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On graded-coherent like properties of commutative graded rings: a survey

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Abstract. Let R be a commutative ring with nonzero identity and Γ (or sometimes G) a commutative monoid. The concept of coherent rings is one of the most significant notions in homological algebra. Because of its importance, there have been many generalizations to the notion of coherent rings. Some of them are linked the context of graded rings. In this paper, we survey known results concerning the different graded generalizations of coherent rings and modules.

Key Words: graded-Noetherian, graded-valuation domain, graded-valuation ring, graded Prüfer domains, graded-coherent, uniformly graded-coherent, graded-v-coherent, graded-quasi-coherent, graded-finite-conductor, graded-n-coherent, graded-S-coherent.

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1 Introduction

We devote this section to some conventions and a review of some standard background material on graded modules and rings, see for instance [19, II, §11, pp. 163–176]. All rings are commutative with unity, Γ or sometimes G will denote a grading commutative monoid (that is, a commutative monoid, written additively, with an identity element denoted by 0), and all the graded rings and modules are graded by Γ. By a *graded ring* R, we mean a ring graded by Γ, that is, a direct sum of subgroups R_α of R such that $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for every $\alpha, \beta \in \Gamma$. A nonzero element $x \in R$ is called *homogeneous* if it belongs to one of the R_α, *homogeneous of degree* α if $x \in R_\alpha$. The element 0 is therefore homogeneous of all degrees; but if $x \neq 0$ is homogeneous, it belongs to only one of the R_α; the index α such that $x \in R_\alpha$ is then called the *degree* of x and is sometimes denoted by $deg(x)$. Every $y \in R$ may be written uniquely as a sum $\sum_\alpha y_\alpha$ of homogeneous elements with $y_\alpha \in R_\alpha$; y_α is called the *homogeneous component* of degree α of y. If Γ is cancellative, that is if every element of Γ is cancellable (An element δ of the grading monoid Γ is called *cancellable* if the relation $\lambda + \delta = \mu + \delta$ implies $\lambda = \mu$, for every $\lambda, \mu \in \Gamma$), then R_0 is a subring of R (and in particular $1 \in R_0$).

By a *graded* R*-module* $M = \bigoplus_{\alpha\in\Gamma} M_\alpha$, we mean an R-module graded by Γ, that is, a direct sum of subgroups M_α of M such that $R_\alpha M_\beta \subseteq M_{\alpha+\beta}$ for every $\alpha, \beta \in \Gamma$. A graded R-module M is called a *graded-free* R*-module* if there exists a basis $(m_\iota)_{\iota\in I}$ of M consisting of homogeneous elements. Note that, any graded-free R-module is a free R-module; the converse is false [48, p. 21]. When Γ is cancellative, the M_α are R_0-modules. Clearly, R is a graded R-module. We can form a new graded R-module by twisting the grading on M as follows: if $\Gamma \ni \alpha_0$ is cancellable (that is, the relation $\lambda + \alpha_0 = \mu + \alpha_0$ implies $\lambda = \mu$, for every $\lambda, \mu \in \Gamma$), define $M(\alpha_0) = \bigoplus_{\alpha\in\Gamma} M(\alpha_0)_\alpha$ (read "M *twisted by* α_0"), by $M(\alpha_0)_\alpha = M_{\alpha+\alpha_0}$. When Γ is a group, the underlying R-module of the graded R-module $M(\alpha_0)$ is identified with M.

Let R and R' be two graded rings, a ring homomorphism $h : R \to R'$ is called *graded* if $h(R_\alpha) \subseteq R'_\alpha$ for all $\alpha \in \Gamma$. A *graded ring isomorphism* is a bijective graded ring homomorphism. Let M and M' be two graded R-modules and let $u : M \to M'$ be an R-module homomorphism and $\delta \in \Gamma$; u is called *graded of degree* δ if $u(M_\alpha) \subseteq M'_{\alpha+\delta}$ for all $\alpha \in \Gamma$. An R-module homomorphism $u : M \to M'$ is called *graded* if there exists $\delta \in \Gamma$ such that u is graded of degree δ. A *graded R-module isomorphism* is a bijective graded R-module homomorphism of degree 0. If $u \neq 0$ and Γ is cancellative, the degree of u is then determined uniquely. An *exact sequence of graded R-modules* is an exact sequence where the R-modules and the R-module homomorphisms in question are graded.

A submodule N of M is called *homogeneous* if $N = \bigoplus_{\alpha\in\Gamma}(N \cap M_\alpha)$. It is well known that the following are equivalent for a submodule N of M: (1) N is homogeneous; (2) the homogeneous components of every element of N belong to N; (3) N is generated by homogeneous elements. A homogeneous submodule of R is called a *homogeneous ideal* of R. If N is a homogeneous submodule of a graded R-module M, then M/N is a graded R-module, where $(M/N)_\alpha := (M_\alpha + N)/N$. If I is a homogeneous ideal of a graded ring R, then R/I is a graded ring, where $(R/I)_\alpha := (R_\alpha + I)/I$.

Let R be a graded ring and let M be a graded R-module. If the grading monoid Γ is a group and if S is a multiplicatively closed set of homogeneous elements of R, then $S^{-1}R$ is a graded ring and $S^{-1}M$ is a graded $S^{-1}R$-module, where $(S^{-1}R)_i = \{\frac{r}{s} \mid r \in R_j, s \in R_k \textit{ and } j-k = i\}$ and $(S^{-1}M)_i = \{\frac{m}{s} \mid m \in M_j, s \in R_k \textit{ and } j-k = i\}$.

A *direct system $(R_\lambda, \phi_{\mu\lambda})$ of graded rings* is a direct system of rings such that each R_λ is graded and each $\phi_{\mu\lambda}$ is a homomorphism of graded rings. If $(R^\alpha_\lambda)_{\alpha\in\Gamma}$ is the graduation of R_λ and if we write $R = \varinjlim R_\lambda$, $R^\alpha = \varinjlim R^\alpha_\lambda$, then $(R^\alpha)_{\alpha\in\Gamma}$ is a graduation of R and R is a graded ring. If $\phi_\lambda : R_\lambda \to R$ is the canonical mapping, ϕ_λ is a homomorphism of graded rings.

Let R be a graded ring and let M and N be graded R-modules. Define $(M \otimes_R N)_\lambda$ as the additive group of $M \otimes_R N$ generated by the $x_\mu \otimes y_\nu$, where $x_\mu \in M_\mu$, $y_\nu \in N_\nu$ and $\mu + \nu = \lambda$. Then $((M \otimes_R N)_\lambda)_{\lambda\in\Gamma}$ is a graduation of $M \otimes_R N$ and $M \otimes_R N$ is a graded R-module. Let M' and N' be other graded R-modules and $u : M \to M'$, $v : N \to N'$ be graded homomorphisms of respective degrees α and β. Then $u \otimes v$ is a graded R-module homomorphism of degree $\alpha + \beta$.

Let R be a graded ring and let M and N be graded R-modules. If the grading monoid Γ is a group and if M is finitely generated, then $Hom_R(M,N)$ is a graded R-module, where $(Hom_R(M,N))_\alpha$ denotes the additive group of graded homomorphisms of degree α of M into N.

2 Commutative graded-coherent rings

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a commutative ring with unity graded by an arbitrary grading commutative monoid Γ. We say that R is a graded-coherent ring if every finitely generated homogeneous ideal of R is finitely presented. In this section, all the results of which, unless otherwise stated are taken from [14], where the authors generalized several facts on coherent rings to graded-coherent rings. Let R be a ring. An R-module M is called a *finitely presented R-module* if there is a *finite presentation of M*, that is, an exact sequence $F_1 \to F_0 \to M \to 0$ of R-modules such that both F_0 and F_1 are finitely generated free R-modules. Any finitely presented R-module is a finitely generated R-module; while the converse holds for Noetherian rings R, even though the converse is false in general. A finitely generated R-module M is said to be a *coherent R-module* if every finitely generated submodule of M is finitely presented; and a ring R is said to be a *coherent ring* if R is coherent as an R-module. An excellent summary of the works done on coherence up to 1989 can be found in [31]. See for instance [2, 16, 21, 24, 25, 26, 27, 29, 32, 33, 37, 56, 57].

Among the many generalizations of the concept of coherence has known, there is a version linked the notion of graded modules and graded rings. The notion of graded-coherent rings, for rings graded by $\mathbb{Z}$, was introduced by Cohen [22]. Let R be a graded ring. A finitely generated graded

R-module M is said to be a *graded-coherent R-module* if every finitely generated homogeneous submodule of M is finitely presented; and a graded ring R is said to be a *graded-coherent ring* if it is graded-coherent as a graded R-module.

2.1 Graded-coherent modules

Definition 2.1. [14, Definition 2.1] Let R be a graded ring. A finitely generated graded R-module M is called a graded-coherent R-module if every finitely generated homogeneous submodule of M is finitely presented.

The next remark collects some immediate classes of graded-coherent modules.

Remark 2.2. [14, Remark 2.2] Let R be a graded ring. Then:

1. Every finitely generated homogeneous submodule of a graded-coherent R-module is a graded-coherent R-module.
2. Any coherent graded R-module is a graded-coherent R-module.

Recall from the introduction that if R is a graded ring and F is a graded R-module, then F is called a *graded-free R-module* if there exists a basis of F consisting of homogeneous elements. The next proposition characterizes graded-coherent modules, in the case where the grading monoid is a group.

Proposition 2.3. *[14, Proposition 2.3] Assume that the grading monoid Γ is a group. Let R be a graded ring and let M be a finitely generated graded R-module. The following assertions are equivalent:*

1. *M is graded-coherent.*
2. *The kernel of every graded R-module homomorphism $F \to M$, where F is a finitely generated graded-free R-module, is finitely generated.*
3. *The kernel of every graded R-module homomorphism $N \to M$, where N is a finitely generated graded R-module, is finitely generated.*

The next result yields most of the elementary properties of graded-coherent modules.

Theorem 2.4. [14, Theorem 2.4] Let R be a graded ring and let $0 \to P \xrightarrow{\alpha} N \xrightarrow{\beta} M \to 0$ be an exact sequence of graded R-modules. Then:

1. If P is finitely generated, N is graded-coherent and β has a cancellable degree, then M is graded-coherent.
2. If M and P are graded-coherent and α has a cancellable degree, then N is graded-coherent.
3. If M and N are graded-coherent , then so is P.

The monoid Γ is said to be *cancellative* if every element of Γ is cancellable. Here are some corollaries of Theorem 2.4.

Corollary 2.5. *[14, Corollary 2.5] Assume that the grading monoid Γ is cancellative. Let R be a graded ring and let $0 \to P \to N \to M \to 0$ be an exact sequence of graded R-modules. If any two of the modules P, N and M are graded-coherent, then so is the third.*

Corollary 2.6. *[14, Corollary 2.6] Let R be a graded ring and let $f : M \to N$ be a graded R-module homomorphism with M and N graded-coherent. Then:*

1. *$Im(f)$ and $Coker(f)$ are graded-coherent R-modules.*
2. *If f has a cancellable degree, then $Ker(f)$ is a graded-coherent R-module.*

Corollary 2.7. *[14, Corollary 2.7] Every finite direct sum of graded-coherent R-modules is a graded-coherent R-module.*

Corollary 2.8. *[14, Corollary 2.8] If M and N are two finitely generated homogeneous submodules of a graded-coherent R-module E, then $M+N$ and $M \cap N$ are graded-coherent.*

Corollary 2.9. *[14, Corollary 2.9] Assume that the grading monoid Γ is a group. Let R be a graded ring and let M and N be two graded-coherent R-modules. Then $M \otimes_R N$ and $Hom_R(M,N)$ are graded-coherent R-modules.*

Recall from the Introduction that if R is a graded ring, N is a graded R-module and α_0 is a cancellable element of Γ, then we can form a new graded R-module $N(\alpha_0)$ by *twisting* the grading on N as follows: define $N(\alpha_0) = \bigoplus_{\alpha \in \Gamma} N(\alpha_0)_\alpha$ where $N(\alpha_0)_\alpha = N_{\alpha+\alpha_0}$. To prove the previous Corollary 2.9, the authors established the following lemma.

