Hamiltonian properties and the bipartite independence number

Oscar Ordaz\textsuperscript{a}, Denise Amar\textsuperscript{b},*, André Raspaud\textsuperscript{b}  
\textsuperscript{a}Mathematics Department, Faculty of Science, Universidad Central de Venezuela, Ap. 47567, Caracas 1041-A, Venezuela  
\textsuperscript{b}LaBRI, Université de Bordeaux 1, 351, Cours de la Libération, 33405 Talence Cedex, France  
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Abstract

By using the notion of compatibility of subgraphs with a perfect matching developed for digraphs in [1], we show that if, in a balanced bipartite graph $G$ of minimum degree $\delta$, the maximum cardinality $\alpha_{\text{bip}}$ of a balanced independent subset satisfies $\alpha_{\text{bip}} \leq 2\delta - 4$, then $G$ is hamiltonian-biconnected, and if $\alpha_{\text{bip}} \leq 2\delta - 2$, $G$ contains a hamiltonian path. Moreover, we give some properties of balanced bipartite graphs which are not hamiltonian, and which satisfy $\alpha_{\text{bip}} \leq 2\delta - 2$.

1. Introduction

A simple bipartite graph with vertex set $V(G) = X \cup Y$ and edge set $E(G)$ is denoted by $G = (X, Y, E)$; $X$ and $Y$ are the partite sets.  
If $|X| = |Y|$, the bipartite graph is said to be balanced.  
The minimum degree of a graph $G$ is denoted by $\delta(G)$.  
Let $G = (X, Y, E)$ be a balanced bipartite graph of order $2n$, that is such that $|X| = |Y| = n$, with minimum degree $\delta(G) \geq 2$.  
A bipartite independent set $S$ is a balanced independent subset of $V(G)$ that is such that $|S \cap X| = |S \cap Y|$.  
The bipartite independence number $\alpha_{\text{bip}}(G)$ of a balanced bipartite graph is the maximum cardinality of a balanced independent set of $G$.  
This parameter has been introduced by Ash [2], Jackson and Ordaz [6]; its relation with hamiltonism has been studied by Fraisse [4], and Favaron et al. [3].

* Corresponding author.
If \( H \) is a subgraph of \( G \), we denote by \( N(H) \) the set of the neighbours of the vertices of \( H \) in \( G \). For any vertex \( x \in V(G) \), \( N_H(x) \) denotes the set of the neighbours of \( x \) which are in \( H \).

We denote by \( \langle G \setminus H \rangle \) the induced subgraph with vertex-set \( V(G) \setminus V(H) \).

If \( P = x_1y_1 \ldots x_py_p \) is a path in \( G \) joining two vertices \( b \in Y \) and \( a \in X \), we denote by \( bx_1y_1 \ldots x_py_a \) the path \( P^* = bx_1y_1 \ldots x_py_a \).

If \( u_i \in V(P) \), and \( v_j \in V(P) \), \( u_iPv_j \) denotes the segment of the path \( P \) with end-vertices \( u_i \) and \( v_j \).

We denote by \( N^+_P(x) \) (resp. \( N^-_P(x) \)) the set of the successors (resp. the predecessors) of the vertices of \( N_P(x) \) on \( P \), following a chosen direction.

If \( C \) is a cycle we choose arbitrarily an orientation on \( C \); if \( a \in V(C) \), and \( b \in V(C) \), \( aCb \) denotes the segment of \( C \) with end-vertices \( a \) and \( b \), following the chosen orientation, \( bCa \) the segment of \( C \) with end-vertices \( b \) and \( a \), following the opposite orientation.

We denote by \( N^+_C(x) \) (resp. \( N^-_C(x) \)) the set of the successors (resp. the predecessors) of the vertices of \( N_C(x) \) on \( C \), following the chosen orientation.

If \( G_1 = (X_1, Y_1, E_1) \) and \( G_2 = (X_2, Y_2, E_2) \) are two bipartite graphs, we denote by \( G_1 \oplus G_2 \) the bipartite graph \( G = (X, Y, E) \) where \( X = X_1 \cup X_2, Y = Y_1 \cup Y_2, \) and \( E = E_1 \cup E_2 \cup \{(a_1,b_2) \mid a_1 \in X_1, b_2 \in Y_2\} \cup \{(a_2,b_1) \mid a_2 \in X_2, b_1 \in Y_1\} \).

