Let $M$ be an o-minimal structure or a $p$-adically closed field. Let $S'_n(M)$ be the space of complete $n$-types over $M$ equipped with the following topology: The basic open sets of $S'_n(M)$ are of the form $\tilde{U} = \{ p \in S_n(M) : U \in p \}$ for $U$ an open definable subset of $M^n$. $S'_n(M)$ is a spectral space. (For $M = K$ a real closed field, $S'_n(M)$ is precisely the real spectrum of $K[X_1, \ldots, X_n]$; see [CR].) We will equip $S'_n(M)$ with a sheaf of $LM$-structures (where $LM$ is a suitable language). Again for $M$ a real closed field this corresponds to the structure sheaf on $S'_n(M)$ (see [S]).

Our main point is that when $\text{Th}(M)$ has definable Skolem functions, then if $p \in S'_n(M)$, it follows that $M(p)$, the definable ultrapower of $M$ at $p$, can be factored through $M_p$, the stalk at $p$ with respect to the above sheaf. This depends on the observation that if $M \prec N$, $a \in N^n$ and $f$ is an $M$-definable (partial) function defined at $a$, then there is an open $M$-definable set $U \subset N^n$ with $a \in U$, and a continuous $M$-definable function $g : U \to N$ such that $g(a) = f(a)$.

In the case that $M$ is an o-minimal expansion of a real closed field (or $M$ is a $p$-adically closed field), it turns out that $M(p)$ can be recovered as the unique quotient of $M_p$ which is an elementary extension of $M$. (So this gives a sheaf-theoretic construction of definable ultrapowers.)

§1. In this section we work in a more general setting. Specifically, let $M$ be a first order topological structure in the sense of [P], namely $M$ has an “explicitly definable” topology which we assume to be Hausdorff. Assume in addition that every definable set $X \subset M^n$ is a Boolean combination of closed definable sets (where $M^n$ is equipped with the product topology).

Let $S_n(M)$ be the set of complete $n$-types over $M$. We equip $S_n(M)$ with a topology, coarser than the usual Stone space topology, as follows: a basic open set is of the form $\tilde{U} = \{ p \in S_n(M) : U \in p \}$, where $U$ is an open definable subset of $M^n$. It is easy to check that the $\tilde{U}$ form the basis for a topology. We call the resulting space $S'_n(M)$.

**Lemma 1.1.** $S'_n(M)$ is a spectral space, i.e. it has a basis of quasicompact open sets and every irreducible closed set is the closure of a unique point.

**Proof.** Easily each set $\tilde{U}$, where $U$ is open definable in $M$, is quasicompact. Conversely if $V \subset S_n(M)$ is open quasicompact then $V = \tilde{U}_1 \cup \cdots \cup \tilde{U}_m = (U_1 \cup \cdots \cup U_m)^\sim$. This proves the first part.
Now let \( F \subset S'_a(M) \) be closed and irreducible. Let \( \Phi = \{ X \subset M^n: X \text{ is closed, definable and } X \in q \forall q \in F \} \). Let \( \Phi^1 = \Phi \cup \{ \neg Y: Y \text{ closed, definable, } Y \notin \Phi \} \). By irreducibility of \( F, \Phi^1 \) is consistent and thus determines a type \( p \in S_n(M) \). Easily \( F \) is the closure of \( p \) and only of \( p \).

We now point out how one can construct a sheaf of structures on \( S'_a(M) \). These structures will be \( L_M \)-structures, where \( L_M \) is a language we now describe. Let \( L \) be the language of \( M \). For each \( L \)-formula \( \varphi(x, y) \) and \( m \subset M \), such that \( \varphi(x, m) \) defines a closed set in \( M^n \), let \( R_{\varphi(x, m)} \) be a new \( n \)-ary relation symbol. Then \( L_M \) is the set of all these \( R_{\varphi(x, m)} \). Note that in particular \( R_{x=m} \) and \( R_{x=y} \) are in \( L_M \). Note first that any elementary extension \( N \) of \( M \) (in particular \( M \) itself) can be canonically construed as an \( L_M \)-structure. Namely, interpret \( R_{\psi} \) in \( N \) as \( \{ b \in N^n: N \models \psi(b) \} \). Let \( T_M \) be the theory of \( M \) as an \( L_M \)-structure. Then moreover any model of \( T_M \) can be canonically construed as an \( L \)-structure which is an elementary extension of \( M \), by virtue of our assumption that any definable set in \( M \) is a Boolean combination of closed sets.