Lemma 2.10. *[14, Lemma 2.10] Assume that the grading monoid Γ is a group. If N is a graded-coherent R-module, then the twisting $N(\lambda)$ is a graded-coherent R-module for every $\lambda \in \Gamma$.*

The next proposition clarifies the situation for localizations of graded-coherent modules.

Proposition 2.11. *[14, Proposition 2.11] Assume that the grading monoid Γ is a group. Let R be a graded ring and let S be a multiplicatively closed set of homogeneous elements of R. If M is a graded-coherent R-module, then $S^{-1}M$ is a graded-coherent $S^{-1}R$-module.*

This section is closed with a result concerning scalar restrictions.

Theorem 2.12. [14, Theorem 2.12] Let $\phi : R \to S$ be a graded ring homomorphism and let M be a graded S-module. Then:

1. If the R-module S is finitely generated and M is graded-coherent over R, then M is graded-coherent over S
2. If ϕ is surjective and the R-module S is finitely presented and M is graded-coherent over S, then M is graded-coherent over R.

2.2 Graded-coherent rings

This section initiates the study of graded-coherent rings.

Definition 2.13. [14, Definition 3.1] A graded ring R is called a graded-coherent ring if it is graded-coherent as a graded R-module, that is, if every finitely generated homogeneous ideal of R is finitely presented.

Obviously, every coherent graded ring is a graded-coherent ring. The converse is not true in general, that is, there exist graded-coherent rings which are not coherent, as shown by the following example.

Example 2.14. [14, Example 3.2] If A is a countable direct product of $\mathbb{Q}[[t,u]]$'s, then the polynomial ring $A[X]$ is graded-coherent but not coherent (where $A[X]$ is graded by $\mathbb{N}$ via $(A[X])_n = AX^n$ for every $n \in \mathbb{N}$).

The next result characterizes graded-coherent rings.

Theorem 2.15. [14, Theorem 3.3] Let R be a graded ring. Consider the following assertions:

1. R is a graded-coherent ring.
2. Every finitely presented graded R-module is graded-coherent.
3. Every finitely generated homogeneous submodule of a graded-free R-module is finitely presented.
4. $(I : a)$ is finitely generated, for every finitely generated homogeneous ideal I of R and for every homogeneous element $a \in R$.
5. $(0 : a)$ is finitely generated, for every homogeneous element $a \in R$ and the intersection of two finitely generated homogeneous ideals of R is finitely generated.

Then (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). Moreover, if the grading monoid Γ is a group, then the five assertions are equivalent.

Example 2.16. [14, Example 3.4] Armed with Theorem 2.15, the authors in [14] were able to produce many examples of graded-coherent rings:

1. Graded-Noetherian rings, that is, graded rings in which every homogeneous ideal is finitely generated, see [20].
2. Graded-valuation domains, see [4].
3. Graded Prüfer domains, see [5].

The next result is an analogue of the fact that the coherence property for rings is stable under direct limit for directed families of rings whose transition maps are flat, [18, I, Exercice 12, e), p. 63].

Theorem 2.17. [14, Theorem 3.5] Let $(R_\lambda)_{\lambda\in S}$ be a direct system of graded rings and let $R := \varinjlim R_\lambda$. Assume that R is a flat R_λ-module and that R_λ is a graded-coherent ring for every $\lambda \in S$. Then R is a graded-coherent ring.

Let R be a ring and let $\{x_1,...,x_d\}$ be (commuting) algebraically independent indeterminates over R. For $m = (m_1,...,m_d) \in \mathbb{N}^d$, let $x^m = x_1^{m_1}...x_d^{m_d}$. Then the polynomial ring $S = R[x_1,...,x_d]$ is graded by $\mathbb{N}$, via

$$S_n = \left\{ \sum_{m\in\mathbb{N}^d} r_m x^m \Big| r_m \in R \text{ and } \sum_{i=1}^{d} m_i = n \right\}.$$

As an application of Theorem 2.17, we have:

Corollary 2.18. *[14, Corollary 3.6] Let R be a ring and let $\{x_1,x_2,...\}$ be (commuting) algebraically independent indeterminates over R such that $R[x_1,...,x_d]$ is a graded-coherent ring for all positive integers d. Then $R[x_1,x_2,...]$ is a graded-coherent ring.*

The following result clarifies the situation for factor rings of graded-coherent rings.

Theorem 2.19. [14, Theorem 3.7] Let I be a homogeneous ideal of a graded ring R. Then:

1. If R is a graded-coherent ring and I is finitely generated, then R/I is a graded-coherent ring.
2. If R/I is a graded-coherent ring and I is a graded-coherent R-module, then R is a graded-coherent ring.

If A is a ring, then the polynomial ring $S = A[X]$ is graded by $\mathbb{N}$ via $S_n = AX^n$ for every $n \in \mathbb{N}$. New examples of graded-coherent rings may stem from Theorem 2.19 and Example 2.14, as shown by the following constructions.

Example 2.20. [14, Example 3.8] If A is a countable direct product of $\mathbb{Q}[[t,u]]$'s, then $A[X]/(X^s)$ is an $\mathbb{N}$-graded-coherent ring for all $s \in \mathbb{N}$.

The next result, concerning localisation of graded-coherent rings, is straightforward, by Proposition 2.11.

Proposition 2.21. *[14, Proposition 3.9] Assume that the grading monoid Γ is a group. If R is a graded-coherent ring and S is a multiplicatively closed set of homogeneous elements of R, then $S^{-1}R$ is a graded-coherent ring.*

If A is a ring, then the polynomial ring $S = A[X]$ is graded by $\mathbb{Z}$ via $S_n = AX^n$ for $n \geq 0$ and $S_n = 0$ for $n < 0$. Another example of a graded-coherent ring is provided by Proposition 2.21 and Example 2.14.

Example 2.22. [14, Example 3.10] If A is a countable direct product of $\mathbb{Q}[[t,u]]$'s, then the Laurent polynomial ring $A[X,X^{-1}]$ is a $\mathbb{Z}$-graded-coherent ring.

This section is closed by a result concerning the product of graded-coherent rings.

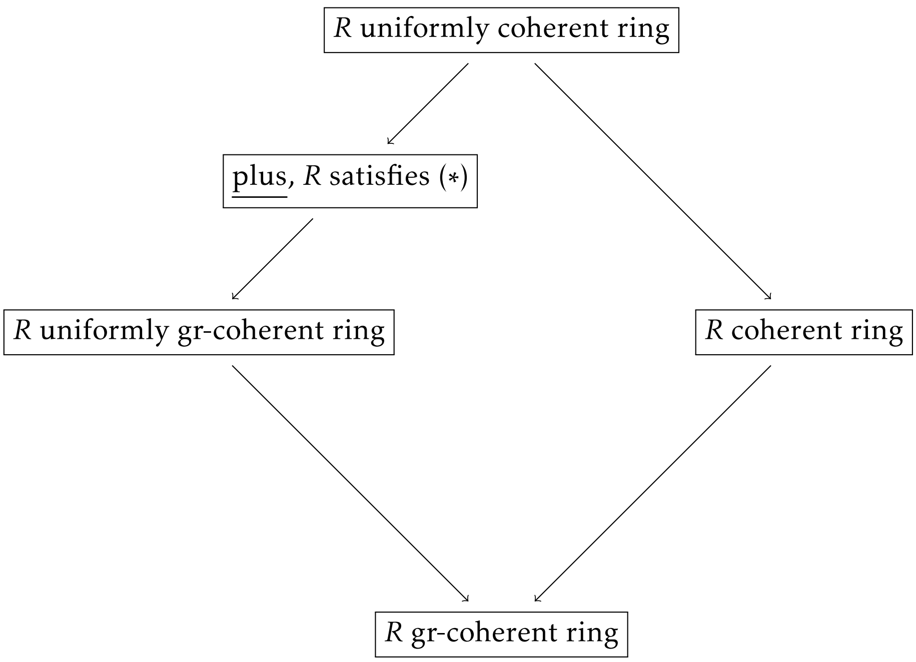
Proposition 2.23. *[14, Proposition 3.11] Let R_1 and R_2 be graded rings. Then R_1 and R_2 are graded-coherent rings if and only if $R_1 \times R_2$ is a graded-coherent ring.*

3 Uniformly graded-coherent rings

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a ring graded by an arbitrary grading abelian group Γ. R is called a uniformly graded-coherent ring if there is a map $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any nonzero graded R-module homomorphism $f : \bigoplus_{i=1}^{n} R(-\lambda_i) \to R$ of degree 0, where $\lambda_1,\ldots,\lambda_n$ are degrees in Γ, $\ker f$ can be generated by $\phi(n)$ homogeneous elements. In this this section, all the results of which, unless otherwise stated, are shown in [15], the authors provide many elementary properties of uniformly graded-coherent rings.

Let R be a ring. A finitely generated R-module M is said to be a *uniformly coherent R-module* if there is a map $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any nonzero R-module homomorphism $f : R^n \to M$, $\ker f$ can be generated by $\phi(n)$ elements; and a ring R is said to be a *uniformly coherent ring* if R is uniformly coherent as an R-module.

It is interesting to note that in [15], the authors generalized the concept of uniform coherence to the context of Γ-graded rings and modules, as follows. Let R be a graded ring. A finitely generated graded R-module M is said to be a *uniformly gr-coherent R-module* if there is a map $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any nonzero graded R-module homomorphism $f : \bigoplus_{i=1}^{n} R(-\lambda_i) \to M$ of degree 0, where $\lambda_1,\ldots,\lambda_n$ are degrees in Γ, $\ker f$ can be generated by $\phi(n)$ homogeneous elements; the map ϕ is called a *uniformity map* of M. The graded ring R is said to be a *uniformly gr-coherent ring* if it is uniformly gr-coherent as a graded R-module. The diagram bellow (Figure 1), summarizes the relations between the (graded-) coherent like notions involved in this section.



$(*)$ $R_\alpha = 0$ for all but finitely many $\alpha \in \Gamma$.

Figure 1: The relations between the (graded-) coherent like notions for a graded ring R.

3.1 Uniformly gr-coherent modules

This section introduces the notion of uniformly gr-coherent modules and provides some of their properties.

Definition 3.1. [15, Definition 2.1] Let R be a graded ring. A finitely generated graded R-module M is called a uniformly gr-coherent R-module if there is a map $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any nonzero graded R-module homomorphism $f : \bigoplus_{i=1}^{n} R(-\lambda_i) \to M$ of degree 0, where $\lambda_1, \ldots, \lambda_n$ are degrees in Γ, $\ker f$ can be generated by $\phi(n)$ homogeneous elements. The map ϕ is called a uniformity map of M.

For a graded ring R and a graded R-module M, let $\mu_R(M)$ denote the minimal number of homogeneous generators of M. The next proposition characterizes uniformly gr-coherent modules.