If \( p \in N^*_G \) and \( G \) is a graph, we use \( pG_1 \) to represent a set of \( p \) independent graphs isomorphic to \( G \).

**Definition 1.** If \( M_0 \) is a perfect matching in \( G \), we say that a cycle \( C \) (resp. a subgraph \( H \)) in \( G \) is \( M_0 \)-compatible, if \( E(M_0) \cap E(C) \) (resp. \( E(M_0) \cap E(H) \)) is a perfect matching of \( C \) (resp. \( H \)).

**Definition 2.** We say that a cycle \( C \), subgraph \( H \) resp. in \( G \) is \( M \)-compatible if there exists a perfect matching \( M_0 \) in \( G \), for which \( C, H \), resp. is \( M_0 \)-compatible.

**2. A condition for hamiltonicity**

**Theorem** (Favaron et al. [3]). If \( G \) is a balanced bipartite graph such that

\[
\alpha_{\text{bip}}(G) \leq \delta(G)
\]

then \( G \) is hamiltonian, except in two cases: \( G = 3K_{1,1} \oplus K_{1,1} \) or \( G = 3K_{1,1} \oplus K_{1,1} \).

We prove the following result which, for \( \delta(G) \geq 3 \), is an improvement of the previous one.

**Proposition 1.** If \( G \) is a balanced bipartite graph such that

\[
\alpha_{\text{bip}}(G) \leq 2\delta(G) - 4
\]

then \( G \) is hamiltonian.
We give a proof of this theorem based on the techniques of perfect matchings and cycles or subgraphs which are $M$-compatible.

**Lemma 1.** If $G$ is a balanced bipartite graph such that $\alpha_{\text{bip}}(G) \leq 2\delta(G)$, then there exists in $G$ a perfect matching.

**Proof.** By a theorem of König-Hall, it suffices to prove that for every subset $A \subset X$, $|N(A)| \geq |A|$. Assume by contradiction that for some subset $A$ of $X$, $|N(A)| < |A|$. Set $N(A) = B$. Since $|N(A)| \geq \delta$, then $|A| \geq \delta + 1$. Let $B' = Y \setminus B$ and $A' = X \setminus A$, then $N(B') \subset A'$, and we have $\delta \leq |A'| = n - |A| < n - |B| = |B'|$. This implies that $|B'| \geq \delta + 1$, and $A \cup B'$ is an independent bipartite set which contains a balanced independent set with $2\delta + 2$ vertices. □

**Proof of Proposition 1.** Assume by contradiction that $G = (X, Y, E)$ is a balanced bipartite graph of order $2n$, such that $\alpha_{\text{bip}}(G) \leq 2\delta(G) - 4$ and that $G$ is not hamiltonian.

By Lemma 1, $G$ admits a perfect matching $M$.

Let $C$ be a longest $M$-compatible cycle of $G$. The cycle $C$ is $M$-compatible; so there exists a perfect matching $M_0$ in $G$ such that $E(C) \cap M_0$, resp. $E(G \setminus C) \cap M_0$ are perfect matchings in $C$, resp. $G \setminus C$. $G$ being connected there exists a path $P$ which satisfies the following three conditions:

(i) $V(P) \subset V(G) \setminus V(C)$, and one of its end-vertex is adjacent to $C$,
(ii) $C \cup P$ is $M$-compatible,
(iii) subject to (i) and (ii), $P$ is as long as possible.

The cycle $C$ and the path $P$ are denoted by $a_1b_1a_2b_2 \ldots a_kb_k$, with $k < n$, and $x_1y_1x_2y_2 \ldots x_py_p$, respectively.

Without loss of generality, we can suppose that $x_1$ is adjacent to $b_k$.

We set $\delta = \delta(G)$.

**Case 1:** Suppose that for every cycle $C$ and path $P$ satisfying the conditions (i)–(iii), $P$ has only one end-vertex which is adjacent to $C$.

Then $N(y_p) \subset V(P)$ and $p \geq \delta(G)$. Moreover, if $\{x_p, x_{i_2}, \ldots, x_{i_d}\} \subset N(y_p)$, with $p > iu_2 > \ldots > i_d$, the vertices $y_p, y_{i_2}, \ldots, y_{i_d}$ are not adjacent to any vertex of $C$, elsewhere one can find a path $P'$ as long as $P$ with both ends adjacent to $C$. It is easy to see that $C \cup P'$ is $M$-compatible:

By definition if $C \cup P$ is $M$-compatible, there exists a perfect matching $M_0$ in $G$ such that:

$M_0 = \{(x_1, y_1) \cdots (x_p, y_p)\} \cup M'_0 \cup M''_0$, where $M'_0$ and $M''_0$ are perfect matchings of $C$ and $\langle G \setminus (C \cup P) \rangle$, respectively.

