Soliton shedding from Airy pulses in Kerr media

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Abstract: We simulate and analyze the propagation of truncated temporal Airy pulses in a single mode fiber in the presence of self-phase modulation and anomalous dispersion as a function of the launched Airy power and truncation coefficient. Soliton pulse shedding is observed, where the emergent soliton parameters depend on the launched Airy pulse characteristics. The Soliton temporal position shifts to earlier times with higher launched powers due to an earlier shedding event and with greater energy in the Airy tail due to collisions with the accelerating lobes. In spite of the Airy energy loss to the shed Soliton, the Airy pulse continues to exhibit the unique property of acceleration in time and the main lobe recovers from the energy loss (healing property of Airy waveforms).

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References and links
1. Introduction

Airy pulses [1], whose electric field temporal profile is defined by an Airy function which is a one-sided, oscillating function having infinite energy, are a solution to the linear dispersion equation

\[
i \frac{\partial A}{\partial z} = \beta_2 \frac{\partial^2 A}{\partial T^2},
\]

and exhibit two interesting features: during propagation the waveform maintains its shape in the presence of dispersion and its wavefront accelerates in time (or travels along a ballistic trajectory) in a time frame moving at the group velocity. However, true Airy pulses are impractical as they contain an infinite amount of energy. By apodizing the Airy pulse, i.e. truncating the semi-infinite oscillations, in our case with a decaying exponential envelope, the waveform maintains its two unique properties over an extended propagation range despite its finite energy (Fig. 1(a)) [2]. Truncated Airy pulses occur naturally if a Gaussian pulse is propagated in a fiber at the zero dispersion point, under the influence of cubic dispersion.

In complete analogy to the Airy pulse solution to the dispersion Eq. (1), spatial Airy beams are a solution to the paraxial equation. Spatial Airy beams have been investigated extensively in the last few years, and found to be useful for various applications such as optical micromanipulation [3], optical switching [4], plasma channel generation [5], and laser filamentation [6]. More recently, temporal Airy pulses are being investigated, in the context of spatiotemporal light bullets in linear conditions [7] and in nonlinear conditions [8], and in the context of one dimensional Airy pulse propagation, under the influence of strong nonlinearity giving rise to supercontinuum and solitary wave generation [9].

In this study, we analyze temporal Airy pulse propagation in media exhibiting Kerr nonlinearity as occurring in single mode silica fibers, leading to the phenomena of self-phase modulation (SPM) and anomalous dispersion. The influence of the Kerr nonlinear effect on spatial Airy beams was investigated under relatively weak parameters and transient narrowing of the Airy main lobe—caused by SPM—was observed [10]; however, we are interested in operating under much higher intensities where the nonlinear effect results in soliton shedding from the Airy pulse and not just a small perturbation of the Airy beam. Although we analyze temporal Airy pulse propagation in fiber, our results are also valid for spatial Airy beams diffracting in Kerr media on account of the isomorphism between the dispersion Eq. (1) and the paraxial diffraction equation.

![Fig. 1. (a) Intensity distribution as a function of time and propagation distance for truncated Airy pulse in the linear regime (or low launch power). (b) Launched Airy pulse in time (blue solid curve), compared to a soliton pulse (red dashed curve).](image)

The evolution of light pulses in single-mode dispersive-nonlinear medium is governed by the Nonlinear Schrödinger Equation (NLSE),

\[
i \frac{\partial A}{\partial z} = \beta_2 \frac{\partial^2 A}{\partial T^2} - \gamma |A|^2 A
\]

where \(\beta_2\) is the dispersion coefficient, \(\gamma\) is the nonlinear coefficient and \(A\) is the wave amplitude that depends on local time-\(T\), and distance-\(z\). Due to the addition of the nonlinear
potential (or SPM term) in the NLSE, the Airy function is no longer a valid solution and we cannot predict analytically the Airy pulse evolution. The Soliton, on the other hand, is a well-known solution of the NLSE. For the canonical first order case, its profile is $P_0 \cdot T_0^2 = \frac{[\beta_2]}{\gamma}$, where $P_0$ is peak power and $T_0$ is duration and it is obtained only when there is equilibrium between the dispersion and the nonlinear effect, leading to the condition

$$P_0 \cdot T_0^2 = \frac{[\beta_2]}{\gamma}$$

The soliton then maintains its form and power level, provided no losses are present. Cases of perturbed soliton propagation (i.e. when there are small deviations from the condition set in Eq. (3) were extensively investigated [11–15], which help us interpret the emergent soliton behavior in our simulations.