Proposition 3.2. *[15, Proposition 2.2] Let R be a graded ring and let M be a finitely generated graded R-module. The following assertions are equivalent:*

1. *M is uniformly gr-coherent.*
2. *For every $n \in \mathbb{N}$, $\sup_f \mu_R(\ker f) < \infty$, where f runs over the set of nonzero graded R-module homomorphisms $\bigoplus_{i=1}^{n} R(-\lambda_i) \to M$ of degree 0, with degrees $\lambda_1, \ldots, \lambda_n$ in Γ.*
3. *There is a map $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any nonzero graded R-module homomorphism $f : \bigoplus_{i=1}^{n} R(-\lambda_i) \to M$, where $\lambda_1, \ldots, \lambda_n$ are degrees in Γ, $\ker f$ can be generated by $\phi(n)$ homogeneous elements.*

The next proposition collects some classes of uniformly gr-coherent modules.

Proposition 3.3. *[15, Proposition 2.3] Let R be a graded ring. Then:*

1. *Every finitely generated homogeneous submodule of a uniformly gr-coherent R-module is a uniformly gr-coherent R-module.*
2. *Assume that $R_\alpha = 0$ for all but finitely many $\alpha \in \Gamma$. Then any uniformly coherent graded R-module is a uniformly gr-coherent R-module.*

Recall from the introduction that, if R is a graded ring and M is a finitely generated graded R-module, then M is called a *gr-coherent R-module* if every finitely generated homogeneous submodule of M is finitely presented. The following result gives a necessary condition for a graded module to be uniformly gr-coherent.

Proposition 3.4. *[15, Proposition 2.4] Let R be a graded ring. Then every uniformly gr-coherent R-module is a gr-coherent R-module.*

The next theorem yields most of the elementary properties of uniformly gr-coherent modules.

Theorem 3.5. [15, Theorem 2.5] Let R be a graded ring and let $0 \to P \xrightarrow{\alpha} N \xrightarrow{\beta} M \to 0$ be an exact sequence of graded R-modules. Then:

1. If P is finitely generated and N is uniformly gr-coherent, then M is uniformly gr-coherent.
2. If M and P are uniformly gr-coherent, then so is N.
3. If M is finitely presented and N is uniformly gr-coherent, then P is uniformly gr-coherent.

Before proving Theorem 3.5, the authors had established the following lemma.

Lemma 3.6. *[15, Lemma 2.6] Let R be a graded ring, N a graded R-module and $\lambda \in \Gamma$. Then, N is uniformly gr-coherent if and only if $N(\lambda)$ is uniformly gr-coherent.*

Here are some corollaries of Theorem 3.5.

Corollary 3.7. *[15, Corollary 2.7] Let R be a graded ring and let $0 \to P \to N \to M \to 0$ be an exact sequence of graded R-modules. If any two of the modules P, N and M are uniformly gr-coherent, then so is the third.*

Corollary 3.8. *[15, Corollary 2.8] Let R be a graded ring and let $f : M \to N$ be a graded R-module homomorphism. If M and N are uniformly gr-coherent, then so are $\ker(f)$, $im(f)$ and $coker(f)$.*

Corollary 3.9. *[15, Corollary 2.9] Every finite direct sum of uniformly gr-coherent R-modules is a uniformly gr-coherent R-module.*

Corollary 3.10. *[15, Corollary 2.10] If M and N are two finitely generated homogeneous submodules of a uniformly gr-coherent R-module E, then $M + N$ and $M \cap N$ are uniformly gr-coherent.*

Corollary 3.11. *[15, Corollary 2.11] Let R be a graded ring, M a finitely presented graded R-module and N a uniformly gr-coherent R-module. Then $M \otimes_R N$ and $Hom_R(M,N)$ are uniformly gr-coherent R-modules.*

The next proposition clarifies the situation for localizations of uniformly gr-coherent modules.

Proposition 3.12. *[15, Proposition 2.12] Let R be a graded ring and S a multiplicatively closed set of homogeneous elements of R. If M is a uniformly gr-coherent R-module, then $S^{-1}M$ is a uniformly gr-coherent $S^{-1}R$-module.*

This subsection is closed by a result concerning scalar restrictions.

Theorem 3.13. [15, Theorem 2.13] Let $\phi : R \to S$ be a graded ring homomorphism and M a graded S-module. If the graded R-module S is finitely generated and M is uniformly gr-coherent over R, then M is uniformly gr-coherent over S.

3.2 Uniformly gr-coherent rings

This subsection initiates the study of uniformly gr-coherent rings.

Definition 3.14. [15, Defintion 3.1] A graded ring R is called a uniformly gr-coherent ring if it is uniformly gr-coherent as a graded R-module, that is, if there is a map $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any nonzero graded R-module homomorphism $f : \bigoplus_{i=1}^{n} R(-\lambda_i) \to R$ of degree 0, where $\lambda_1,...,\lambda_n$ are degrees in Γ, $\ker f$ can be generated by $\phi(n)$ homogeneous elements. The map ϕ is called a uniformity map of R.

The next proposition presents the relation between the notion of "uniform gr-coherence" and that of "uniform coherence".

Proposition 3.15. *[15, Proposition 3.2] Let R be a graded ring such that $R_\alpha = 0$ for all but finitely many $\alpha \in \Gamma$. If R is a uniformly coherent ring, then R is a uniformly gr-coherent ring.*

Every ring R can be graded trivially by Γ, via $R_0 = R$ and $R_\alpha = 0$ for $\alpha \neq 0$. The following Corollary shows that, for trivially graded rings, the concepts of "uniform gr-coherence" and "uniform coherence" coincide.

Corollary 3.16. *[15, Corollary 3.3] Let R be a trivially graded ring. Then, R is a uniformly coherent ring if and only if R is a uniformly gr-coherent ring.*

Let A be a ring and let $\{X_1,...,X_n\}$ be indeterminates over A. For $m = (m_1,...,m_n) \in \mathbb{Z}^n$, let $X^m = X_1^{m_1}...X_n^{m_n}$. Then the Laurent polynomial ring $R = A[X_1, X_1^{-1},...,X_n, X_n^{-1}]$ is graded by $\mathbb{Z}^n$, via $R_m = \{aX^m | a \in A\}$ for every $m \in \mathbb{Z}^n$. The next result gives an example of a uniformly gr-coherent ring which is not uniformly coherent.

Example 3.17. [15, Example 3.4] Let $R = K[X_1, X_1^{-1},...,X_n, X_n^{-1}]$ be the Laurent polynomial ring over a field K with $n > 2$ indeterminates. Then R is a graded-field (so a uniformly gr-coherent ring, by Example 3.25 (1)) which is not uniformly coherent.

Let A be a ring and let $\{X_1,...,X_d\}$ be indeterminates over A. For $m = (m_1,...,m_d) \in \mathbb{N}^d$, let $X^m = X_1^{m_1}...X_d^{m_d}$. Then the polynomial ring $B = A[X_1,...,X_d]$ is graded by $\mathbb{Z}$, via

$$B_n = \left\{ \sum_{m \in \mathbb{N}^d} a_m X^m \middle| a_m \in A \text{ and } \sum_{i=1}^{d} m_i = n \right\}$$

for $n \geq 0$ and $B_n = 0$ for $n < 0$. Our next two examples illustrate Proposition 3.15.

Example 3.18. [15, Example 3.5] Let D be a PID and X an indeterminate over D. Then $D[X]/(X^n)$ is a uniformly gr-coherent ring, for every $n \in \mathbb{N}$.

Example 3.19. [15, Example 3.6] Let K be a field and let X and Y be indeterminates over K. Then $K[X,Y]/(X^n,Y^m)$ is a uniformly gr-coherent ring, for every $n,m \in \mathbb{N}$.

Recall from the second section that, a graded ring R is called a *gr-coherent ring* if every finitely generated homogeneous ideal of R is finitely presented. The next result presents the relation between the notion of "uniform gr-coherence" and that of "gr-coherence".

Proposition 3.20. *[15, Proposition 3.7] Every uniformly gr-coherent ring is a gr-coherent ring.*

The converse of Proposition 3.20 fails: the easiest example is any trivially graded ring which is coherent but not uniformly coherent; a nontrivially graded example is provided by Example 3.29. The next Theorem characterizes uniformly gr-coherent rings.

Theorem 3.21. [15, Theorem 3.8] Let R be a graded ring. The following assertions are equivalent:

1. R is a uniformly gr-coherent ring.
2. Every finitely presented graded R-module is uniformly gr-coherent.
3. $\bigoplus_{i=1}^{n} R(-\lambda_i)$ is a uniformly gr-coherent R-module, for every $n \in \mathbb{N}$ and any degrees $\lambda_1,\ldots,\lambda_n \in \Gamma$.
4. There is a map $\psi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any homogeneous ideal I of R generated by n homogeneous elements and any homogeneous element $a \in R$, $(I : a)$ can be generated by $\psi(n)$ homogeneous elements.
5. (a) There is an integer $s \in \mathbb{N}$ such that for every homogeneous element $a \in R$, $(0 : a)$ can be generated by s homogeneous elements, and
   (b) There is a map $\chi : \mathbb{N}^2 \to \mathbb{N}$ such that for every $(n,m) \in \mathbb{N}^2$, and any homogeneous ideals I and J of R generated respectively by n and m homogeneous elements, $I \cap J$ can be generated by $\chi(n,m)$ homogeneous elements.

Before proving Theorem 3.21, the authors had established the following lemmas.

Lemma 3.22. *[15, Lemma 3.9] Let R be a graded ring and $u_1,\ldots,u_{n+1}$ some homogeneous elements of R of degrees $\lambda_1,\ldots,\lambda_{n+1}$ respectively. Set $I = (u_1,\ldots,u_n)$ and $J = I + Ru_{n+1}$. Consider the following exact sequences of graded R-modules:*

$$0 \to \ker f \hookrightarrow \bigoplus_{i=1}^{n+1} R(-\lambda_i) \xrightarrow{f} J \to 0,$$

$$0 \to \ker g \hookrightarrow \bigoplus_{i=1}^{n} R(-\lambda_i) \xrightarrow{g} I \to 0,$$

with $f(e_j) = g(e_j) = u_j$, $1 \le j \le n$ and $f(e_{n+1}) = u_{n+1}$ where $\left(e_j\right)_{j=1}^{n+1}$ is the canonical basis of R^{n+1}. Then there exists a graded R-module homomorphism $\alpha : \ker f \to (I : u_{n+1})$ such that the sequence of graded R-modules $0 \to \ker g \hookrightarrow \ker f \xrightarrow{\alpha} (I : u_{n+1}) \to 0$ is exact.

Lemma 3.23. *[15, Lemma 3.10] Let R be a graded ring and $u_1,...,u_{n+m}$ some homogeneous elements of R of degrees $\lambda_1,...,\lambda_{n+m}$ respectively. Set $I=(u_1,...,u_n)$ and $J=(u_{n+1},...,u_{n+m})$. Consider, as in Lemma 3.22, the following exact sequences of graded R-modules:*

$$0 \to \ker f \hookrightarrow \bigoplus_{i=1}^{n} R(-\lambda_i) \xrightarrow{f} I \to 0,$$

$$0 \to \ker g \hookrightarrow \bigoplus_{i=1}^{m} R(-\lambda_{n+i}) \xrightarrow{g} J \to 0,$$

$$0 \to \ker h \hookrightarrow \bigoplus_{i=1}^{n+m} R(-\lambda_i) \xrightarrow{h} I+J \to 0.$$

Then there exists a graded R-module homomorphism $\beta : \ker h \to I \cap J$ such that the sequence of graded R-modules $0 \to \ker f \times \ker g \hookrightarrow \ker h \xrightarrow{\beta} I \cap J \to 0$ is exact.