Let \( M_0, M_1 \) be \( L_M \)-structures. By a homomorphism \( h: M_0 \to M_1 \) we mean a map \( h \) such that \( M_0 \models R(a) \) implies \( M_1 \models R(h(a)) \) for each of the (basic) \( R \in L_M \). We call \( h \) a strict homomorphism if moreover \( M_1 \not\models R(h(a)) \) implies there is \( b \in M_0 \) with \( h(b) = h(a) \) and \( M_0 \models R(b) \). By a quotient of \( M_0 \) we mean a strict homomorphic image of \( M_0 \).

Now, let \( U \) be a definable open subset of \( M^n \). We define an \( L_M \)-structure \( M_U \) as follows: the elements of \( M_U \) are continuous functions \( U \to M \) which are definable (with parameters) in \( M \). If \( R_\varphi \) is a basic \( k \)-ary relation of \( L_M \) and \( f_1, \ldots, f_k \in M_U \) we say \( R_\varphi(f_1, \ldots, f_k) \) if for all \( x \in U, M \models \varphi(f_1(x), \ldots, f_k(x)) \). If \( U \subset V \) are open definable subsets of \( M^n \) then \( h_{U,V}: M_V \to M_U \) is the map defined by \( h_{U,V}(f) = f \upharpoonright U \). Clearly \( h_{U,V} \) is a homomorphism of \( L_M \)-structures.

Returning to \( S'_a(M) \), we can regard \( S'_a(M) \) as a special kind of site whose objects are the quasicompact open sets and whose morphisms are the inclusion maps. A sheaf \( F \) of \( L_M \)-structures over \( S'_a(M) \) is then the assignment to each quasicompact open \( \tilde{U} \subset S'_a(M) \) of an \( L_M \)-structure \( F(\tilde{U}) \), and to each pair \( \tilde{U} \subset \tilde{V} \) of a homomorphism \( F_{\tilde{U}, \tilde{V}}: F(\tilde{V}) \to F(\tilde{U}) \), satisfying the usual compatibility conditions—in particular, if \( U_i, i \in I \), is a covering of \( \tilde{U} \) and \( s_i \in F(\tilde{U}) \) for each \( i \in I \), and if, for \( i, j \in I, F_{\tilde{U}, \tilde{V}}(s_i, s_j) = F_{\tilde{U}, \tilde{V}}(s_j, s_i) \) for each \( i \in I \), then there is a unique \( s \in F(\tilde{U}) \) such that \( F_{\tilde{U}, \tilde{V}}(s) = s_i \forall i \in I \). Then for \( p \in S'_a(M) \) we can form the stalk \( F_p \), an \( L_M \)-structure which is the limit of the \( L_M \)-structures \( F(\tilde{U}) \) for \( p \in \tilde{U} \). We will be interested in the sheaf \( \mathcal{M} \) over \( S'_a(M) \), where, for \( \tilde{U} \subset S'_a(M), \mathcal{M}(\tilde{U}) = M_U \) and \( \mathcal{M}_{\tilde{U}, \tilde{V}} = h_{U,V} \). We denote the stalk \( \mathcal{M}_p \) by \( M_p \). \( M_p \) can be described more concretely as follows: the elements of \( M_p \) are germs of continuous definable functions at \( p \); namely for \( f \) a continuous definable function \( U \to M \), and \( g \) continuous definable \( V \to M \), where \( U, V \subset M^n \) are open definable and both are in \( p \), put \( f \sim g \) if there is an open definable \( W \subset M^n \), \( W \subset U \cap V \), \( W \in p \), such that \( f \upharpoonright W = g \upharpoonright W \). Then \( M_p \) consists of the equivalence classes \( f \sim \) of such \( f \), and for \( R_\varphi \) a basic \( k \)-ary relation in \( L_M \) we put \( R_\varphi(f_1, \ldots, f_k) \) if for some open definable \( U \in p, U \subset M^n \), on which each \( f_i \) is continuous, we have \( M \models \varphi(f_1(x), \ldots, f_k(x)) \forall x \in U \).