In this paper, we propagate Airy pulses with different intensities and apodization values and investigate both the resulting 'emergent soliton' parameters, as well as the behavior of the residual Airy pulse. All our simulations are based on numerical solutions of the NLSE, using the split-step Fourier method (SSFM). This numerical method was chosen due to its efficiency in simulating one-dimensional pulse propagation [16].

1.1. Normalization terms

In our simulations we used the normalized NLSE form [16]

$$i \frac{\partial A}{\partial z} = \text{sgn}(\beta_2) \frac{1}{2} \frac{\partial^2 A}{\partial T^2} - |A|^2 A,$$

where $|\beta_2| = \gamma = T_0 = 1$, and the launched Airy pulse profile is defined as:

$$A(T, z=0) = \sqrt{R \cdot K_p(a) \cdot \text{Ai}(T \cdot r) \cdot \text{Exp}(a \cdot T)}$$

where $0 < a < 1$ is the truncation coefficient, and $K_p(a)$ is a truncation-dependent factor that sets the pulse peak intensity to 1 for any $a$ value. This factor was numerically calculated and found to be in parabolic dependence with the truncation coefficient. $T$ is the time variable in a frame of reference that moves with the wave group velocity, i.e. $T = t - z/v_g$, and $R$ is a dimensionless parameter we vary for scaling the Airy power. At $R=1$ the Airy main lobe intensity profile looks quite similar to the fundamental soliton, as shown in Fig. 1(b).

We measure the propagation distance in $L_d$ units, defined as $L_d = T_0^2 / \beta_2$, which in our normalized coordinates equals 1.

2. Effects of launched Airy power

In order to investigate the influence of Airy launched power on its evolution, we varied the scaling parameter $R$ in the range 0.1-2 and for every $R$ value we propagated the pulse using the SSFM algorithm. Figure 2 shows pulse evolution examples for select $R$ values. At low launched power, the Airy pulse performs the acceleration in time and subsequently it succumbs to dispersion. However, when $R$ is sufficiently large (above 0.9) a stationary soliton pulse is formed out of the centered energy about the Airy main lobe. The soliton exhibits periodic oscillations in the soliton amplitude and width as a function of propagation distance. In addition, we witness the resilience of the temporal Airy waveform to shedding of a fraction of the energy as a soliton; the wavefront continues to propagate along a parabolic trajectory. Similar resilience has been shown in main lobe masking for spatial Airy beams [17] and in supercontinuum generation for temporal Airy pulses propagation [9].
2.1. The emergent soliton

Unsurprisingly, the shed pulse profile well conforms to a hyperbolic-secant function, or that of a soliton with background radiation. We fit a $\text{sech}(\cdot) + \text{background radiation}$ profile at every propagation distance and track the emergent soliton peak power, duration and time position along the propagation distance. We find that the power × duration$^2$ product oscillates about the equilibrium condition (= 1) defined in Eq. (2). These oscillations about the stable soliton are known to arise as a result of interference between dispersive background radiation and the formed soliton [11,12].

We examined the relations between the soliton oscillations and the launched Airy peak power. In Fig. 3(a) the oscillations of soliton width are shown as a function of propagation distance for select $R$ values. The pulse width narrows and the oscillations period decreases with higher launch power. The decreasing oscillation period with increasing launch power is depicted in Fig. 3(b). Similar behavior was reported in [12], where the amount of excess energy that was supplied to the launched soliton was expressed in the evolved soliton oscillations period. Another property of the oscillations is the modulation depth that sharply decreases with increased initial peak power (Fig. 3(c)). We can relate the low modulation depth to the greater stability of the formed soliton and conclude that high launched peak power is required for stable soliton formation.

Additional soliton parameters as soliton peak time position and phase also oscillate in similar manner as the peak power and width. Figures 4(a, c) show the evolution of time position and phase as a function of propagation distance (phase fluctuations are plotted after subtracting the soliton’s accumulated linear phase term). These oscillations are the result of interaction with the background radiation as explained in [13] and demonstrated in [14] for the problem of background radiation that is formed by soliton amplification in optical communication.