Remark 3.24. [15, Remark 3.11] Theorem 3.21 remains valid if the condition $(5)(b)$ is weakened to: there is a map $\chi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any homogeneous element $a \in R$ and any homogeneous ideal I of R generated by n homogeneous elements, $I \cap Ra$ can be generated by $\chi(n)$ homogeneous elements.

Armed with Theorem 3.21, the authors in [15] were able to produce many examples of uniformly gr-coherent rings.

Example 3.25. [15, Example 3.12] By Theorem 3.21(5), the following graded rings are uniformly gr-coherent with uniformities of graded-coherence $s=1$ and $\chi(n,m)=1$:

1. Graded-fields, that is, graded rings in which every nonzero homogeneous element is invertible, see [61].
2. Graded-PIDs, that is, graded rings in which every homogeneous ideal is generated by one homogeneous element, see [9].
3. Graded-valuation domains, that is, graded integral domains in which the set of homogeneous ideals is totally ordered under inclusion, see [4].

By applying Theorem 3.21 to trivially graded rings and using Corollary 3.16, the following corollary characterizes uniformly coherent rings (cf. [57, p. 165]).

Corollary 3.26. *[15, Corollary 3.13] Let R be a ring. The following assertions are equivalent:*

1. *R is a uniformly coherent ring.*
2. *There is a map $\psi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any element $a \in R$ and any ideal I of R generated by n elements, $(I : a)$ can be generated by $\psi(n)$ elements.*
3. *(a) There is an integer $s \in \mathbb{N}$ such that for every element $a \in R$, $(0 : a)$ can be generated by s elements, and*

(b) *There is a map $\chi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any element $a \in R$ and any ideal I of R generated by n elements, $I \cap Ra$ can be generated by $\chi(n)$ elements.*

The next proposition determines when the group ring $A[X;\Gamma]$ over a ring A, is (uniformly) gr-coherent.

Proposition 3.27. *[15, Proposition 3.14] Let $R = A[X;\Gamma]$ be the group ring of Γ over a ring A graded by $deg(aX^{\alpha}) = \alpha$ for every $0 \neq a \in A$ and $\alpha \in \Gamma$. Then, R is a (uniformly) gr-coherent ring if and only if A is a (uniformly) coherent ring.*

Lemma 3.28. *[15, Lemma 3.15] Let $R = A[X;\Gamma]$ be the group ring of Γ over a ring A. Then the extension mapping $\mathcal{E} : I \mapsto IR$ induces a bijection from the ideals of A to the homogeneous ideals of R. Moreover, an ideal I of A is generated by n elements if and only if IR is generated by n homogeneous elements.*

Armed with Proposition 3.27, the authors in [15] were able to produce examples of gr-coherent rings which are not uniformly gr-coherent.

Example 3.29. [15, Example 3.16] If A is a coherent ring which is not uniformly coherent, then the group ring $A[X;\Gamma]$ is gr-coherent but not uniformly gr-coherent.

The next theorem examines direct limits of uniformly gr-coherent rings (cf. [18, I, Exercice 12, e), p. 63]).

Theorem 3.30. [15, Theorem 3.17] Let $(R_\lambda)_{\lambda \in S}$ be a direct system of graded rings and $R := \varinjlim R_\lambda$. Assume that R is a flat R_λ-module and that R_λ is a uniformly gr-coherent ring with uniformity map $\chi_\lambda = \chi$ for every $\lambda \in S$. Then R is a uniformly gr-coherent ring.

Let R be a ring and let $\{X_1, ..., X_d\}$ be indeterminates over R. For $m = (m_1, ..., m_d) \in \mathbb{N}^d$, let $X^m = X_1^{m_1} ... X_d^{m_d}$. Then the polynomial ring $S = R[X_1, ..., X_d]$ is graded by $\mathbb{Z}$, via

$$S_n = \left\{ \sum_{m \in \mathbb{N}^d} a_m X^m \middle| a_m \in R \text{ and } \sum_{i=1}^{d} m_i = n \right\}$$

for $n \geq 0$ and $S_n = 0$ for $n < 0$. As an application of Theorem 3.30, we have:

Corollary 3.31. *[15, Corollary 3.18] Let R be a ring and let $\{X_1, X_2, ...\}$ be indeterminates over R such that $R[X_1, ..., X_d]$ is a uniformly gr-coherent ring with uniformity map $\chi_d = \chi$ for all positive integers d. Then $R[X_1, X_2, ...]$ is a uniformly gr-coherent ring.*

The following result concerns factor rings of uniformly gr-coherent rings.

Theorem 3.32. [15, Theorem 3.19] If R is a uniformly gr-coherent ring and I is a finitely generated homogeneous ideal of R, then R/I is a uniformly gr-coherent ring.

The next result clarifies the situation for localisations of uniformly gr-coherent rings.

Proposition 3.33. *[15, Proposition 3.20] If R is a uniformly gr-coherent ring and S a multiplicatively closed set of homogeneous elements of R, then $S^{-1}R$ is a uniformly gr-coherent ring.*

We close by a result about product of uniformly gr-coherent rings.

Proposition 3.34. *[15, Proposition 3.21] Let R_1 and R_2 be graded rings. Then R_1 and R_2 are uniformly gr-coherent rings if and only if $R_1 \times R_2$ is a uniformly gr-coherent ring.*

4 Weakly uniformly graded-coherent rings

It is interesting to note that in [53], the author generalized the concept of weakly uniform coherence to the context of Γ-graded rings and modules, as follows. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a ring graded by an arbitrary grading abelian group Γ. R is called weakly uniformly graded-coherent ring if there is a map $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any nonzero graded R-module homomorphism $f : \bigoplus_{i=1}^{n} R(-\lambda_i) \to R$ of degree 0, where $\lambda_1,\ldots,\lambda_n$ are degrees in Γ, $\ker f$ can be generated by $\phi(n)$ elements (not necessary homogeneous).

In this section, we survey the elementary properties of weakly uniformly graded-coherent rings provided and shown by the author in [53].

The author investigated a particular class of uniformly gr-coherent rings (and modules) that he called weakly uniformly gr-coherent rings (and modules). Let R be a graded ring. A finitely generated graded R-module M is said to be a *weakly uniformly gr-coherent R-module* if there is a map $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any nonzero graded R-module homomorphism $f : \bigoplus_{i=1}^{n} R(-\lambda_i) \to M$ of degree 0, where $\lambda_1,\ldots,\lambda_n$ are degrees in Γ, $\ker f$ can be generated by $\phi(n)$ elements (not necessary homogeneous); the map ϕ is called a *uniformity map* of M. The graded ring R is said to be a *weakly uniformly gr-coherent ring* if it is weakly uniformly gr-coherent as a graded R-module. The diagram bellow (Figure 2) summarizes the relations between the (graded-) coherent like notions involved in this section.

4.1 Weakly uniformly gr-coherent modules

This subsection defines weakly uniformly gr-coherent modules and provides their properties.

Definition 4.1. [53, Definition 2.1] Let R be a graded ring. A finitely generated graded R-module M is called a weakly uniformly gr-coherent R-module if there is a map $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any nonzero graded R-module homomorphism $f : \bigoplus_{i=1}^{n} R(-\lambda_i) \to M$ of degree 0, where $\lambda_1,\ldots,\lambda_n$ are degrees in Γ, $\ker f$ can be generated by $\phi(n)$ elements (not necessary homogeneous). The map ϕ is called a uniformity map of M.

Another way to look at this definition is the following. For a ring R and an R-module M, let $\mu_R(M)$ denote the minimal number of generators of M. A finitely generated graded R-module M is weakly uniformly gr-coherent $\Leftrightarrow$ for every $n \in \mathbb{N}$, $\sup_f \mu_R(\ker f) < \infty$, where f runs over the set of nonzero graded R-module homomorphisms $\bigoplus_{i=1}^{n} R(-\lambda_i) \to M$ of degree 0, with degrees $\lambda_1,\ldots,\lambda_n$ in Γ.

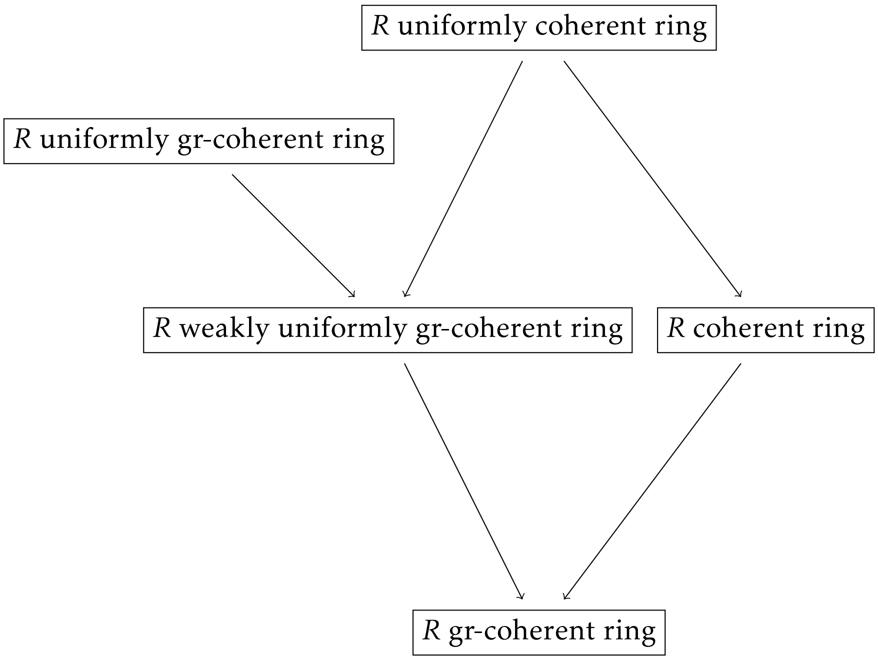


Figure 2: The relations between the (graded-) coherent like notions for a graded ring R.

Remark 4.2. [53, Remark 2.2] Every finitely generated homogeneous submodule of a weakly uniformly gr-coherent R-module is a weakly uniformly gr-coherent R-module.

Let R be a Γ-graded ring. By $sup(R) = \{\alpha \in \Gamma, R_\alpha \neq 0\}$ we denote the *support* of the graded ring R. In case $sup(R)$ is a finite set we will write $sup(R) < \infty$ and then R is said to be a Γ-graded ring *of finite support* [48].

The next proposition collects some classes of weakly uniformly gr-coherent modules.

Proposition 4.3. *[53, Proposition 2.3] Let R be a graded ring. Then:*

1. *Every uniformly coherent graded R-module is weakly uniformly gr-coherent.*
2. *(a) Every uniformly gr-coherent R-module is weakly uniformly gr-coherent.*
   (b) Assume that R is of finite support.
   Then any weakly uniformly gr-coherent R-module is uniformly gr-coherent.
3. *Every weakly uniformly gr-coherent R-module is a gr-coherent R-module.*

Theorem 4.4. [53, Theorem 2.5] Let R be a graded ring and let $0 \to P \xrightarrow{\alpha} N \xrightarrow{\beta} M \to 0$ be an exact sequence of graded R-modules. Then:

1. If P is finitely generated and N is weakly uniformly gr-coherent, then M is weakly uniformly gr-coherent.
2. If M is uniformly gr-coherent and P is weakly uniformly gr-coherent, then N is weakly uniformly gr-coherent.

