FUZZY COMPROMISE PROGRAMMING FOR PORTFOLIO SELECTION

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Summary: This paper presents an application of Fuzzy Compromise Programming to Portfolio Selection using Sharpe’s single index model. Estimations of subjective or imprecise future beta for every asset can be represented through fuzzy numbers constructed on the basis of statistical data and the relevant knowledge of financial analyst; the model, therefore, works with data that contain more information than any classical model and dealing with it does not involve a great extra computation effort.

In order to solve the Portfolio Selection problem we have formulated a Fuzzy Compromise Programming problem. For this task we have introduced the fuzzy ideal solution concept based on soft preference and indifference relationships and on canonical representation of fuzzy numbers by means of their $\alpha$-cuts. The accuracy between the ideal solution and the objective values is evaluated handling the fuzzy parameters through their expected intervals and by using discrepancy between fuzzy numbers in our analysis.

A major feature of this model is its sensitivity to the analyst’s opinion as well as to the decision-maker’s preferences. This allows interaction with both when it comes to design the best portfolio.

1. Introduction

Portfolio selection has been one of the most important research fields in modern finance. In this context Markowitz’s mean-variance model (Markowitz (1952)) has been unanimously recognized as a pioneering work (see e. g. Constantinides et al. (1995) for a brief historical account). A lot of models and extensions have been proposed to improve the performance of portfolio investment which have led to a great number of papers, monographs, and textbooks with the aim of formulating risk and returns of economic agents and understanding diversification in investment strategies (Yusen, X. et al.).

Markowitz’s portfolio optimization model presents two main difficulties for being applied. First one data required: if we could accurate expectations about future mean returns for each asset and the correlation of returns between each pair of assets then the Markowitz’s model under certain conditions and supposed known the investors’ utility function, would produce optimum portfolios. The obtaining of accurate forecast of input data needed for this model is a difficult task, particularly in the case of the variance-covariance matrix between securities (Elton and Gruber, (1995)). Second one, there is a computational difficulty associated to the resolution of large-scale quadratic programming problems with a dense covariance matrix. Several efforts to transform the quadratic problem into a linear one have been outlined in Konno et al. (1991). These authors have proposed a measure $L_1$ of risk instead of a $L_2$ measure of the Markowitz’s model and then they have formulated a linear problem. If it seems to be not convenient or not possible to reduce the quadratic initial problem into a linear one, other solving methods as simulate annealing, tabu search and genetic algorithm, have been developed in order to solve the problem (Ehrgott et al. (2003), Chang et al. (2000)).

In this paper the portfolio selection problem has been handled in a soft framework. In this sense, this work can be regarded as a contribution to the determination of risk handling imprecise information in Beta estimation. Imprecision will be quantified by means of fuzzy numbers that represent the continuous possibility distributions for fuzzy
parameters and hence place a constraint on the possible values the parameter may assume. On the basis of fuzzy sets theory (Zadeh (1962)) and Compromise Programming (Yu (1972) and Zeleny (1973)), we have proposed a new model for portfolio selection which implies solving linear programming problems and that includes the imprecision and subjectivity inherent to some data in the model.

Compromise Programming (CP) is a mathematical programming technique with the capability of handling multiple objectives in those situations where the existence of a high level of conflict between criteria does not allow the simultaneous optimization of all the considered objectives. In those situations it seems rational to find compromise solutions between objectives. The pioneering applications of CP for Portfolio selection are due to Ballestero and Romero (1996). When attributes and constraints are in an imprecise environment and they cannot be stated with precision, we work with Fuzzy CP. We will apply the Fuzzy CP approach developed by Arenas et al. (2004) to the resolution of a real portfolio selection problem.

In next section we shall present the portfolio selection model proposed in this paper, introducing a discussion about data selection, objectives, constraints and also about the fuzzy representation of the parameters of the problem. In section 3 we describe the Fuzzy CP optimization technique to solve the proposed portfolio selection problem. In section 4, in order to illustrate the above mentioned ideas, we present a real portfolio selection problem. We have considered 26 Spanish mutual funds of the type called Fondos de Inversión Mobiliaria (FIM) whose quarterly returns data are referred to the period 1996-2000 and we have obtained two fuzzy compromise portfolios, one of them of maximum efficiency and the other of maximum equilibrium between the level of achievement of the considered objectives, risk and returns. In the last section we present the conclusions of the paper.

2. Portfolio selection model

Markowitz’s mean-variance model of portfolio selection is a biobjective problem which the author solves using the Constraint Method (Margling, (1967)). The basic model determines the composition of a portfolio of $n$ assets which minimizes risk while achieving a predetermined level of expected return, $E_0$. The risk of each asset is measured by the variance of its return. If each component $x_i$ of the $n$-vector $x$ represents the proportion of an investors’ wealth allocated to asset $i$, then the total return of the portfolio is given by the scalar product of $x$ by the vector of individual asset returns. Therefore, if $R = (R_1, R_2, \ldots, R_n)$ denotes the $n$-vector of returns of the assets and $C$ the $n \times n$ covariance matrix of the returns, we obtain the expected portfolio return by the expression $\sum_{i=1}^{n} E(R_i)x_i$ and its level of risk by $\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}x_i x_j$. Mathematically, the problem can be formulated as follows:

$$\begin{align*}
\text{min} & \quad \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}x_i x_j \\
\text{s.t.} & \quad \sum_{i=1}^{n} E(R_i)x_i \geq E_0 \\
& \quad \sum_{i=1}^{n} x_i = 1 \\
& \quad x_i \geq 0 \quad i = 1, \ldots, n
\end{align*}$$

(1)

The first constraint expresses the requirement placed on expected return. The second constraint, budget constraint, requires that 100% of the budget be invested in the portfolio. The nonnegativity condition expresses that no short sales are allowed.