From the results in Fig. 4(a) we see that the position of the emergent soliton is also dependent on launch power. We plot the mean time position of the emergent soliton in Fig. 4(b). More intense excitation results in the soliton appearing at an earlier time. This phenomena is explained by the fact that for low values of $R$ a relatively long time is required
for accumulation of enough energy by SPM for the soliton formation and shedding, and
during this time the Airy pulse is accelerating and ‘carries’ the accumulating energy with it to 
later times. For larger $R$ values there is enough energy in the Airy main lobe for soliton 
formation and shedding at an early point.

Fig. 4. (a) Soliton peak time position along propagation distance, (b) mean soliton peak time 
position as a function of launched power. Note that Airy peak time position at launch is at $t = 
-1$. (c) soliton peak phase oscillations along propagation distance for select launched powers.

2.2. The accelerating wavefront

As seen in Fig. 2, the Airy wavefront continues to exhibit the parabolic acceleration in time,
even under the influence of Kerr effect and after shedding energy to the soliton. To study 
whether this acceleration continues with the properties of the linear propagation we compared 
the nonlinear propagations to linear, as the intensity is scaled with the $R$ parameter. Note that 
the linear Airy pulse evolution is identical for every intensity value.

These linear propagation results are compared to the nonlinear ones by tracking the main 
lobe acceleration trajectory for each case and extracting information about its peak power and 
position. Furthermore, we calculate the accelerating energy distribution along propagation 
distance.

Figure 5 shows the Airy main lobe parabolic trajectory and peak power as a function of 
propagation distance, under linear and nonlinear propagation, for three select launched power 
cases. We see that the wavefront continues to exhibit the parabolic trajectory in time (blue 
curves), which is almost identical in the linear and the nonlinear propagation cases, although 
the nonlinear peak slightly trails the linear peak, on account of a delay associated with the 
energy shedding to the soliton. The intensity evolution of the accelerating wavefront is shown 
in green. We can see that in the nonlinear propagation its peak power performs decaying 
oscillations, as opposed to the monotonic decay in the linear case. The oscillations of the peak 
power in the nonlinear case are known to be a result of the interplay between the SPM and the 
dispersion. Similar influence of SPM on the Airy accelerating main lobe was already observed 
in [10]. However, the peak power oscillations there exhibit faster decay due to a relatively 
large truncation coefficient, 0.1-0.3 vs. 0.0335 in the current simulations.

Fig. 5. – Airy accelerating tail trajectories in time-distance space(blue) and in intensity-distance 
space (green) for (a) $R = 1$, (b) $R = 1.3$ and (c) $R = 2$.

Next, we investigate the energy distribution of the accelerating wavefront. It is important 
to note that the simulations preserve the launched pulse energy along the propagation 
distance, as well as preservation of ‘center of gravity’ (first order moment) position according
to the finite pulse energy and the uniformity of the media [2]. The power spectrum of the Airy pulse is symmetric about the central frequency, and upon propagation in anomalous dispersive media the high frequencies components are delayed (low frequency components are advanced) with respect to central frequency group delay (in anomalous media), such that the pulse total energy is eventually divided to two equal fractions about $T = 0$- half of the energy at each direction. In the presence of Kerr nonlinearity, considerable part of the pulse energy is shed to the soliton that propagates at the group velocity, and the remaining energy disperses in opposite directions with less than a half of the launched energy dispersing to each side (due to soliton shedding).

The energy that is carried in the accelerating wavefront (delayed components) was found by summing the energy over positive time at every distance sample. These calculations were performed with both the linear and nonlinear propagations.

Figure 6(a) shows the delayed energy evolution of the accelerated Airy wavefront along the propagation distance for various Airy launched powers. The energy is normalized by the launched pulse energy, such that we can see the relative energy portion of the accelerating wavefront for linear and nonlinear cases. For all $R$ values, the energy evolution of the linear propagations coincides to one curve that asymptotically approaches the value of half launched pulse energy, according to its linear nature. For the nonlinear propagations we clearly see that as $R$ grows the fractional energy amount that is delayed is decreasing, where the oscillatory behavior is due to the soliton oscillations which take place in the boundary of the right half propagation plane. Those curves and those of Fig. 6(b), which chart the energy evolution of the formed soliton for different $R$ values, show the fact that the formed soliton not only has more intensity when $R$ is growing, but also carries a larger energy fraction from the whole pulse. This can also be seen in Fig. 6(c), where the mean soliton relative energy was calculated for every $R$ value. From Figs. 6(b-c) we also see the energy preservation—the normalized delayed energy is missing energy that is about half of the shed soliton energy, where the other half originates from the faster propagating energy components. When $R = 2$, for example, the soliton energy fraction is about 0.39 and the missing fractional energy amount from the delayed energy is about 0.19, half of 0.39.

