NOTE

RATIONAL $\omega$-LANGUAGES ARE NON-AMBIGUOUS

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Abstract. We prove that every rational $\omega$-language can be recognized by a non-ambiguous automaton, i.e., an automaton which accepts every infinite word in at most one way.

One knows (see [1] for example) that a rational $\omega$-language cannot be recognized by a deterministic automaton. However, one can ask whether it can be recognized by a non-ambiguous automaton which, although nondeterministic, accepts a word in the $\omega$-language in only one way. We answer this question by proving the following proposition.

Proposition. Every rational $\omega$-language is recognized by a non-ambiguous automaton.

Notation. An automaton over a finite alphabet $A$ is a 4-uple $\mathcal{A} = (Q, Q_0, Q_{inf}, \delta)$ where

$Q$ is a finite set of states,

$Q_0 \subseteq Q$ is the set of initial states,

$Q_{inf} \subseteq Q$ is a set of designated states,

$\delta : Q \times A \rightarrow \mathcal{P}(Q)$ is the transition mapping.

For every infinite word $u = u(1)u(2) \cdots u(n) \cdots \in A^\omega$ and for every state $q \in Q$, we define a computation of $u$ from $q$ in $\mathcal{A}$ as being an infinite sequence $\{q_i\}$ of states such that $q_0 = q$ and $q_i \in \delta(q_{i-1}, u(i))$ for $i \geq 1$. We say that a computation $\{q_i\}$ of $u$ is successful if $q_0 \in Q_0$ and $\{i \mid q_i \in Q_{inf}\}$ is infinite. The $\omega$-language recognized by $\mathcal{A}$ is the set $L(\mathcal{A})$ of all infinite words $u$ which have a successful computation in $\mathcal{A}$. The automaton $\mathcal{A}$ is said to be non-ambiguous if for every $u$ in $L(\mathcal{A})$ there exists only one successful computation of $u$ in $\mathcal{A}$. Finally, an $\omega$-language $L$ is said to be rational if it is recognized by an automaton.
The starting point of the proof of the proposition is the following version of
the Büchi–MacNaughton Theorem.

**Theorem.** Every rational ω-language can be recognized by a deterministic Muller
automaton.

Here a deterministic Muller automaton is a 4-uple $A = (Q, Q_0, \mathcal{C}, \delta)$ where $Q,$ $Q_0$ and $\delta$ are as above but $\text{Card}(\delta(q, a))$ is always less than 1, and $\mathcal{C} \subseteq \mathcal{P}(Q)$; the
set of infinite words recognized by $A$ is the set of words $u$ such that the (unique
if it exists) computation of $u$ in $A$ satisfies

$$q_0 \in Q_0 \quad \text{and} \quad \{q \in Q \mid \{i \mid q_i = q\} \text{ is infinite} \} \in \mathcal{C}.$$ 

Now we are ready for the proof. Let $L$ be any rational ω-language and let
$A = (Q, Q_0, \mathcal{C}, \delta)$ be a deterministic Muller automaton recognizing it.

**Proof of the Proposition.** First, let us define, for every $T$ in $\mathcal{C}$, the deterministic
Muller automaton $A_T = (Q, Q_0, \{T\}, \delta)$. Obviously, $L(A)$ is the disjoint union of
the $L(A_T)$, since if $u \in L(A_T) \cap L(A_T')$, the unique computation of $u$ in $A_T$ satisfies
$T = \{q \in Q \mid \{i \mid q_i = q\} \text{ is infinite} \} = T'$. Now the disjoint union of ω-languages recog-
nized by non-ambiguous automata is recognized by the disjoint union of these
automata which is still non-ambiguous. Thus it remains to prove that $L(A_T)$ is
recognized by a non-ambiguous automaton.

Let us remark that any word $u$ in $L(A_T)$ can be written in a unique way in the
form $vw$ for $w$ such that

$$u \in A^*.$$

Thus, assuming $T = \{s_0, s_1, \ldots, s_n \}$, we consider the automaton $\mathcal{A}' = (Q', Q_0', Q_0', \delta')$ where

$$Q' = Q \cup \{w \times \{0, 1, \ldots, n\} \}, \quad Q_0' = Q_0 \cup \{(T \cap Q_0) \times \{0\}\},$$

and $\delta'$ is defined by

if $q' = \delta(q, a)$, then

$$q' \in \delta'(q, a),$$

$(q', 0) \in \delta'(q, a)$ iff $q \in T$ and $q' \in T$.
Rational $\omega$-languages are not ambiguous

\[(q', i) \in \delta'( (q, i), a) \quad \text{iff} \quad q \in T \text{ and } q' \neq s_i,\]
\[(s_n, i + 1) \in \delta'( (q, i), a) \quad \text{iff} \quad q \in \Gamma, q' = s_i \text{ and } i < n,\]
\[(q', 0) \in \delta'( (q, n), a).\]

It is just an exercise to prove that $L(\mathcal{A}') = L(\mathcal{A})$. Moreover, to every successful computation \{\[q_i\] \} of \(u\) in \(\mathcal{A}'\) we have either

1. \(\forall i \geq 0: q_i = (q_i, n_i)\) with \(q_i \in T\), or
2. \(\exists k > 0: \bar{q}_k \in Q - T \text{ and } \forall i > k + 1, q_i = (q_i, n_i)\) with \(q_i \in T\).

In both cases we get a decomposition of \(u\) in \(w\) or \(vww\) which satisfies \((*)\). Since this decomposition is unique, \(u\) has only one successful computation in \(\mathcal{A}'\) and \(\mathcal{A}'\) is non-ambiguous. \(\square\)

Some other properties of rational $\omega$-languages can be derived from the previous construction of $\mathcal{A}'$.

1. Like the automaton constructed by Karpinski in [2], $\mathcal{A}'$ is of 'nondeterministic rank' 2 and we get Theorem 2 of [2].
2. More important is the following improvement of a part of the Büchi MacNaughton theorem:

Every rational $\omega$-language $L$ has a non-ambiguous decomposition in the form $\bigcup_{i=1, \ldots, n} U_i V_i^\omega$

which means that every word $u$ in $L$ has a unique decomposition in the form $uw_1w_2 \cdots w_n \cdots$ with $u \in U_i$ and $v_n \in V_i$.

References