3. If M is finitely presented and N is weakly uniformly gr-coherent, then P is weakly uniformly gr-coherent.

Corollary 4.5. *[53, Corollary 2.6] If $f : M \to N$ is a graded R-module homomorphism and M and N are weakly uniformly gr-coherent, then so are $\ker(f)$, $im(f)$ and $coker(f)$.*

The next theorem clarifies the situation for scalar restrictions.

Theorem 4.6. [53, Theorem 2.7] Let $\phi : R \to S$ be a graded ring homomorphism making S a finitely generated graded R-module. Let M be a graded S-module. If M is weakly uniformly gr-coherent over R, then M is weakly uniformly gr-coherent over S.

This subsection is closed with a result concerning localizations of weakly uniformly gr-coherent modules.

Proposition 4.7. *[53, Proposition 2.8] Let S be a multiplicatively closed set of homogeneous elements of a graded ring R. If M is a weakly uniformly gr-coherent R-module, then $S^{-1}M$ is a weakly uniformly gr-coherent $S^{-1}R$-module.*

4.2 Weakly uniformly gr-coherent rings

This subsection defines weakly uniformly gr-coherent rings and provides their properties.

Definition 4.8. [53, Definition 3.1] A graded ring R is called a weakly uniformly gr-coherent ring if it is weakly uniformly gr-coherent as a graded R-module, that is, if there is a map $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any nonzero graded R-module homomorphism $f : \bigoplus_{i=1}^{n} R(-\lambda_i) \to R$ of degree 0, where $\lambda_1,...,\lambda_n$ are degrees in Γ, $\ker f$ can be generated by $\phi(n)$ elements (not necessary homogeneous). The map ϕ is called a uniformity map of R.

The next theorem characterizes weakly uniformly gr-coherent rings.

Theorem 4.9. [53, Theorem 3.2] Let R be a graded ring. The following assertions are equivalent:

1. R is a weakly uniformly gr-coherent ring.
2. There is a map $\psi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any homogeneous ideal I of R generated by n homogeneous elements and any homogeneous element $a \in R$, $(I : a)$ can be generated by $\psi(n)$ elements.
3. (a) There is an integer $s \in \mathbb{N}$ such that for every homogeneous element $a \in R$, $(0 : a)$ can be generated by s elements, and
   (b) There is a map $\chi : \mathbb{N}^2 \to \mathbb{N}$ such that for every $(n,m) \in \mathbb{N}^2$, and any homogeneous ideals I and J of R generated respectively by n and m homogeneous elements, $I \cap J$ can be generated by $\chi(n,m)$ elements.
4. (a) There is an integer $s \in \mathbb{N}$ such that for every homogeneous element $a \in R$, $(0 : a)$ can be generated by s elements, and

(b) There is a map $\chi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, and any homogeneous ideal I of R generated by n homogeneous elements and any homogeneous element $a \in R$, $I \cap aR$ can be generated by $\chi(n)$ elements.

The next proposition compares the concepts of "weak uniform gr-coherence" and "uniform gr-coherence" for graded rings.

Proposition 4.10. *[53, Proposition 3.5] Every uniformly gr-coherent ring is a weakly uniformly gr-coherent ring.*

The author in [53] was unable to determine whether the converse of Proposition 4.10 is true. However, he gave some cases where the converse holds true (see Proposition 4.11 and Proposition 4.15).

Proposition 4.11. *[53, Proposition 3.6] Let R be a graded ring of finite support. If R is weakly uniformly gr-coherent, then R is uniformly gr-coherent.*

The next proposition presents the relation between the notion of "weak uniform gr-coherence" and that of "uniform coherence".

Proposition 4.12. *[53, Proposition 3.7] Let R be a graded ring. If R is uniformly coherent, then it is weakly uniformly gr-coherent.*

Every ring R can be graded trivially by Γ, via $R_0 = R$ and $R_\alpha = 0$ for $\alpha \neq 0$. It is easy to see that, for trivially graded rings, the concepts of "weak uniform graded-coherence", "uniform graded-coherence" and "uniform coherence" coincide.

Let A be a ring and let $\{X_1, ..., X_n\}$ be indeterminates over A. For $m = (m_1, ..., m_n) \in \mathbb{Z}^n$, let $X^m = X_1^{m_1} ... X_n^{m_n}$. Then the Laurent polynomial ring $R = A[X_1, X_1^{-1}, ..., X_n, X_n^{-1}]$ is graded by $\mathbb{Z}^n$, via $R_m = \{aX^m | a \in A\}$ for every $m \in \mathbb{Z}^n$. The converse of Proposition 4.12 fails; in fact, there exist (weakly) uniformly gr-coherent rings which are not uniformly coherent, as shown by the following example.

Example 4.13 ([15, Example 3.4])**.** Let $R = K[X_1, X_1^{-1}, ..., X_n, X_n^{-1}]$ be the Laurent polynomial ring over a field K with n indeterminates ($n > 2$).

Then R is a graded-field so it is (weakly) uniformly gr-coherent; but R is not uniformly coherent.

As an immediate consequence of Proposition 4.3 (3), the next Proposition compares the concepts of "weak uniform gr-coherence" and "gr-coherence".

Proposition 4.14. *[53, Proposition 3.9] Any weakly uniformly gr-coherent ring is a gr-coherent ring.*

The converse of Proposition 4.14 fails: the easiest example is any trivially graded ring which is coherent but not uniformly coherent; a nontrivially graded example is provided by Example 4.17.

The next proposition determines when the group ring $A[X;\Gamma]$ over a ring A, is (weakly uniformly) gr-coherent.

Proposition 4.15. *[53, Proposition 3.10] Let $R = A[X;\Gamma]$ be the group ring of Γ over a ring A graded by $deg(aX^{\alpha}) = \alpha$ for every $0 \neq a \in A$ and $\alpha \in \Gamma$. Then:*

1. *The following statements are equivalent.*
   (a) *R is a gr-coherent ring.*
   (b) *A is a coherent ring.*
2. *The following statements are equivalent.*
   (a) *R is a weakly uniformly gr-coherent ring.*
   (b) *R is a uniformly gr-coherent ring.*
   (c) *A is a uniformly coherent ring.*

Lemma 4.16. *[53, Lemma 3.11] Let $R = A[X;\Gamma]$ be the group ring of Γ over a ring A. Then:*

1. *The extension mapping $\mathcal{E} : I \mapsto IR$ induces a bijection from the ideals of A to the homogeneous ideals of R.*
2. *The following statements are equivalent for an ideal I of A.*
   (a) *IR is generated by n elements of R.*
   (b) *IR is generated by n homogeneous elements of R.*
   (c) *I is generated by n elements of A.*

Armed with Proposition 4.15, the author in [53] was able to produce examples of gr-coherent rings which are not weakly uniformly gr-coherent.

Example 4.17. [53, Example 3.12] If A is a coherent ring which is not uniformly coherent, then the group ring $A[X;\Gamma]$ is gr-coherent but not weakly uniformly gr-coherent.

The following result concerns quotients of weakly uniformly gr-coherent rings.

Theorem 4.18. [53, Theorem 3.13] Let I be a finitely generated homogeneous ideal of a graded ring R. If R is a weakly uniformly gr-coherent ring, then so is R/I.

The next proposition examines the product of weakly uniformly gr-coherent rings.

Proposition 4.19. *[53, Proposition 3.14] Let R_1 and R_2 be graded rings. Then $R_1 \times R_2$ is a weakly uniformly gr-coherent ring if and only if so are R_1 and R_2.*

The next result clarifies the situation for direct limits of weakly uniformly gr-coherent rings (see for instance [18, I, Exercice 12, e), p. 63]).

Theorem 4.20. [53, Theorem 3.15] Let $(R_\lambda)_{\lambda \in S}$ be a direct system of graded rings and $R := \varinjlim R_\lambda$. Assume that R is a flat R_λ-module for all $\lambda \in S$. If the R_λ's are weakly uniformly gr-coherent rings with the same uniformity map χ, then so is R.

Let $\{X_1,...,X_d\}$ be indeterminates over a ring R. For $m=(m_1,...,m_d)\in\mathbb{N}^d$, let $X^m = X_1^{m_1}...X_d^{m_d}$. Then the polynomial ring $S=R[X_1,...,X_d]$ is graded by $\mathbb{Z}$, via $S_n = \left\{\sum_{m\in\mathbb{N}^d} a_m X^m \middle| a_m\in R \text{ and } \sum_{i=1}^d m_i = n\right\}$ for $n\geq 0$ and $S_n=0$ for $n<0$.

Corollary 4.21. *[53, Corollary 3.16] Let R be a ring and let $\{X_1,X_2,...\}$ be indeterminates over R. If the polynomial rings $R[X_1,...,X_d]$, d a positive integer, are weakly uniformly gr-coherent with the same uniformity map, then so is the polynomial ring $R[X_1,X_2,...]$.*

This subsection is closed by a result about localisations of weakly uniformly gr-coherent rings.

Proposition 4.22. *[53, Proposition 3.17] Let S be a multiplicatively closed set of homogeneous elements of a graded ring R. If R is a weakly uniformly gr-coherent ring, then so is $S^{-1}R$.*

5 On Graded Coherent-like properties in Trivial ring extensions

Let A be a ring and let E be an A-module. We can make Cartesian product $R:=A\times E$ into a ring with respect to componentwise addition and multiplication defined by $(a_1,e_1)(a_2,e_2):=(a_1a_2,a_1e_2+a_2e_1)$. This is a commutative ring with identity $(1,0)$, called the idealization of E or the trivial extension of A by E, denoted by $A\propto E$. Note that A naturally embeds into $A\propto E$ via $a\to(a,0)$. If N is a submodule of E, then $0\propto N$ is a nilpotent ideal of $A\propto E$, and that $(A\propto E)/(0\propto E)\simeq A$. Idealization of a module was introduced by Nagata [46]. We usually use trivial extension to reduce results concerning submodules to the ideal case, generalizing results from ring to modules and constructing examples of commutative rings with zero divisors. An excellent introduction to the subject is [36] and an excellent summary of work done on trivial extension can be found in [8]. See, for instance [1, 13, 28, 31, 38, 39, 42, 43, 44, 45, 51, 52, 54].

The construction of trivial extension has many generalizations, see e.g., [7, 59, 60]. Among these generalizations, we have the graded trivial extension. The notion of graded trivial extension, for rings graded by $\mathbb{N}$, was introduced by Nagata [46, Exercice 1, page 24], or also as a theorem by Anderson [8, Theorem 4.5]. And recently in [59] for rings graded by an abelien group. Let G be a commutative monoid, $A=\oplus_{\alpha\in G}A_\alpha$ be a graded ring and $E=\oplus_{\alpha\in G}E_\alpha$ a graded A-module. Then $R=A\propto E=\oplus_{\alpha\in G}R_\alpha$ is a graded ring where $R_\alpha:=A_\alpha\oplus E_\alpha$. In this section, unless otherwise stated, all the results are taken from [10], in the first section, the authors improved some results on the graded idealization and they characterized some homogeneous properties of R in terms of those of A and E. Then, in another subsequent section, they introduced the notions of graded-v-coherent, graded-quasi coherent and graded-finite conductor rings. Then they studied the transfer of these properties to the graded idealization.

5.1 On the graded trivial extension ring

The following result is an extension of a result given in [59, Proposition 3.1] to the context of rings graded by a commutative grading monoid G.