Suppose $y_{i_u}$ is adjacent to a vertex $a_s \in V(C)$. Let $P' = x_1y_1 \cdots x_{i_u}y_px_py_{p-1} \cdots y_{i_u}$; we consider the perfect matching:

$M_1 = M'_0 \cup M''_0 \cup \{(x_1, y_1) \cdots (x_{i_{u-1}}, y_{i_{u-1}}), (x_{i_u}, y_p)(x_p, y_{p-1}) \cdots (x_{i_{u+1}}, y_{i_u})\}$. 
$C \cup P'$ is $M_1$-compatible. Then it is $M$-compatible.

Then the set $S = \{a_1, a_2, \ldots, a_b, y_p, y_{i_2}, \ldots, y_{i_b}\}$ is a balanced independent set of cardinality $2\delta$, and thus we have a contradiction.

Case 2: Suppose that $y_p$ is adjacent to $C$.

Remark. The set $N(x_1) \subset C \cup P$, elsewhere we could find a path $P'$ longer than $P$ which satisfies conditions (i)-(iii).

Let $a_1 b_1 \ldots a_b b_k a_1$ be an arbitrary orientation of $C$.

Let $N_p(y_p) = \{x_p, x_{i_2}, \ldots, x_{i_b}\}$.

$|N_c(y_p)| \geq \sup(1, \delta - |N_p(y_p)|) \Rightarrow |N_c(y_p)| \geq \delta - r$.

We can find $a_j \in N_c(y_p)$, such that

$$(N_c(x_1) \cup N_c(y_p)) \cap V(a_1, C, b_{j-1}) = \emptyset.$$ 

Then $2p \leq |V(a_1 C b_{j-1})| = 2(j - 1)$, and $r \leq p \leq j - 1$.

Claim. The set $S = (\{a_1, a_2, \ldots, a_p\} \cup N_c(x_1)^+) \cup (\{y_p, y_{i_2}, \ldots, y_{i_b}\} \cup N_c(y_p)^+)$ is an independent subset of $G$.

It is easy to see that every edge between vertices of $S$ creates a cycle which is $M$-compatible and longer than $C$.

The set $S$ contains a balanced independent set with at least $2(r + (\delta - r - 1)) = 2\delta - 2$ vertices, a contradiction. $\square$

Remark. Theorem 1 is the best possible, in the following sense:

Although the graphs $G = K_{p, p} \oplus K_{1, 1}$, or $G = 3K_{p, p} \oplus K_{1, 1}$ and the graphs $G = (2p + 1)K_{1, 1} \oplus \Gamma$, where $\Gamma$ is a balanced bipartite graph with $2p$ vertices, are not hamiltonian, each of them satisfies:

$\alpha_{bip}(G) = 2p = 2\delta(G) - 2$.

3. Non-Hamiltonian bipartite graphs satisfying $\alpha_{bip}(G) = 2\delta(G) - 2$

We want to prove that bipartite graphs satisfying the condition $\alpha_{bip}(G) = 2\delta(G) - 2$ are hamiltonian except for some families of graphs we can describe and we obtain the following result.

Proposition 2. If $G$ is a balanced bipartite graph such that $\alpha_{bip}(G) = 2\delta(G) - 2$, then $G$ is hamiltonian or $G$ contains a cycle $C$ of length $2n - 2$, such that $\langle G \setminus C \rangle$ is an edge, or $G$ is isomorphic to $3K_{p, p} \oplus K_{1, 1}$ or to $3K_{p, p} \oplus \bar{K}_{1, 1}$.