3. Truncation coefficient effect

The ability of Airy pulses to exhibit their unique features is strongly related to the degree of truncation in the apodization function. As the truncation is stronger, the Airy pulse quickly loses the unique features of the Airy pulse and disperses. Here we wish to examine how the truncation degree influences the soliton shedding and pulse propagation under the Kerr effect.

We employ the same pulse profile defined in Eq. (4), fixing the intensity scaling parameter $R$ to 1.5 while varying the truncation coefficient in the range 0.01-0.1, as shown in Fig. 7(a), and propagate the apodized Airy for every truncation value. Figures 7(b-c) show two examples of the Airy pulse evolution in time-distance space. We see that when the truncation is small the Airy original features as self-similarity and acceleration in time are more noticeable. The influence of the truncation degree on emergent soliton properties and on the accelerating wavefront was examined in the same manner as in the previous section.
3.1. The emergent soliton

Larger truncation coefficient values make the exponential apodization of the Airy function stronger and the Airy tail is shortened; there is a negligible effect on the main Airy lobe, as shown in Fig. 7(a). Hence the emergent soliton, which forms from the main lobe, achieves stability faster (after a shorter propagation distance) in cases of larger truncation coefficients, as the newly formed soliton experiences less collisions with the accelerating Airy tail, as shown in the propagation images in Fig. 7. Therefore, the Sech(\cdot) fit process was started from a different propagation distance for every truncation value.

From the soliton fit data we see that the emergent soliton parameters do not experience significant variations for different truncation values, as shown in the soliton parameters evolution curves in Figs. 8(a-b). However, the soliton mean peak time position does shift considerably from the launched Airy peak position, and this shift increases for smaller truncation values (see Fig. 8(c)). This behavior is explained by the interaction between the formed soliton from the main lobe and the accelerating lobes of the Airy tail, which constitute collision perturbations to the soliton and cause temporal shift of the soliton in the direction opposed to the accelerating lobes [15]. This temporal shift to earlier times depends on the perturbation energy, which increases for small truncation coefficient values. It is important to note that even without perturbing lobes (i.e. while propagating Airy with strong truncation), the soliton is not necessarily formed at the launched Airy peak position because of the acceleration that the original pulse undergoes before the soliton is shed. Also, the launched Airy peak time position is not constant with different truncation coefficients (dashed red line in Fig. 8(c)), as a result of a shift from the multiplication by the exponential apodization function.

3.2. The accelerating wavefront

The extent to which the truncated Airy maintains its form and continues to accelerate before dispersing strongly depends on the truncation coefficient. As in the previous section, we compared the linear and the nonlinear propagations in order to investigate the Airy’s accelerating wavefront behavior for different truncation values. In the linear propagation
regime, the truncation coefficient determines both the distance at which the accelerating wavefront is still distinguishable, and the total Airy energy according to \( E_{\text{Airy}} = (8\pi a)^{3/2} \) [2].

In our investigation range for truncation coefficient, the linear Airy varies widely.

After tracking the accelerating wavefront trajectory for every truncation value, we compare the main lobe trajectory and peak power under the linear and the nonlinear propagation regimes (Fig. 9). The main finding here is that the intensity of the accelerating main lobe in the nonlinear regime (green curves) first experiences SPM and focuses to the same peak power (with no dependence on truncation value). This peak is then shed to the soliton and the remaining accelerating wavefront immediately after the soliton shedding is at lower power compared to the linear propagation case. However, as a consequence of chromatic dispersion, the high frequency components travel slower and eventually the leading wavefront main lobe re-emerges and matches the main-lobe power of the linear propagation case (the Airy self-healing property). In spite of this wavefront matching between the linear and nonlinear propagations we see that in the nonlinear propagation the accelerating main lobe remains distinguishable for longer distances than in linear propagation for a given truncation value. This finding is related to the differences between the radiation energy distribution in the nonlinear and in the linear propagations. In the linear propagation (see example in Fig. 1(a)) the dispersed Airy intensity roughly converges to a Gaussian distribution in time with propagation distance that eventually (after a certain distance) engulfs the accelerating main lobe. In the nonlinear propagation the dispersive radiation intensity is no longer Gaussian distributed due to the soliton formation and the energy centering about it, making the accelerating peak visible for longer propagation distance.

