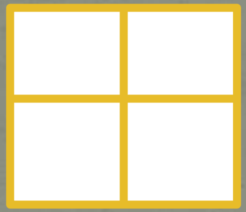
1+1 = 
On the Polysemy of Symbols

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Polysemy

- Tie a tie
- Milk for milk

- In mathematics, a word may be polysemous if its mathematical meaning is different from its everyday, familiar meaning (Durkin and Shire, 1991), or if it has two related, but different, mathematical meanings (Zazkis, 1998).

- E.g., Continuity, function
- E.g, Quotient, divisor

Polysemy

- Tie a tie
- Milk for milk

- In a mathematical discourse, symbols such as +, =, and 1, may also be considered ‘words’

Ambiguity in Mathematics Education

- Gray and Tall (1994) advocated for flexible interpretation of symbols such as 5+4 as processes or concepts, i.e. *procepts*.

"This ambiguous use of symbolism is at the root of powerful mathematical thinking" (p.125).

Ambiguity in Mathematics Education

- Byers (2007) suggested ambiguity in mathematics is “an essential characteristic of the conceptual development of the subject” and “opens the door to new ideas, new insights, deeper understanding” (p.78).

- Durkin and Shire (1991) suggested that enriched learning may ensue from monitoring, confronting and ‘exploiting to advantage’ ambiguity

Ambiguity in Mathematics Education

- A flexible interpretation of a symbol can go beyond process-concept duality to include other ambiguities relating to the diverse meanings of that symbol, which in turn may also be the source of powerful mathematical thinking and learning.

- This presentation will explore cases of ambiguity connected to the context-dependent definitions of symbols, that is, the *polysemy of symbols*. Specifically, polysemy of '+'

A familiar meaning

Symbol	Meaning in context of natural numbers
1	Cardinality of a set containing a single element
2	Cardinality of a set containing exactly two elements
1+2	Cardinality of the union set
+	Binary operation over the set of natural numbers

Polysemy of '+'

- Building on the idea of addition as a domain-dependent binary operation, there are cases where extended meanings of '$a + b$' contribute to results that are inconsistent with the 'familiar'.

- Modular arithmetic with base 3
  - $+_3$ as addition over the set {0, 1, 2}
- Transfinite arithmetic
  - $+_\infty$ as addition over the class of cardinal numbers

  - The symbol $+_N$ will be used to represent addition over the set of natural numbers, $+_Z$ as addition over the set of integers

An extended meaning $+_3$

Symbol	Meaning in context of $\mathbf{Z}_3$
1	Congruence class of 1 modulo 3: {… -5, -2, 1, 4, 7, …}
2	Congruence class of 2 modulo 3: {… -4, -1, 2, 5, 8,…}
1+2	Congruence class of (1+2) modulo 3: {…, -3, 0, 3, …}
+	Binary operation over set {0, 1, 2}; addition modulo 3

An extended meaning $+_3$

- Dummit and Foote (1999) define the sum of congruence classes by outlining its computation:
  - 1+2 (modulo 3), is computed by taking *any representative integer* in the set {… -5, -2, 1, 4, 7, …} and *any representative integer* in the set {… -4, -1, 2, 5, 8,…}, and summing them in the 'usual integer way'.

E.g. $1 +_3 2 = (1 +_Z 2)$ modulo 3

$= (1 +_Z 5)$ modulo 3

$= (-2 +_Z -1)$ modulo 3

Extending meaning

- As with words, the extended meaning of a symbol can be interpreted as a metaphoric use of the symbol, and thus may evoke prior knowledge or experience that is incompatible with the broadened use.

Extending meaning

- Pimm (1987) notes that “the required mental shifts involved [in extending meaning] can be extreme, and are often accompanied by great distress, particularly if pupils are unaware that the difficulties they are experiencing are not an inherent problem with the idea itself” (p.107) but instead are a consequence of inappropriately carrying over meaning.

An extended meaning $+_{\infty}$

- Transfinite arithmetic may be thought of as an extension of natural number arithmetic
  - its addends represent cardinalities of finite or infinite sets
  - a sum is defined as the cardinality of the union of two disjoint sets

- Transfinite arithmetic poses many challenges for learners, not the least of which involves appreciating the idea of 'infinity' in terms of cardinalities of sets (i.e. the transfinite numbers $\aleph_0$, $\aleph_1$, $\aleph_2$, …).

An extended meaning $+_\infty$

- In resonance with Pimm's (1987) observation regarding negative and complex numbers, the concept of a transfinite number "involves a metaphoric broadening of the notion of number itself" (p.107).

- A generic example: the sum $\aleph_0 + 1$
  - It's the cardinality associated with the union set $N \cup \{\beta\}$, where β is not in **N**.
  - The addends are elements of the (generalised) class of cardinals, which includes transfinite cardinals.
  - Between the sets $N \cup \{\beta\}$ and **N** there exists a bijection, which, in line with the definition (Cantor, 1915), guarantees that the two sets have the same cardinality – that is, $\aleph_0 + 1 = \aleph_0$.

An extended meaning $+_\infty$

Symbol	Meaning in context of transfinite arithmetic
1	Cardinality of the set with a single element; class element
$\aleph_0$	Cardinality of **N**; transfinite number; 'infinity'
1 + $\aleph_0$	Cardinality of the set **N**$\cup$ β; equal to $\aleph_0$
+	Binary operation over the class of transfinite numbers

Broadening notions

- $\aleph_0 = \aleph_0 + \upsilon$, for any $\upsilon \in \mathbf{N}$, and $\aleph_0 + \aleph_0 = \aleph_0$.
- Whereas with '$+_N$' adding two numbers always results in a new (distinct) number, with '$+_\infty$' there exist non-unique sums.
  - A consequence: indeterminate differences.
  - Since $\aleph_0 = \aleph_0 + \upsilon$, for any $\upsilon \in$, then $\aleph_0 - \aleph_0$ has no unique resolution.

- The familiar notion that 'anything minus itself is zero' does not extend to transfinite subtraction.

MeaningS of symbols

- Mason, Kniseley, and Kendall in research on literacy suggest that knowledge of language includes “learning a meaning of a word, learning more than one meaning, and learning how to choose the contextually supported meaning” (1979, p.64).

- Similarly, knowledge of mathematics includes
  - learning a meaning of a symbol,
  - learning more than one meaning, and
  - learning how to choose the contextually supported meaning of that symbol.

Further research needed!

- Mason et al. (1979) suggest students "will choose a common meaning [of a word], violating the context, when they know one meaning very well" (p.63).

    To what degree do analogous observations apply as students begin to learn '+' in new contexts?

- Attending to the polysemy of symbols, as a learner, for a learner, or as a researcher, may expose confusion or inappropriate associations that could otherwise go unresolved.

Looking ahead…

- Echoing Pimm's (1987) advice :

  "If … certain conceptual extensions in mathematics [are] not made abundantly clear to pupils, then specific meanings and observations about the original setting, whether intuitive or consciously formulated, will be carried over to the new setting where they are often inappropriate or incorrect" (p.107).

- How to tease out influence of polysemy on student (mis)understanding?
  - Could explicit attention help avoid difficulties?

Thank you!