Let $G = (X, Y, E)$ be a balanced bipartite graph of order $2n$, such that $\alpha_{bip}(G) = 2\delta(G) - 2$. 
By Lemma 1, we know that there exist perfect matchings in $G$.
We suppose that $G$ is not hamiltonian and does not contain a cycle $C$ such that $\langle G \backslash C \rangle$ is an edge.
We consider a longest $M$-compatible cycle $C$, and a path $P$, that satisfy the three conditions (i)-(iii) defined in the proof of the Theorem 1.
In order to prove the Proposition 2, we prove Claims 1–8.
In Section 2, we proved that if $y_p$ is not adjacent to $C$, $\alpha_{bip}(G) \geq 2\delta(G)$.
Without loss of generality, we can suppose that $x_1$ is adjacent to $b_k$ and that $y_p$ is adjacent to a vertex $a_j$ such that $(N_C(x_1) \cup N_C(y_p)) \cap V(a_1 C b_{j-1}) = \emptyset$.
Theorem 2. \[ 2p \leq |V((a_1 C b_{j-1}))| = 2(j - 1). \]

Claim 1. Under the previous hypothesis, $V(G) = V(C \cup P)$ and $p > 1$.

Proof. If $(x, \beta)$ is an edge of a matching in $\langle G \backslash (C \cup P) \rangle$, we consider the set $(\{a_1, \ldots, a_p\} \cup N_C^+(x_1) \cup \{x\}) \cup (N_C^+(y_p) \cup N_C^+(y_p))$; if it is independent, it contains a balanced independent set of cardinality $2\delta(G)$, a contradiction.
We can suppose that $x$ is adjacent to a vertex of the set $N_C^+(y_p)$.
Then, we consider vertices $a_i \in V(C) \cap X$, and $b_s \in V(C) \cap Y$, such that $j < l < s < k$,
$$(a_i, y_p) \in E(G), (x_1, b_s) \in E(G) \quad \text{and} \quad (N_C(x_1) \cup N_C(y_p)) \cap V(b_l, C, a_s) = \emptyset.$$\\
The set $(\{b_l, b_{l+1}, \ldots, b_{l+p-1}\} \cup N_C^+(y_p) \cup \{\beta\}) \cup (N_C^+(x_1) \cup N_C^+(x_1))$ is then independent, and it contains a balanced independent set of cardinality $2\delta(G)$, a contradiction. Then $V(G) = V(C) \cup V(P)$.
Theorem 2. If $G$ is not hamiltonian, and satisfies the condition $\alpha_{bip}(G) \leq 2\delta(G) - 2$, the vertices $x_1$ and $y_p$ satisfy:

$$|N_p(x_1)| = p = |N_p(y_p)|; d(x_1) = \delta(G) = d(y_p).$$\\

Proof. The set $I = (\{a_1, \ldots, a_p\} \cup N_C^+(x_1)) \cup (N_C^+(y_p) \cup N_C^+(y_p))$ is independent, then:
The equality holds, and we can complete the proof of Claim 2.

Claim 3. (1) $\{x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_p\}$ induces a complete bipartite graph.
(2) For $1 \leq i \leq p, d(x_i) = d(y_i) = \delta$, and $N(x_i) = N(x_1)$, and $N(y_i) = N(y_p)$. 
Proof. If \( |\bigcup_{i=1}^{p} N_c(x_i)| > \delta - p \), \( |\bigcup_{i=1}^{p} N_c^+(x_i) \cup \{a_1, \ldots, a_p\}| \geq \delta \), and \( S = (\bigcup_{i=1}^{p} N_c^+(x_i) \cup \{a_1, \ldots, a_p\}) \cup (N_c^+(y_p)) \cup N_c^+(y_p) \) contains a balanced independent set of cardinality \( 2\delta \).

As in Claim 1 we consider vertices \( a_i \in V(C) \cap X \), and \( b_i \in V(C) \cap Y \), such that \( j < l < s < k \), \((a_i, y_p) \in E(G), (x_i, b_s) \in E(G)\), and \((N_c(x_i) \cup N_c(y_p)) \cap V(b_t C a_s) = \emptyset\). □

Claim 4. The vertex \( a_{p+1} \) is adjacent to the set \( \{y_1, \ldots, y_p\} \cup N_c^+(y_p) \).

Proof. The set \( \{\{a_1, \ldots, a_p, a_{p+1}\} \cup N_c^+(x_1)\} \cup \{\{y_1, \ldots, y_p\} \cup N_c^+(y_p)\} \) is a balanced set of cardinality \( 2\delta \), then it is not independent; the only edges that are not excluded are between \( a_{p+1} \) and \( \{y_1, \ldots, y_p\} \cup N_c^+(y_p) \). □

Claim 5. The set \( \{a_1, \ldots, a_p, b_1, \ldots, b_p\} \) induces a complete bipartite graph such that for \( 1 \leq i \leq p \), \( d(a_i) = d(b_i) = \delta \) and \( N(a_i) = N(b_1) \) and \( N(b_1) = N(b_1) \).