As the emergent soliton has roughly the same energy for all truncation values, its relative energy fraction in the launched pulse energy is larger for increasing truncation values (Fig. 10(a)), therefore the relative energy fraction in the accelerating Airy wavefront decreases (Fig. 10(b)) In the linear propagation regime the accelerating Airy energy always asymptotically approaches one half of the whole pulse energy, although its energy growth rate is truncation factor dependent. In the nonlinear case the delayed Airy energy fraction decreases from this value as the truncation is growing, as the nearly constant soliton energy is missing.

![Fig. 9. Airy accelerating wavefront trajectories in time-distance space (blue) and in intensity-distance space (green) for (a) \( a = 0.01 \), (b) \( a = 0.04 \) and (c) \( a = 0.08 \).](image)

![Fig. 10. Examples of energy evolution along propagation distance of (a) the relative energy of the emergent soliton (the soliton energy itself is hardly dependent on truncation coefficient) and (b) accelerating wavefront.](image)
4. Soliton time position for power and truncation

In the previous sections we showed that: (1) the emergent soliton time position is at earlier times when the launched power increases (at fixed truncation) due to quick build-up of a soliton. At lower powers, self-focusing results in the eventual build-up of the soliton, but as the conditions materialize the main lobe is undergoing the ballistic trajectory leading to soliton shedding at a later time position. And (2) the emergent soliton time position is at earlier times when the truncation coefficient decreases (at fixed launched power) due to collision perturbations with the accelerating tail lobes. The time shift associated with collision perturbations depends on the energy; hence higher truncation coefficients result in lower Airy tail energies and reduced soliton time shifts. These two effects are graphically depicted in Fig. 11(a).

To verify that these two effects independently and consistently occur, we varied both the Airy launched power and the truncation coefficient over our investigation range (Fig. 11(b)). Indeed we see this trend continuing; the emergent soliton mean time position shifts to earlier (later) times for smaller (larger) truncation coefficients and for higher (lower) launched power levels. These results reinforce our finding that soliton is shed at an earlier time when the launched power is higher, and that collisions with the accelerating Airy tail lobes shift the position in the direction counter to the acceleration, i.e. towards earlier times.

Fig. 11. (a) Schematic illustration of the sources of temporal shift of the emergent Soliton. (b) Distribution of Soliton mean time position as a function of truncation coefficient and launched power in our investigation range.

5. Summary

In this paper we investigated the propagation of a truncated temporal Airy pulse in nonlinear Kerr media. The phenomena of soliton shedding from the original Airy pulse under sufficiently strong excitation was already identified [8,9], but in this work we investigated in detail the properties of the soliton and the remaining Airy radiation. We characterized the emergent soliton parameters under different truncation and power conditions and identified the mechanisms at play, in accordance to processes known from literature. The soliton parameters perform oscillations due to the presence of background radiation from the dispersed Airy pulse. The temporal position of the emergent soliton depends both on the Airy launched power and truncation coefficient, due to the location of the shedding event and the interaction with the accelerating Airy tail. We also observed the SPM influence on the accelerating Airy main lobe, and we found that the SPM has large effect on the accelerating main lobe visibility in comparison the linear truncated Airy propagation. Finally, we found that the energy distribution of the Airy pulse along the propagation depends on the launched power and the truncation degree.

In this work we studied the soliton shedding phenomena for relatively intense launched Airy pulses. This research avenue can continue to even higher launched pulse powers, however eventually the well-understood phenomena explored here starts to break down. Figure 12 shows the time-space evolution when launching the Airy pulse with a power factor of four ($R = 4$). We see that for such intense excitation three solitons are shed, the main soliton in a consistent manner to that described here, and two additional weaker solitons at...
both higher and lower center frequencies. This result was still obtained with the standard nonlinear Schrödinger equation (Eq. (4)). However, for proper simulation of intense Airy pulse excitation, one should also add additional terms to account for higher-order nonlinear effects such as Raman scattering and self-steepening.

![Graph](image)

**Fig. 12.** Intensity distributions as a function of time and propagation distance for $R = 4$, showing multiple soliton shedding at high launched peak powers.