Proposition 5.1. *[10, Proposition 2] Let $A=\oplus_{\alpha\in G}A_\alpha$ be a graded ring and $E=\oplus_{\alpha\in G}E_\alpha$ a graded A-module. Then $R:=A\propto E$ the trivial extension of A by E is a graded ring $R=\oplus_{\alpha\in G}R_\alpha$ where $R_\alpha=(A\propto E)_\alpha:=A_\alpha\oplus E_\alpha$.*

In [8], the homogeneous ideals of $A \propto E$ are the ideals of the form $I \propto N$ where I is an ideal of R, N is a submodule of E, and $IE \subseteq N$ respecting to its natural $\mathbb{N}$-grading with $(A \propto E)_0 = A \oplus 0$, $(A \propto E)_1 = 0 \oplus E$ and $(A \propto E)_n = 0$ for $n \geq 2$ which can also be viewed as a $\mathbb{Z}_2$-grading since $(0 \oplus E)^2 = 0$. The next result characterizes certain homogeneous ideals in the graded trivial extension rings graded by an arbirary grading commutative monoid G.

Theorem 5.2. [10, Theorem 1] Let A be a graded ring, I an ideal of A, E a graded A-module and N a submodule of E. Then $I \propto N$ is a homogeneous ideal of $A \propto E$ if and only if I is a homogeneous ideal of A, N is a homogeneous submodule of E and $IE \subseteq N$. When $I \propto N$ is a homogeneous ideal, E/N is a graded A/I-module and $(A \propto E)/(I \propto N) \cong (A/I) \propto (E/N)$ is a graded ring isomorphism. In particular, $(A \propto E)/(0 \propto N) \cong A \propto (E/N)$ is a graded ring isomorphism and therefore $(A \propto E)/(0 \propto E) \cong A$ is a graded ring isomorphism. So the homogeneous ideals of $A \propto E$ containing $0 \propto E$ are of the form $J \propto E$ for some homogeneous ideal J of A.

Note that Proposition 3.3 in [59] characterizes the homogeneous-maximal and the homogeneous-prime ideals of $A \propto E$ under the assumption that G is a group. The next result improves this Proposition by taking G a grading commutative monoid (not necessarily a group), and characterizes the homogeneous-maximal, the homogeneous-prime and the homogeneous-radical ideals of the graded trivial extension ring.

Theorem 5.3. [10, Theorem 2] Let A be a graded ring, E a graded A-module and $R := A \propto E$ the graded trivial extension of A by E.

1. The homogeneous-maximal ideals of R have the form $M \propto E$ where M is a homogeneous-maximal ideal of A.
2. The homogeneous-prime ideals of R have the form $P \propto E$ where P is a homogeneous-prime ideal of A.
3. The homogeneous-radical ideals of R have the form $I \propto E$ where I is a homogeneous-radical ideal of A.

The next result determines the homogeneous units and the homogeneous idempotents for the graded trivial extension.

Proposition 5.4. *[10, Proposition 5] Assume that the grading monoid G is a group. Let A be a graded ring and E a graded A-module. Then the homogeneous units of $A \propto E$ are $U(A \propto E) \cap h(A \propto E) = (U(A) \propto E) \cap h(A \propto E)$ and the homogeneous idempotents of $A \propto E$ are $Id(A \propto E) \cap h(A \propto E) = Id(A_0) \propto 0$.*

The following theorem determines when $A \propto E$ is gr-Noetherian.

Theorem 5.5. [10, Theorem 3] Assume that the grading monoid G is a cancellative torsionfree monoid. Let A be a graded ring and E a graded A-module. Then $A \propto E$ is gr-Noetherian, if and only if A is gr-Noetherian, and E is finitely generated.

The next theorem characterises the graded-coherence in the graded trivial extension ring in the case where G is a group. Recall that a graded ring R is said to be graded-coherent if every finitely genereted homogeneous ideal is finitely presented.

Theorem 5.6. [10, Theorem 4] Assume that the grading monoid G is a group. Let (A,M) be a local graded ring and E a graded A-module with $ME = 0$. Let $R := A \propto E$ be the graded trivial extension ring of A by E. Then R is graded-coherent if and only if A is graded-coherent, M is finitely generated, and E is an (A/M)-vector space of finite rank.

To prove this Theorem, the authors established the following lemma which is the graded version of lemma 2.7 in [37].

Lemma 5.7. *[10, Lemma 1] Assume that the grading monoid G is a group. Let (A,M) be a local graded ring and E a graded A-module with $ME = 0$. Let $R := A \propto E$ be the graded trivial extension ring of A by E. $(0 : c)$ is a finitely generated homogeneous ideal of R for each homogeneous element $c \in R$ if and only if $(0 : a)$ is a finitely generated homogeneous ideal of A for each homogeneous element $a \in A$, M is finitely generated, and E is an (A/M)-vector space of finite rank.*

Proposition 5.8. *[10, Proposition 6] Let A_1 and A_2 be two graded rings, and let E_i be a graded A_i-module, $i = 1,2$. Then $(A_1 \times A_2) \propto (E_1 \times E_2) \cong (A_1 \propto E_1) \times (A_2 \propto E_2)$ is a graded ring isomorphism.*

Recall from [60] that the notion of trivial extension can be generalized to what is called a semi-trivial extension. Let A be a ring, E an A-module, and $\varphi : E \otimes_A E \to A$ an A-module homomorphism satisfying $\varphi(e \otimes e') = \varphi(e' \otimes e)$ and $\varphi(e \otimes e')e'' = e\varphi(e' \otimes e'')$. Then $A \propto_\varphi E = A \oplus E$ with coordinate-wise addition and multiplication defined by $(a,e)(a',e') = (aa' + \varphi(e \otimes e'), ae' + a'e)$ is a commutative ring, called a semi-trivial extension of A by E. For $\varphi = 0$, we have trivial extension. See [60] for more details. The next result introduce the notion of graded semi-trivial extension and show that it is a graded ring.

Theorem 5.9. [10, Theorem 5] Let A be a graded ring, E a graded A-module, and $\varphi : E \otimes_A E \to A$ a graded A-module homomorphism of degree 0 satisfying $\varphi(e \otimes e') = \varphi(e' \otimes e)$ and $\varphi(e \otimes e')e'' = e\varphi(e' \otimes e'')$. Then $R := A \propto_\varphi E$ the semi-trivial extension of A by E is a graded ring $R = \oplus_{\alpha \in G} R_\alpha$ where $R_\alpha = (A \propto_\varphi E)_\alpha := A_\alpha \oplus E_\alpha$.

5.2 Graded-v-coherent rings

Let A be a graded ring, and let $Q(A)$ denote the total ring of quotients of A and H the saturated multiplicative set of nonzero regular homogeneous elements of A. Then, by extending some definitions in [3] to the case where rings are with zerodivisors, A_H, called the homogeneous total ring of quotients of A, is a ring graded by $\langle G \rangle$, where $A_H = \oplus_{\alpha \in \langle G \rangle} (A_H)_\alpha$ with:

$$(A_H)_\alpha = \left\{ \frac{a}{b} \mid a \in A_\beta, b \text{ a regular element of } A_\gamma \text{ and } \beta - \gamma = \alpha \right\}.$$

If A is a graded integral domain (An integral domain graded by G), then A_H is called the homogeneous quotient field of A. Clearly, every nonzero homogeneous element of A_H is invertible and $(A_H)_0$ is a field. A fractional ideal I of A is called a homogeneous fractional ideal of A if $I \subseteq A_H$ and $I = \oplus_{\alpha \in \langle G \rangle} (I \cap (A_H)_\alpha)$; equivalently, if $I = \frac{1}{s} J$ for some regular element $s \in H$ and some homogeneous integral ideal J of A, see [3, Proposition 2.4]. It is easy to see that a homogeneous principal fractional ideal of A has the form xA for some homogeneous element x of A_H. Given two nonzero fractional ideals I, J. The set $\{x \in T \mid xJ \subseteq I\}$, where T is an overring of A, is a fractional ideal and denoted by $(I :_T J)$. If $T = Q(A)$, the subscript $Q(A)$ will usually be omitted. The fractional ideal $(A : I)$ is denoted by I^{-1} and $(A : (A : I)) = \left(I^{-1}\right)^{-1}$ by I_v. Also, I is called invertible if $II^{-1} = A$, divisorial ideal or v-ideal if $I = I_v$ and v-finite if $I_v = J_v$ (or, equivalently, if $I^{-1} = J^{-1}$) for some finitely generated fractional ideal J of A. Recall from [37] that a ring A is called v-coherent if $(0 : a)$ and $\cap_{1\leq i\leq n} Aa_i$ are v-finite ideals of A for any finite set of elements a and $a_1, \cdots, a_n$ of A.

The following result extends Proposition 2.5 in [3] to the case where rings are with zerodivisors.

Proposition 5.10. *[10, Proposition 7] Let A be a graded ring and H the saturated multiplivative set of regular homogeneous elements of A. If I and J are two regular homogeneous fractional ideals, then $(I : J)$ is also a homogeneous fractional ideal and $(I : J) = (I :_{A_H} J)$. In particular, if I is a regular homogeneous fractional ideal, then so are I^{-1} and I_v.*

Definition 5.11. [10, Definition 1] Let A be a graded ring and I be a fractional ideal. I is called homogeneous-v-finite ideal if $I_v = J_v$ (which is equivalent to $I^{-1} = J^{-1}$) for some finitely generated homogeneous fractional ideal J of A.

Definition 5.12. [10, Definition 2] Let A be a graded ring. A is called graded-v-coherent ring if $(0 : a)$ and $\cap_{1\leq i\leq n} Aa_i$ are homogeneous-v-finite ideals for any finite set of homogeneous elements a and $a_1, \cdots, a_n$ of A.

Obviously, every v-coherent graded ring is a graded-v-coherent ring which gives us the first and easiest example of graded-v-coherent rings.

Example 5.13. [10, Example 2] Let (A, M) be any local graded ring with $M^2 = 0$. Then A is a graded-v-coherent ring.

Proposition 5.14. *[10, Proposition 8] Let A be a graded ring, and let consider the following statements:*

1. *I^{-1} is homogeneous-v-finite for any finitely generated homogeneous fractional ideal I of A.*
2. *$I_v \cap J_v$ is homogeneous-v-finite for any two finitely generated homogeneous fractional ideals I and J of A.*
3. *$\bigcap_{1\leq i\leq n} Aa_i$ is homogeneous-v-finite for any finite set of homogeneous elements $a_1, \ldots, a_n$ of A.*

Then $(1) \Rightarrow (2)$. Moreover, if A is a graded integral domain, Then the three assertions are equivalent, which each of them characterizes the graded-v-coherence in graded integral domains.

The next result characterizes the graded-v-coherence in the graded trivial extension of a graded integral domain by its homogeneous quotient field.

Theorem 5.15. [10, Theorem 6] Assume that the grading monoid G is a group. Let A be a graded integral domain which is not a field, A_H its homogeneous quotient field graded by G and let $R := A \propto A_H$ be the graded trivial extension of A by A_H, then R is a graded-v-coherent ring if and only if A is graded-v-coherent.

Definition 5.16. [10, Definition 3] Let A be a graded ring. Then A is said to be graded-quasi-coherent (resp., graded-finite conductor) if $(0 : a)$ and $\bigcap_{i=1}^{n} Aa_i$ (resp., $Ab \cap Ac$) are finitely generated for every finite set of homogeneous elements $a, a_1, a_2, \ldots, a_n$ (resp., b, c) of A.