The set \( \{a_s, \ldots, a_{s-1}, b_{s-1}, \ldots, b_{s-p}\} \) induces a complete bipartite graph such that for \( s - p + 1 \leq i \leq s \), \( d(a_i) = d(a_i) = \delta \) and \( N(a_i) = N(a_i) \), and for \( s - p < i \leq s - 1 \), \( d(b_i) = d(b_{s-1}) = \delta \) and \( N(b_i) = N(b_{s-1}) \).

Proof. By Claim 4, \( a_{p+1} \) is adjacent to a vertex of the set \( \{y_1, \ldots, y_p\} \cup N_c^+(y_p) \).

If \( (a_{p+1}, y_i) \in E(G) \), the cycle \( C' = x_1 \ldots y_i a_{p+1} C b_i x_1 \) is as long as \( C \) and \( M \)-compatible (between \( x_1 \) and \( y_i \), there are all the vertices of \( P \)), while if \( (a_{p+1}, b_i) \in E(G) \), with \( a_i \in N_c(y_p) \), the cycle \( C' = x_1 P y_p a_i C a_{p+1} b_i C b_i x_1 \) is as long as \( C \) and \( M \)-compatible.

In both cases, we can consider the cycle \( C' \) and the path \( P' = a_1 b_1 \ldots a_p b_p \) and apply Claim 3.

The proof for the set \( \{a_s, \ldots, a_{s-1}, b_{s-1}, \ldots, b_{s-p}\} \) is similar. □

Claim 6. We have: \( N_c(x_1) = \{b_i, s \leq i \leq k\} \) and \( N_c(y_p) = \{a_i, j \leq i \leq l\} \).

Proof. If we suppose that \( s < k \), and that there exists \( i, s \leq i < i + 1 < k \), such that \((x_1, b_i) \in E(G) \) and \((x_i, b_{i+1}) \notin E(G) \); then \( \{a_1, \ldots, a_p\} \cup N_c^+(x_1) \cup \{a_{i+2}\} \cup \{y_1, \ldots, y_p\} \cup N_c^+(y_p) \) is a bipartite independent set of cardinality \( 2\delta \). (Any edge between \( a_{i+2} \) and \( \{y_1, \ldots, y_p\} \cup N_c^+(y_p) \) creates a \( M \)-compatible cycle of length \( 2n - 2 \).

If we suppose that \( y_p \) is adjacent to \( a_i \) with \( s + 1 \leq i \leq k \), \( x_1 \) being adjacent to \( b_i \), the graph would be hamiltonian. □

Claim 7. The vertex \( y_p \) is adjacent to \( a_{p+1} \), (then \( j = p + 1 \)), and to \( a_{s-p} \), (then \( l = s - p \)),

\( a_1 \) is adjacent to \( b_s \), and \( b_p \) is adjacent to \( a_t = a_{s-p} \),
\( a_s \) is adjacent to \( b_k \), and \( b_{s-p} \) is adjacent to \( a_{p+1} \).

Proof. We suppose \((a_{p+1}, y_p) \notin E(G) \).

By Claim 4, there exists an edge \((a_{p+1}, b_r) \) with \( a_r \in N_c(y_p) \).

\[ |N_c^+(y_p) \cup \{b_1, b_2, \ldots, b_p\}| = \delta, (b_p \notin N_c^-(y_p)), |N_c^+(x_1) \cup \{x_1, x_2, \ldots, x_p\}| = \delta. \]
If \((N_c^{-}(x_1) \cup \{x_1, \ldots, x_p\}) \cup (N_c^{-}(y_p) \cup \{b_1, \ldots, b_p\})\) is not independent, the only edges that are not yet been excluded are between \(N_c^{-}(x_1)\) and \(\{b_1, \ldots, b_p\}\). If \((a_h, b_i) \in E(G)\), with \((x_1, b_i) \in E(G)\), by Claim 6, \(r \leq l \leq s \leq h\).

The cycle \(x_1 P y_p a_r C a_{p+1} b_r C a_b b_l C b_1 x_1\) is an \(M\)-compatible cycle which is longer than \(C\).

Then \((N_c^{-}(x_1) \cup \{x_1, \ldots, x_p\}) \cup (N_c^{-}(y_p) \cup \{b_1, \ldots, b_p\})\) is a bipartite independent set of cardinality \(2\delta\), a contradiction.