§2. An arbitrary structure \( M \) for a language \( L \) is said to have definable Skolem functions if for each formula \( \varphi(x, y) \) of \( L \) (with maybe parameters from \( M \)), there is a formula \( \psi(x, y) \) such that \( M \models \forall x (\exists y \varphi(x, y) \to (\exists y)(\varphi(x, y) \land \psi(x, y))) \) and

\[ \]
We call the (partial) function which takes suitable $a$ to the unique $b$ such that $M \models \psi(a, b)$ an $(M)$-definable function, $f$ or $f_\psi$. The same formula $\psi$ will of course define a function in any $N \succ M$, and we call this function $f^N$.

If $M$ has definable Skolem functions it follows that for any $p \in S_n(M)$, there is a model $M(p) \prec M$ with the feature that there is an $a \in (M(p))^n$ realizing $p$, and for every $b \in M(p)$ there is an $M$-definable function $f$ such that $f^M(p)(a) = b$. $M(p)$ is unique up to isomorphism over $M$, and can also be obtained as a “definable ultrapower” of $M$.

If $M$ is an o-minimal expansion of an ordered divisible abelian group or $M$ is a $p$-adically closed field, then $M$ has definable Skolem functions (see [D] for the latter).

In order to show that in the “topological” cases we are interested in, $(M(p))$ is a quotient of $M$, the stalk at $p$, we clearly have to at least show that there are “enough” continuous definable functions on open sets $U \subset M^n$. To do this we make use of the facts on definable functions and cell decompositions given in [KPS] for o-minimal structures and in [SD] for $p$-adically closed fields.

**Proposition 2.1.** Let $M$ be an o-minimal structure or $p$-adically closed field. Let $M \prec N, a \in N^n$ and $f$ an $M$-definable partial function on $N^n$, defined at $a$. Then there is an open (in $N^n$), $M$-definable set $U$ containing $a$ and an $M$-definable continuous function $g: U \to N$ such that $g(a) = f(a)$.

**Proof.** First note that for any of the structures we are concerned with we have the notion of a basic open neighborhood in 1-space over $M$, i.e. for o-minimal $M$ this is an interval, and for $p$-adically closed fields this is $\{x: \nu(x - a) \geq i\}$ for some $a \in M$ and $i \in$ (value group of $M$). A basic open neighborhood in $n$-space over $M$ will then be a product of such neighborhoods in 1-space.

Now as $f$ is $M$-definable and defined at $a$, there is an $M$-definable $X \subset N^n, a \in X$, such that $f$ is defined on $X$. By [KPS] for the o-minimal case and [SD] for $p$-adically closed fields there is an $M$-definable $X_1 \subset X$ containing $a$ such that:

(i) $X_1$ is homeomorphic by a projection $\Pi$ along certain coordinate axes to an open set in $N^k$ for some $k \leq n$. (We will assume the projection $\Pi$ is along the first $k$ coordinate axes.)

(ii) $f \upharpoonright X_1$ is continuous.

**Claim.** For each $b \in X_1$ there is a basic open neighborhood $I = I_1 \times \cdots \times I_n$ of $b$ in $N^n$ such that for each $(c_1, \ldots, c_n) \in I$ there are (unique) $c_{k+1}, \ldots, c_n \in N$ such that $(c_1, \ldots, c_k, c_{k+1}, \ldots, c_n) \in I \cap X_1$.

**Proof of Claim.** First let $J_1 \times \cdots \times J_n$ be a basic open neighborhood of $b$ in $N^n$. As $\Pi^{-1}: \Pi(X_1) \to N^n$ is continuous, there is a basic open neighborhood $I_1 \times \cdots \times I_k$ of $(b_1, \ldots, b_k)$ in $N^k$ such that for all $(c_1, \ldots, c_k) \in I_1 \times \cdots \times I_k$

$$\Pi^{-1}(c_1, \ldots, c_k) \subset J_1 \times \cdots \times J_k \times \cdots \times J_n.$$ It is then clear that $I = I_1 \times \cdots \times I_k \times J_{k+1} \times \cdots \times J_n$ works for the claim.