Proposition 5.17. *[10, Proposition 9] Let (A, M) be a graded-local ring with $M^2 = 0$. The following statements are equivalent.*

1. *A is a graded-coherent ring.*
2. *A is a graded-quasi-coherent ring.*
3. *A is a graded-finite conductor ring.*
4. *$(0 : a)$ is finitely generated for every homogeneous element a of A.*
5. *M is finitely generated.*

Theorem 5.18. [10, Theorem 7] Assume that the grading monoid G is a group. Let (A, M) be a local graded ring and E a graded A-module with $ME = 0$. Let $R = A \propto E$ be the graded trivial extension. The following statements are equivalent.

1. R is a graded-quasi-coherent (respectively, graded-finite conductor) ring.
2. A is a graded-quasi-coherent (respectively, graded-finite conductor) ring, M is finitely generated and E is an (A/M)-vector space of finite rank.

6 Commutative graded-n-coherent and graded valuation rings

Let $R = \oplus_{\alpha \in G} R_\alpha$ be a commutative ring with unity graded by an arbitrary grading commutative monoid G. In this section, unless otherwise stated, all the results are taken from [11], were the authors generalized several facts on n-coherent rings to graded-n-coherent rings. In the last section, the authors provided necessary and sufficient conditions for the graded trivial extension ring to be a graded-valuation ring.

If n is a nonnegative integer, we say that an R-module M is n-presented if there is an exact sequence $F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$ of R-modules in which each F_i, is finitely generated and free. (Here we follow the terminology of [23]; in [62], such M is said to "have a finite n-presentation"). In particular, "0-presented" means finitely generated and "1-presented" means finitely presented. Following [18], we let $\lambda(M) = \lambda_R(M) = \sup\{n \mid M \text{ is an } n\text{-presented } R\text{-module}\}$, so that $0 \leq \lambda(M) \leq \infty$; the properties of the function λ are recalled in Lemma 6.3. Classically, the “n-presented" concept allows both ideal-theoretic and module-theoretic approaches to coherent rings. Indeed (cf. [18], p. 63, Exercise 12), a ring R is said to be coherent if each finitely generated ideal is finitely presented; equivalently if each finitely presented R-module is 2-presented.

Let n be a positive integer. Recall from [26] that R is n-coherent ring if each $(n-1)$-presented ideal of R is n-presented. Thus, the 1-coherent rings are just the coherent rings. Note that each Bezout (for instance, valuation) domain R is n-coherent for each $n \geq 1$; indeed, each $(n-1)$-presented ideal of R

is principal and hence infinitely-presented (in the obvious sense). Moreover, each Noetherian ring is n-coherent for any $n \geq 1$. An excellent summary of works done on n-coherence can be found in [26]. Accordingly, like it was done in [26], the authors in [11] used the λ-function to introduce both ideal and module theoretic approaches to "graded-n-coherence" for any positive integer n.

6.1 Graded-n-coherent modules

Definition 6.1. [11, Definition 2.1] Let R be a graded ring and let n be a positive integer, we say that a graded R-module M is a graded-n-coherent module if M is n-presented and each $(n-1)$-presented homogeneous submodule of M is n-presented.

It follows from [14] that the graded-1-coherent modules are just the "graded-coherent modules".

Remark 6.2. [11, Remark 2.2] Let R be a graded ring and let n be a positive integer. Then following assertions hold true:

1. Every $(n-1)$-presented homogeneous submodule of a graded-n-coherent R-module is a graded-n-coherent R-module.
2. Any n-coherent graded R-module is a graded-n-coherent R-module.

For reference purposes, it will be helpful to recall the following elementry result which summarizes some properties of λ.

Lemma 6.3. *[18, p.61, Exercise 6] Let R be a ring and let $0 \to P \to N \to M \to 0$ be an exact sequence of R-modules. Then:*

1. $\lambda(N) \geq \inf\{\lambda(P), \lambda(M)\}$
2. $\lambda(M) \geq \inf\{\lambda(N), \lambda(P)+1\}$
3. $\lambda(P) \geq inf\{\lambda(N), \lambda(M)-1\}$
4. *If $N = P \oplus M$ then $\lambda(N) = inf\{\lambda(M), \lambda(P)\}$.*

Theorem 6.4. [11, Theorem 2.4] Let R be a graded ring and let $0 \to P \xrightarrow{u} N \xrightarrow{v} M \to 0$ be an exact sequence of graded R-modules.

(1) If $\lambda(P) \geq n-1$, N is a graded-n-coherent module and v has a cancellable degree then M is a graded-n-coherent module.

(2) If $\lambda(M) \geq n$ and N is a graded-n-coherent module, then P is a graded-n-coherent module.

Theorem 6.5. [11, Theorem 2.5] Let $m \geq n$ be positive integer and let $M_0 \xrightarrow{u_1} M_1 \xrightarrow{u_2} M_2 \to \cdots \xrightarrow{u_m} M_m$ be an exact sequence of graded-n-coherent R-modules such that the degree of every u_i is cancellable. Then $\mathrm{Im}(u_i), \mathrm{Ker}(u_i)$ and $\mathrm{Coker}(u_i)$ are graded-n-coherent R-modules for each $i = 1, 2, \ldots, m$.

Theorem 6.6. [11, Theorem 2.6] Let M be a graded R-module and I be a homogeneous ideal of R such that $IM = 0$. Let $n \geq 1$ and let the canonical graded ring homomorphism $R \to R/I$ satisfy $\lambda_R(R/I) \geq n$. Then M is graded-n-coherent as a graded R/I-module if and only if M is graded-n-coherent as a graded R-module.

To prove this theorem, the authors first had established the following three Lemmas.

Lemma 6.7. *[11, Lemma 2.7] Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) \geq n$ and let M be an n-presented graded S-module. Then M is an n-presented graded R-module*

Lemma 6.8. *[11, Lemma 2.8] Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be a graded S-module. If M is n-presented as a graded R-module, then it is n-presented as a graded S-module.*

Lemma 6.9. *[11, Lemma 2.9] Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be a graded S-module. If M is graded-n-coherent as a graded R-module, then it is graded-n-coherent as a graded S-module.*

Remark 6.10. [11, Remark 2.10] Let the canonical graded ring homomorphism $R \to R/I$ satisfy $\lambda_R(R/I) \geq n-1$, and let M be a graded R-module such that $IM = 0$, where I is a homogeneous ideal of R. If M is graded-n-coherent as a graded R-module, then it is graded-n-coherent as an R/I-module by Lemma 6.9.

Theorem 6.11. [11, Theorem 2.11] Let $R \to S$ be a graded ring homomorphism making S a faithfully flat R-module and let M be a graded R-module. If $M \otimes_R S$ is a graded-n-coherent S-module, then M is a graded-n-coherent R-module.

6.2 Graded-n-coherent rings

Definition 6.12. [11, Definition 3.1] A graded ring R is called graded-n-coherent if it is graded-n-coherent as a graded R-module, that is, if each $(n-1)$-presented homogeneous ideal of R is n-presented.

Remark 6.13. [11, Remark 3.2] Obviously, every n-coherent graded ring is a graded-n-coherent ring. The converse is not true in general, example 3.2 in [14] gives an example of graded-1-coherent ring which is not 1-coherent.

The next result shows that we have already many examples of graded-n-coherent rings.

Example 6.14. [11, Example 3.3]

1. Every graded-valuation domain is a graded-n-coherent ring for each $n \geq 1$, see [4].

2. Every graded-Noetherian ring is a graded-n-coherent for each $n \geq 1$, see [20].

Proposition 6.15. *[11, Proposition 3.4] Let R be a graded-n-coherent ring and let I be an $(n-1)$-presented homogeneous ideal of R. Then R/I is a graded-n-coherent ring.*

Remark 6.16. [11, Remark 3.5] The case $n = 1$ recovers the known fact that if I is a finitely generated homogeneous ideal of a graded-1-coherent ring R, then R/I is a graded-1-coherent ring ([14], Theorem 3.7(1)).

Theorem 6.17. [11, Theorem 3.7] Let $(R_i)_{i=1,2,...,m}$ be a family of graded rings. Then $\prod_{i=1}^{m} R_i$ is a graded-n-coherent ring if and only if R_i is a graded-n-coherent ring, for each $i = 1, \ldots, m$.

To establish this Theorem, the authors had proved the following Lemma.

Lemma 6.18. *[11, Lemma 3.8] Let R_1 and R_2 be two graded rings. Then R_i is an infinitely presented homogeneous ideal of $R_1 \times R_2$, for $i = 1, 2$*

6.3 Graded-valuation property in graded trivial extension

Assume that the grading monoid G is torsionless, that is a commutative, cancellative monoid and the quotient group of G is a torionfree abelian group. Let A be a graded ring, and let H denote the saturated multiplicative set of regular homogeneous elements of A. Then, by extending some definitions to the case where rings are with zero divisors, A_H, called the homogeneous total ring of quotients of A, is a ring graded by $\langle G \rangle$, where $A_H = \oplus_{\alpha \in \langle G \rangle} (A_H)_\alpha$ with $(A_H)_\alpha = \left\{ \frac{a}{b} \mid a \in A_\beta, b \right.$ a regular element of A_γ and $\beta - \gamma = \alpha\}$. If A is a graded integral domain (An integral domain graded by G), then A_H is called the homogeneous quotient field of A. Clearly, every nonzero homogeneous element of A_H is invertible and $(A_H)_0$ is a field. We say that A is a graded-valuation ring (gr-valuation ring for short) if either $x \in A$ or $x^{-1} \in A$ for every nonzero homogeneous element $x \in A_H$. Recall that if A is a graded ring and E is a graded A-module, then $A \propto E$ is a graded ring where $A \propto E = \oplus_{\alpha \in G} (A \propto E)_\alpha = \oplus_{\alpha \in G} (A_\alpha \oplus E_\alpha)$. This section gives a result of the transfer of gr-valuation property to graded trivial extension ring. We begin with the following result extending Theorem 1.2 in [4] to the case where rings are with zero divisors and which characterizes gr-valuation rings.

Theorem 6.19. [11, Theorem 4.1] Let $A = \oplus_{\alpha \in G} A_\alpha$ be a graded ring. The following statements are equivalent:

1. A is a gr-valuation ring.
2. Either $a \mid b$ or $b \mid a$ for every nonzero homogeneous elements $a, b \in A$, one at least of which is regular.
3. Every pair of homogeneous (fractional) ideals of A, one at least of which is regular, are totally ordered under inclusion.
4. Every pair of principal homogeneous ideals of A, one at least of which is regular, are totally ordered under inclusion.

Definition 6.20. [11, Definition 4.3] Let A be a graded ring, a graded A-module E is said to be graded-uniserial (gr-uniserial for short) if the set of its homogeneous submodules is totally ordered by inclusion.

The next result gives a characterization for the graded trivial extension to be a gr-valuation ring. Note that here the assumption "G is a torsionfree abelian group" is necessary since the grading monoid of A and E must be the same and the fact that its "torsion-freeness" is used by Lemma 6.22.

Theorem 6.21. [11, Theorem 4.4] Assume that the grading monoid is a torsionfree abelian group. Let A be a graded ring and E a nonzero graded A-module. Let $R := A \propto E$ be the graded trivial extension ring of A by E. Assume that E is a torsionfree graded A-module. Then R is a gr-valuation ring if and only if A is a gr-valuation domain and E is isomorphic to A_H, the homogeneous quotient field of fractions of A.

To prove Theorem 6.21, the authors had established the following lemma.

Lemma 6.22. *[11, Lemma 4.5] Assume that the grading monoid is a torsionfree abelian group. Let A be a graded ring, E be a nonzero graded A-module, and $R := A \propto E$ be the graded trivial extension ring of A by E. If R is a gr-valuation ring, then A is a gr-valuation domain and E is a gr-uniserial A-module.*

The previous Theorem enriches the literature with new examples of gr-valuation rings.