A similar proof applies for the other cases. \(\square\)

Let \(T\) be the union of the two segments \(a_l C a_l\) and \(b_l C b_k\) of \(C\).

**Claim 8.** We have: \(N_T(a_1) = N_T(x_1) = N_T(a_1)\) and \(N_T(b_p) = N_T(y_p) = N_T(b_1)\).

**Proof.** It is an immediate consequence of Claims 6 and 7. \(\square\)

**Proof of Proposition 2.** We suppose that \(s < k\); let \(i, s + 1 \leq i \leq k\), then, by Claim 6, \(a_i \in N_c(x_1)\).

The vertex \(a_i\) is independent of any vertex of \(N_c^{-}(y_p) \cup \{y_1, \ldots, y_p\} \cup \{b_1, \ldots, b_p\} \cup \{b_{s-p}, \ldots, b_{s-1}\}\); by Claim 8, \(a_i\) is independent of \(\{y_1, \ldots, y_p\} \cup \{b_1, \ldots, b_{s-1}\}\), and \(N(a_i) \subseteq \{b_s, \ldots, b_k\}\).

Then \(|\{b_s, \ldots, b_k\}| \geq \delta\); that implies \(|N(x_1)| \geq \delta + p\), a contradiction with Claim 2; we can conclude that \(s = k\); by a similar proof, we can conclude that \(j = l\). \(\square\)

As a corollary of the previous result, we can obtain the following result.

**Corollary.** If \(G\) is a balanced bipartite graph such that

\[
\alpha_{bip} \leq 2\delta(G) - 2
\]

then \(G\) has a hamiltonian path.

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**4. A condition to be hamilton-biconnected**

**Theorem 1.** Let \(G\) be a balanced bipartite graph such that \(\alpha_{bip}(G) \leq 2\delta(G) - 4\). Then \(G\) is hamiltonian-biconnected.

**Proof.** Let \(G\) be a balanced bipartite graph such that \(\alpha_{bip}(G) \leq 2\delta(G) - 4\).

We suppose that it is not hamilton-biconnected. Then, there exist \(u \in X, v \in Y\), such that there is no hamiltonian path in \(G\) with end-vertices \(u\) and \(v\).

Let \(\Gamma = \langle G \setminus \{u, v\} \rangle\); we have: \(\delta(\Gamma) \geq \delta(G) - 1\), \(\alpha_{bip}(\Gamma) \leq \alpha_{bip}(G) = 2\delta(G) - 4\).
Claim. The graph $\Gamma$ is not hamiltonian, and then $\alpha_{bip}(\Gamma) = \alpha_{bip}(G) \leq 2\delta(G) - 4$ and $\delta(\Gamma) = \delta(G) - 1$.

Proof. If $\Gamma$ is hamiltonian, let $K$ be an hamiltonian cycle in $\Gamma$; we give it an arbitrary orientation;

$N^+_K(u) \cup N^+_K(v)$ contains a balanced subset of cardinality $2\delta(G) - 2$ vertices. Then it is not independent, and any edge between $N^+_K(u)$ and $N^+_K(v)$ creates an hamiltonian path with end-vertices $u$ and $v$.

Then $\alpha_{bip}(\Gamma) \geq 2\delta(\Gamma) - 2$.

We have:

$$2\delta(G) - 2 \leq \alpha_{bip}(\Gamma) \leq \alpha_{bip}(G) \leq 2\delta(G) - 4 \leq 2(\delta(\Gamma) + 1) - 4 = 2\delta(\Gamma) - 2.$$  

Then $\alpha_{bip}(G) = \alpha_{bip}(\Gamma) = 2\delta(\Gamma) - 2$, and $\delta(G) = \delta(\Gamma) + 1$. □

Proof of Theorem 1. The graph $\Gamma$ satisfies the hypothesis of Proposition 1.

If $\Gamma$ is isomorphic to $3K_{p,p} \oplus K$ with $K = K_{1,1}$ or $K = K_{1,1}$, the degrees of the vertices of the subgraphs $K_{p,p}$ are $\delta(\Gamma) = \delta(G) - 1$, then $u$ and $v$ are adjacent to every vertex of the subgraphs $K_{p,p}$, not belonging to the same partite set. It is then easy to find an hamiltonian path in $G$ joining $u$ and $v$.