Now let $U$ be the union of all such $I$ in the claim as $b$ varies over $X_1$. Clearly $U \subset N^n$ is open $M$-definable and includes $X_1$. Define $g$ on $U$ as follows: let $(c_1, \ldots, c_n) \in U$. There will be unique $c_{k+1}, \ldots, c_n \in N$ such that $(c_1, \ldots, c_k, c_{k+1}, \ldots, c_n) \in X_1$. Put $g(c_1, \ldots, c_n) = f(c_1, \ldots, c_k, c_{k+1}, \ldots, c_n)$.

It is easily checked that $g$ is continuous on $U$ and $M$-definable. Moreover $g$ agrees with $f$ on $X_1$, so $g(a) = f(a)$, completing the proof.
Note that $U \cap M^n$ is a definable open set which is in $p = \text{tp}(a/M)$, and that the graph of $g$ intersected with $M^{n+1}$ is the graph of a continuous definable function $U \cap M^n \to M$.

**Proposition 2.2.** Let $M$ be an o-minimal structure with definable Skolem functions or a $p$-adically closed field. Let $p \in S_n(M)$. Then $M(p)$ is a quotient of $M_p$.

**Proof.** Let $a$ realize $p$ in $M(p)$ such that $M(p) = \text{dcl}(M \cup a)$; that is, $\forall b \in M(p), b = f(a)$ for some $M$-definable function $f$. Put $N = M(p)$.

First some notation. Suppose $f$ is a definable (partial) $M$-valued function on $M^n$ and let $f^N$ denote the function on $N^n$ defined by the same formula defining $f$.

Define $h: M_p \to N$ as follows.

If $f^- \in M_p$ then $h(f^-) = f^N(a)$. $h$ is clearly well-defined and is a homomorphism of $L_M$-structures. ($R_\varphi(f_1^-, \ldots, f_k^-)$ in $M_p$ is for some open $U \subset M^n$, $U \in p = \text{tp}(a)$, $M \models \forall x \in U \varphi(f_1(x), \ldots, f_k(x)) \Rightarrow N \models \forall x \in U \varphi(f_1(x), \ldots, f_k(x))$.

As in the proof of 2.1 we can choose $X$ homeomorphic by projection along certain coordinate axes to an open set in $M^r$ for some $r \leq n$, and such that $f_i|X$ is continuous for each $i$. As in the proof of 2.1 again, we can find open definable $U \subset M^n$, $X \subset U$, continuous definable $g_i: U \to M$ such that $g_i|X = f_i|X$ and $\forall x \in U \exists x^1 \in X (h_i(x) = f_i(x^1))$ for all $i$. Then clearly $h^- \in M_p$, $h^N_i(a) = f^N_i(a) = d$, and $R_\varphi(h_1^-, \ldots, h_k^-)$ in $M_p$. This completes the proof of 2.2.

§3. Finally we consider the question of whether $M_p$ has a unique quotient which is canonically an elementary extension of $M$. We first point out a situation in which this fails.

**Proposition 3.1.** Let $M$ be o-minimal with definable Skolem functions. Let $a \in M$ be such that for every $c$ in $M$ with $c > a$ and every definable $f: (a, c) \to M$, $\lim_{x \to a^+} f(x) \in M$ (i.e. this limit is not $\pm \infty$), let $p = \text{tp}(a/M)$ (an isolated type) and let $q \in S_1(M)$ be determined by $\{a < x < b: b \in M, a < b\}$. Then $M(p) \not\cong_M M(q)$, but nevertheless $M(q)$ is a strict homomorphic image of $M_p$.

**Proof.** Note first that $M(p) = M$, and $M(q) \supset M$ and $M(q) \neq M$; so $M(p) \not\cong_M M(q)$.

Let $c$ realize $q$ in $M(q)$ such that $M(q) = \text{dcl}(M \cup c)$. Again write $N = M(q)$.

Define $h: M_p \to N$ by $h(f^-) = f^N(c)$ for $f^- \in M_p$.

(a) $h$ is a homomorphism of $L_M$-structures. Note $M_p = \{f^-: \text{for some } b_1 < a < b_2 \in M, f \text{ is definable and continuous } (b_1, b_2) \to M\}$. So clearly if $f_1^-, \ldots, f_k^- \in M_p$ and $R_\varphi \in L_M$ then $R_\varphi(f_1^-, \ldots, f_k^-) \Rightarrow \forall x \in (b_1, b_2) \varphi(f_1(x), \ldots, f_k(x))$.