Example 6.23. [11, Example 4.6] Let K be a field graded by an arbitrary torsionfree group. Let K_H be its homogeneous quotient field of fractions. Then the trivial ring extension of K by K_H, $K \propto K_H$ is a gr-valuation ring.

The next theorem characterizes the gr-valuation property in the graded trivial extension rings in a general case. Recall from [47, page 179] that a graded A-module is said to be gr-divisible if $ax = b$ with $a \in h(A)$, $b \in h(E)$ has a solution in E.

Theorem 6.24. [11, Theorem 4.8] Assume that the grading monoid is a torsionfree abelian group. Let A be a graded ring and E be a nonzero graded A-module. Then $R := A \propto E$ is a gr-valuation ring if and only if A is a gr-valuation domain and E a gr-divisible gr-uniserial A-module.

7 Commutative Graded-S-coherent rings

Recently, motivated by Anderson and Dumitrescu's S-finiteness, Bennis and El Hajoui introduced the notion of S-coherent rings which is the S-version of coherent rings. Let R be a ring, M an R-module and S a multiplicatively closed subset of R. Then M is called S-finite if there exists a finitely generated submodule N of M such that $sM \subseteq N$ for some $s \in S$. This notion was introduced by Anderson and Dumitrescu, see [6]. According to [17], E is called an S-finitely presented R-module if there exist an exact sequence $0 \to F_1 \to F_0 \to E \to 0$ of R-modules such that F_0 is a finitely generated free R-module and F_1 is S-finite. Any finitely presented R-module is an S-finitely presented R-module; while the converse is false in general (for more results and details, the reader can refer to [17, Section 2]). A finitely generated R-module M is said to be an S-coherent R-module if every finitely generated R-submodule of M is an S-finitely presented R-module; and a ring R is said to be an S-coherent ring if R is S-coherent as an R-module. In this Section 7, unless otherwise stated, all the results are taken from [12], where the authors introduced and studied the notions of graded-S-coherent rings and graded-S-coherent modules (over an arbitrary graded ring).

Definition 7.1. [12, Definition 3.1] A graded R-module M is said to be graded-S-coherent, if it is finitely generated and every finitely generated homogeneous submodule of M is S-finitely presented. And a graded ring R is said to be graded-S-coherent, if it is graded-S-coherent as a graded R-module; that is, if every finitely generated homogeneous ideal of R is S-finitely presented.

The next remark collects some immediate classes of graded-S-coherent modules and rings.

Remark 7.2. [12, Remark 3.2] Let R be a graded ring. Then the following statements hold true:

(1) Every finitely generated homogeneous R-submodule of a graded-S-coherent R-module is graded-S-coherent.

(2) Recall from [14] that a graded R-module M is said to be graded-coherent if it is finitely generated and every finitely generated homogeneous submodule of M is finitely presented. Clearly, any graded-coherent R-module is a graded-S-coherent R-module since every finitely presented module is S-finitely presented. Hence, the following rings are graded-S-coherent: graded-Noetherian ring ([20]), graded-valuation domain ([4]) and graded Prüfer domain ([5]).

(3) Obviously, every S-coherent graded R-module is a graded-S-coherent R-module and every S-coherent graded ring is a graded-S-coherent ring. But the converse is not true in general, as shown by the following example.

Example 7.3. [12, Example 3.3] If A is a countable direct product of $\mathbb{Q}[[t,u]]$'s, consider the polynomial ring graded by $\mathbb{N}$ via $(A[X])_n = AX^n$ for every $n \in \mathbb{N}$, and let $S = \{1\}$. Then $A[X]$ is graded-S-coherent but not S-coherent.

The following result studies the behavior of graded-S-coherence of graded modules in short exact sequences. It is the graded version of [17, Proposition 3.2].

Proposition 7.4. *[12, Proposition 3.4] Let $0 \longrightarrow P \xrightarrow{\alpha} N \xrightarrow{\beta} M \longrightarrow 0$ be an exact sequence of graded R-modules. Then the following statements hold:*

1. If P is finitely generated, N is graded-S-coherent and β has a cancellable degree, then M is graded-S-coherent.

2. If M is graded-coherent, P is graded-S-coherent and α has a cancellable degree, then N is graded-S-coherent.

3. If N is graded-S-coherent and P is finitely generated, then P is graded-S-coherent.

Remark 7.5. [12, Remark 3.5] 1. Let G be an abelian group, R be a graded ring and S be a multiplicatively closed subset of R_0. Then, recall from [41], that R is said to be graded-S-Noetherian if every homogeneous ideal of R is S-finite. Note that every graded-S-Noetherian ring is graded-S-coherent. Indeed, this follows by applying [17, Proposition 2.3] and from the fact that when R is graded-S-Noetherian, every finitely generated graded R-module is graded-S-Noetherian.

The next definition generalizes the definition given in [41].

Definition 7.6. [12, Definition 3.6] Let R be a graded ring and S a multiplicatively closed subset of homogeneous elements of R (not necessarily of R_0). Then, R is said to be graded-S-Noetherian if every homogeneous ideal of R is S-finite.

The next result presents the graded version of Proposition 2(f) in [6]. For an ideal I of R, $IR_S \cap R$ means the S-saturation of I.

Proposition 7.7. *[12, Proposition 3.7] Let G be a group, R be a graded ring and $S \subseteq R$ be a multiplicatively closed subset of homogeneous elements of R. Then, R is graded-S-Noetherian if and only if R_S is graded-Noetherian and for every finitely generated homogeneous ideal I of R, $IR_S \cap R = (I : s)$ for some $s \in S$.*

The following proposition investigates the behavior of the graded-S-coherence under some change of rings.

Proposition 7.8. *[12, Proposition 3.8] Let $\phi : R \longrightarrow L$ be a graded ring homomorphism, M be a graded L-module and S be a multiplicatively subset of homogeneous elements of R such that $0 \notin \phi(S)$. If the graded R-module L is finitely generated and M is graded-S-coherent as an R-module, then M is graded-$\phi(S)$-coherent as an L-module.*

Proposition 7.9. *[12, Proposition 3.9] Let R be a graded ring, I be an S-finite homogeneous ideal of R where S is a multiplicatively closed subset of homogeneous elements of R and M be a graded R/I-module. Assume that $I \cap S = \emptyset$ so that $V := \{s + I \in R/I; s \in S\}$ is a multiplicatively closed subset of homogeneous elements of R/I. Then the following statements hold true:*

1. M is graded-V-coherent as an R/I-module if and only if it is graded-S-coherent as an R-module.

2. Let I be a finitely generated ideal of R, if R is graded-S-coherent ring, then R/I is graded-V-coherent ring. The converse holds if R/I is a graded-coherent ring and I is graded-S-coherent as an R-module.

The following theorem clarifies the situation for the product of graded-S-coherent modules.

Theorem 7.10. [12, Theorem 3.10] Let M_i be a graded R_i-module and S_i a multiplicatively closed set of homogeneous elements of R_i for $i = 1, 2, \ldots, n$. Suppose that $R = R_1 \times R_2 \times \cdots \times R_n$, $M = M_1 \times M_2 \times \cdots \times M_n$ and $S = S_1 \times S_2 \times \cdots \times S_n$. The following statements are equivalent.

1. M is a graded-S-coherent R-module.
2. M_i is a graded-S_i-coherent R_i-module for each $i = 1, 2, \ldots, n$.

As a consequence of the previous theorem, we have the following result.

Corollary 7.11. *[12, Corollary 3.11] Let $R = \prod_{i=1}^{n} R_i$ be a direct product of graded rings $R_i (n \in \mathbb{N})$ and $S = \prod_{i=1}^{n} S_i$ be a cartesian product of multiplicatively closed sets S_i of homogeneous elements of R_i. Then, R is graded-S-coherent if and only if R_i is graded-S_i-coherent for every $i \in \{1, \ldots, n\}$.*

Remark 7.12. [12, Remark 3.12] (1). As a particular case of Remark 7.2(2), any graded-coherent ring is a graded-S-coherent ring, see [14]. The converse is not true in general. Consider the graded ring $A = \mathbb{Z} \propto (\mathbb{Z}_2)^{(\mathbb{N})}$ with its natural $\mathbb{Z}_2$-grading; $A_0 = \mathbb{Z} \propto 0$ and $A_1 = 0 \propto (\mathbb{Z}_2)^{(\mathbb{N})}$ and consider the multiplicative set of homogeneous elements $S = \{2^n : n \in \mathbb{N}\} \propto 0$. Since $(2,0)$ is a homogeneous element and $(0 : (2,0)) = 0 \propto (\mathbb{Z}_2)^{(\mathbb{N})}$ is not a finitely generated ideal, then, according to [14, Theorem 3.3] A is not graded-coherent. Then $(2,0)I$ is finitely generated for any homogeneous ideal I of A. Hence, A is graded-S-Noetherian and so graded-S-coherent by Remark 7.5.

(2). Recall from [17, Remark 3.4(3)] that if M is an S-finitely presented R-module, then M_S is a finitely presented R_S-module. Thus, if R is a graded-S-coherent ring, R_S is a graded-coherent ring.

The following theorem is the graded version of the S-counterpart of the classical Chase's result [24, Theorem 2.2]. For reference purpose, it will be helpful to recall the following elementry lemma.

Lemma 7.13. *[31, Lemma 2.3.1] Let R be a ring, let $I = (u_1, u_2, \ldots, u_r)$ be a finitely generated ideal of $R (r \in \mathbb{N})$ and let $a \in R$. Set $J = I + Ra$. Let F be a free module on generators $x_1, x_2, \ldots, x_{r+1}$ and let $0 \longrightarrow K \longrightarrow F \xrightarrow{f} J \longrightarrow 0$, be an exact sequence with $f(x_i) = u_i (1 \leq i \leq r)$ and $f(x_{r+1}) = a$. Then there exists an exact sequence $0 \longrightarrow K \cap F' \longrightarrow K \xrightarrow{g} (I : a) \longrightarrow 0$, where $F' = \bigoplus_{i=1}^{r} Rx_i$.*

Theorem 7.14. [12, Theorem 3.14] Let R be a graded ring. The following assertions are equivalent:

1. R is graded-S-coherent.
2. $(I : a)$ is an S-finite ideal of R, for every finitely generated homogeneous ideal I of R and for every homogeneous element $a \in R$.
3. $(0 : a)$ is an S-finite ideal of R for every homogeneous element $a \in R$ and the intersection of two finitely generated homogeneous ideals of R is an S-finite ideal of R.

Recall from [48] that a homogeneous ideal M of a graded ring R is said to be a homogeneous-maximal ideal if it is maximal among proper homogeneous ideals; equivalently, if every nonzero homogeneous element of R/M is invertible. A graded ring is said to be graded-local if it has a unique homogeneous-maximal ideal.

Proposition 7.15. *[12, Proposition 3.15] Assume that the grading monoid G is cancellative and let (R, M) be a graded-local ring such that $M^2 = 0$. Let S be a multiplicatively closed set of homogeneous elements of R. Then the following statements are equivalent.*

1. *R is a graded-S-coherent ring.*
2. *$(0 : x)$ is an S-finite ideal for every homogeneous element $x \in R$.*
3. *M is S-finite.*

We end this section with the following result which studies the transfer of graded-S-coherence under localizations.

Proposition 7.16. *[12, Proposition 3.16] Assume that the grading monoid G is a group and let R be a graded ring. If R is a graded-S-coherent ring, then R_V is a graded-S_V-coherent ring for every multiplicative set V of homogeneous elements of R.*

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