u_{j',k'}^{\phi(p^{w_{j'}})-1} p^{r_{j',k'}-s_{j',k'}},$$

for $1 \le j \le j' \le m$, and $1 \le k \le k' \le q_{j'}$. The formula in (b) follows directly from Lemma 2.

We observe that, $|a| \ge |b|$ is a necessary, but not sufficient condition for $(a, b) \in E(P(G_p))$. In $C_4 \times C_4 = \langle a \rangle \times \langle b \rangle$ we have $|ab| = 4$ and $|a^2| = 2$ but (ab, a^2) does not belong to $E(P(C_4 \times C_4))$. Moreover, $P(G_p)$ is not the direct product of the power graphs of its components as in Lemma 2 for $(a, a^2) \in E(P(C_4))$ and $(b, b^3) \in E(P(C_4))$, but $(ab, a^2b^3) \notin E(P(C_4 \times C_4))$.

Next, we count the number of cliques in $P(G)$. Suppose $|a| = p^t$ in G_p. At least one $g_{j,k}$ has order p^t in $(C_{p^{w_j}})^{q_k}$, where $w_j \ge t$. Assume w_h is the least such w_j. Then $|g_{j,k}| \le p^t$ for all other j, k.

For $j < h$ rewrite G_p in the form

$$(C_{p^1})^{f_1} (C_{p^2})^{f_2} \dots (C_{p^{h-1}})^{f_{h-1}},, \text{ where } f_j \ge 0.$$

The number of elements of order p^t is obtained by summing over all possible combinations of elements of order up to p^t in G_p. For ease in notation define $y = q_h + \cdots + q_m$ and $e_i = f_{h-1} + \cdots + f_{h-i}$, for $i = 1, \dots h-1$. Then the number of elements of order p^t to be

$$\begin{aligned} &\sum_{z_1=1}^{y} \binom{y}{z_1} (p^t - p^{t-1})^{z_1} . \Bigg(\sum_{z_2=0}^{e_1+y-z_1} \binom{e_1 + y - z_1}{z_2} (p^{t-1} - p^{t-2})^{z_2} . \Bigg(\sum \cdots \\ &\cdots \sum_{z_h=0}^{e_{h-1}+y-(z_1+\cdots+z_{h-1})} \binom{e_{h-1} + y - (z_1 + \cdots + z_{h-1})}{z_h} (p-1)^{z_h} \Bigg) \dots \Bigg) \Bigg) \end{aligned} \tag{3}$$

Now (3) is a series of nested binomial expansions and simplifies to

$$(p^1)^{f_1} (p^2)^{f_2} \dots (p^{h-1})^{f_{h-1}} ((p^t)^{q_h+\cdots+q_m} - (p^{t-1})^{q_h+\cdots+q_m}). \tag{4}$$

Resubstituting, (4) becomes

$$(p^{w_1})^{q_1} (p^{w_2})^{q_2} \dots (p^{w_{h-1}})^{q_{h-1}} ((p^t)^{q_h+\cdots+q_m} - (p^{t-1})^{q_h+\cdots+q_m})$$

Each element of order p^t in G_p belongs to a clique of order $(p^t - p^{t-1})$. Therefore $P(G_p)$ has exactly

$$\frac{(p^{w_1})^{q_1}(p^{w_2})^{q_2}\dots(p^{w_{h-1}})^{q_{h-1}}((p^t)^{q_h+\dots+q_m} - (p^{t-1})^{q_h+\dots+q_m})}{(p^t - p^{t-1})}$$

cliques containing elements of order $(p^t - p^{t-1})$ for $t = 1, 2, \dots w_m$. When $t = 0$, the identity forms a clique of order 1. Lemma 2 yields the formula in (c) and completes the proof. □

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