(b) $h$ is onto $N$. Let $d \in N$. So $d = f^N(c)$ for some $M$-definable $f$, and moreover we can choose $f$ to be continuous on some interval $I = (b_1, b_2)$, $b_1, b_2 \in M$, which contains $c$ (in $N$). If $a \in (b_1, b_2)$ then $f^- \in M_p$ and $d = h(f^-)$. Otherwise $I = (a, b_2)$. Now by our assumptions $\lim_{x \to a} f(x) = d^1 \in M$. Let us define $f'$ on $M$ as follows:
for $x \in M$, $x \leq a$, $f'(x) = d'$, and for $x \in (a, b_2)$, $f'(x) = f(x)$. Thus $f'$ is continuous on an open set containing $a$, and so $(f')^N(c) = f^N(c) = d$. So $h((f')^N) = d$.

(c) $h$ is strict. Let $R_\phi \in L_M$, $d_1, \ldots, d_k \in N$ and $N \models \phi(d_1, \ldots, d_k)$. We must find $g_1^\sim, \ldots, g_k^\sim \in M_p$ such that $d_i = g_i^N(c)$ and $R_\phi(g_1^\sim, \ldots, g_k^\sim)$ in $M_p$.

First we can easily obtain (see [KPS]) $M$-definable functions $f_1, \ldots, f_k$ and $e \in M$, $e > a$, such that $d_i = f_i(c) \forall i$ and $M \models \forall x \in (a, e) \phi(f_1(x), \ldots, f_k(x))$ with moreover each $f_i$ constant or an order preserving or reversing isomorphism on $(a, e)$. (*)

Let $d'_i = \lim_{x \to a^+} f_i(x)$ in $M$. Define $g_i$ on $M$ as follows: $g_i(x) = d'_i$ if $x \leq a$, and $g_i(x) = f_i(x)$ if $x > a (x < e)$. So $g_i$ is continuous on its domain, which is an open interval in $M$ containing $a$. So $g_i^\sim \in M_p \forall i$. Moreover as $g_i \upharpoonright (a, e) = f_i \upharpoonright (a, e)$, $g_i^N(c) = d_i \forall i$ (**)

To show that $R_\phi(g_1^\sim, \ldots, g_k^\sim)$ in $M_p$, it suffices to see that $M \models \phi(d'_1, \ldots, d'_k)$. But this follows immediately from (*) and (**) by the continuity of the $g_i$ and the fact that $\phi$ defines a closed set in $M^k$.

Note that the hypotheses of 3.1 hold for any ordered divisible abelian group (e.g. $(\mathbb{R}, +, 0)$). On the other hand:

**Proposition 3.2.** Let $M$ be an o-minimal expansion of a real closed field, or a $p$-adically closed field. Let $p \in S'_q(M)$. Then $M_p$ is a local ring. More precisely, $0, 1$, and the graphs of addition and multiplication on $M$ are closed definable relations and thus in $L_M$, and with respect to these relations $M_p$ is a local ring.

**Proof.** It is first easy to see that $M_p$ is a commutative ring with respect to the relations $0_M, 1_M, +_M, \cdot_M$ in $L_M$. Now let $c$ realize $p$ in $N > M$. Let $I = \{ f^\sim \in M_p : N \models f(c) = 0 \}$. Clearly $I$ is an ideal in $M_p$. If $f^\sim \in I$ then clearly $f^\sim$ is a nonunit in $M_p$. On the other hand, if $f^\sim \notin I$ then $N \models f(c) \neq 0$. So there is $U \subset M^n$ open, $U \in p$, and $g$ definable continuous $U \to M$ such that $M \models g(x) \neq 0 \forall x \in U$ and $g^\sim = f^\sim$. Then $(1/g)^\sim$ shows that $f^\sim$ is a unit in $M_p$. So $I$ is the unique maximal ideal of $M_p$.

**Corollary 3.3.** With the hypotheses of 3.2, let $p \in S(M)$. Then $M(p)$ is the unique quotient of $M_p$ which is an elementary extension of $M$. Moreover $M(p) \cong M_p/m_p$, where $m_p$ is the maximal ideal of $M_p$.

**References**


DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NOTRE DAME
NOTRE DAME, INDIANA 46556