If $\Gamma$ contains a cycle $C$ of length $2|V(\Gamma)| - 2$, such that $\langle G \setminus C \rangle$ is an edge $(x_1, y_1)$, by Claim 5, $d(x_1) = d(y_1) = \delta(\Gamma) = \delta(G) - 1$, and with the same notations than in the proof of Proposition 1, $d(a_1) = \delta(G) - 1$; then $(u, y_1) \in E(G)$ and $(a_1, v) \in E(G)$; the path $u y_1 x_1 a_1 v$ is an hamiltonian path joining $u$ and $v$. □

5. Cycles through specified paths

In [5] Häggkvist and Thomassen proved the following theorem in general graphs:

Theorem (Häggkvist, and Thomassen [5]). Let $\alpha$ be the stability number of a graph $G$. If $G$ is $(\alpha + k)$-connected, then any system of disjoint paths in $G$ of total length at most $k$ can be extended into a hamiltonian cycle.

We have an analogous result for bipartite graphs to the Theorem of Häggkvist and Thomassen.

If $G$ is a bipartite graph, we consider disjoint paths $P_i$, $1 \leq i \leq s$ in $G$, of odd length (then $|V(P_i)|$ is an even integer, and the end-vertices of the different paths are in distinct partite sets).

Let $x_i \in X$ and $y_i \in Y$ be the end-vertices of $P_i$.

We can suppose that the end-vertices of different paths $(P_i)$ are independent: if $P_i$ and $P_j$ have adjacent end-vertices, for example if $(y_i, x_j) \in E(G)$, we can replace $P_i \cup P_j$ by the path $x_i P_i y_i x_j P_j y_j$. 

**Theorem 2.** If \( P_i, 1 \leq i \leq s \), is a system of disjoint paths of odd lengths, with independent end-vertices, such that \( \sum_{1 \leq i \leq s} |V(P_i)| = 2t \), and \( G \) satisfies the relation:

\[
\alpha_{\text{bip}}(G) \leq 2\delta(G) - 2t - 4
\]

then, there exists in \( G \) a hamiltonian cycle which contains every path \( P_i \).

**Proof.** We can suppose \( \alpha_{\text{bip}}(G) \geq 2 \), then \( \delta(G) - t \geq 3 \).

Let \( G' \) be the induced subgraph \( G' = (G\setminus(\bigcup_{1 \leq i \leq s} P_i)) \).

\( G' \) is a balanced bipartite graph; its minimum degree satisfies \( \delta(G') \geq \delta(G) - t \); its independence bipartite number satisfies \( \alpha_{\text{bip}}(G') \leq \alpha_{\text{bip}}(G) \).

Then \( \alpha_{\text{bip}}(G') \leq 2\delta(G') - 4 \), and \( G' \) is hamiltonian.

Let \( C = a_1b_1 \ldots a_nb_a \) be a hamiltonian cycle in \( G' \).

We choose an orientation on \( C \).

If \( P_1 \) is one of the paths, \( |N_C(x_1)| \geq \delta(G) - t \) and \( |N_C(y_1)| \geq \delta(G) - t \).

Let \( S_1 = N_C(x_1) + N_C(y_1) \); \( S_1 \) contains a balanced set of cardinality \( 2(\delta(G) - t) \).

As \( \alpha_{\text{bip}}(G') < 2(\delta(G) - t) \), \( S_1 \) is not independent:

\[ \exists b \in N_C(x_1) \text{ and } \exists a \in N_C(y_1) \text{ such that } (a^+,b^+) \in E(G). \]

The cycle \( C_1 = x_1P_1y_1aCb+a^+Cbx_1 \) is a cycle in \( G \) containing the path \( P_1 \).

If we suppose that \( C_h \) is a cycle in \( G \) which contains the paths \( P_i \), for \( 1 \leq i \leq h \leq s \).

If \( h = s \), the theorem is proved.

If \( h < s \), we choose an orientation on \( C_h \), and if \( P_{h+1} \) is a path with end-vertices \( x_{h+1} \) and \( y_{h+1} \), we consider \( S_{h+1} = N_C(x_{h+1})^+ \cup N_C(y_{h+1}^+) \).

The set \( S_{h+1} \) is not independent and, as in the previous case, with an edge between a vertex of \( N_C(x_{h+1})^+ \) and a vertex of \( N_C(y_{h+1}^+) \), we create a new cycle \( C_{h+1} \) which contains the paths \( P_i \) for \( 1 \leq i \leq h + 1 \). \( \square \)

**References**