The portfolio selection model to be solved in this paper is an extension of Markowitz’s basic mean-variance model. We are considering, as well as the constraints included in the basic Markowitz’s model, other type of restrictions limiting the number of assets included in the portfolio and establishing the upper fuzzy bounds concerning the amount invested in each security which ensure diversification of portfolio.

In order to solve the multiobjective problem we have applied Compromise Programming which provides the Decision Maker with a subset of efficient portfolios obtained following Zeleny’s rational axiom which establishes that preferred solutions are those nearest to the ideal point.
2.1 Data

Input data of model (1) are the expected returns and variance-covariance matrix of expected returns of the portfolio assets. Xia et al. (2000) pointed out three main methods to determine these data:

Technique 1: According to the historical data, considering historical mean as the forecast of the expected return of the asset and the variance-covariance historical matrix itself (Markowitz (1959)).

Technique 2: Assuming that the only reason of the assets’ correlation is the common response to market changes. The measure of their correlation can be obtained by relating the returns of a general market index (Elton and Gruber (1995)). The returns of a stock can be separate into two components, one part is due to the market and the other is independent of the market:

\[ R_i = a_i + \beta_i I_m + \epsilon_i \]  

where \( I_m \) is a general market index, \( a_i, \beta_i \) are obtained through time series regression, thus, \( a_i \) represents the component of \( R_i \) independent from the index variation \( I_m \) and \( \beta_i \) is the slope on the straight line of \( R_i \) return with regard to the changes in the \( I_m \) general market index and \( \epsilon_i \) is the random error component. Thus, the expected return and variance of the return of a portfolio \( P \) can be, respectively, simplified as \( E(R_P) = a_P + \beta_P E(I_m) \) and \( \sigma_P^2 = \beta_P^2 \sigma_m^2 + \sum_{i=1}^{n} \sigma_{e_i}^2 x_i^2 \), where \( \sigma_{e_i}^2 \) is the variance of the random error component \( \epsilon_i \), \( \sigma_m^2 \) is the variance of \( I_m \), \( a_P = \sum_{i=1}^{n} a_i x_i \) is the independent term of portfolio \( P \) and \( \beta_P = \sum_{i=1}^{n} \beta_i x_i \) is Beta of portfolio \( P \).

Technique 3: Using a multi-index model (Elton and Gruber (1995)) to determine some of the non-market influences. The standard formulation of a multi-index model is:

\[ R_i = a_i + \sum_{k=1}^{m} b_{ik} I_k + c_i \quad i = 1, \ldots, n \]  

where \( a_i \) is the expected value of the returns not related to any index, \( c_i \) is the random component satisfying \( E(c_i) = 0 \), \( I_k \) is index \( k \) assumed to be pairwise independent and \( b_{ik} \) is the measure of sensitivity of \( R_i \) to index \( k \).

In this work we have applied a modified version of technique 2. Elton et al. (1978) have compared the ability of the following models to forecast the correlation structure between securities: historical correlation matrix, forecast of the correlation matrix prepared by estimating Betas from the prior historical period or forecast of the correlation matrix estimating Betas from the prior historical period and updating them via some techniques such as Blume technique, (Blume (1975)) or Vasicek Bayesian technique (Vasicek (1973)). The authors found that the historical correlation matrix itself was the poorest of all techniques. In most cases it was outperformed by all of the Beta forecasting techniques at a statistically significant level. This indicates that a large part of the observed correlation structure between securities, not captured by the single-index model, represents random noise with respect to forecasting. The point to note, following Elton et al., is that the single-index model, developed to simplify the inputs to portfolio analysis and thought to lose information due to the simplification involved, actually does a better job of forecasting than the full set of historical data. Although the multi-index model can better describe historical data than the single index model, it often contains more noise than the real information when forecasting. Hence, it is not surprising that the single index model usually outperforms the more complicated multi-index model (Elton and Gruber (1995)).

2.2 Objectives and constraints.

Markowitz’s basic model is based on crisp objectives and constraints. Considering that the parameters of the model arise from financial market and are usually established by the Decision Maker, some degree of subjective imprecision is included. To study this imprecise nature possibility theory might be helpful.

Possibility theory has been proposed by Zadeh (1978) and advanced by Dubois and Prade (1985); in this theory fuzzy variables are associated with possibility distributions. Possibility distributions are represented as normal convex fuzzy...
sets, such as L-R fuzzy numbers, quadratic or exponential functions. As an application of possibility theory to portfolio analysis, possibility portfolio selection models were initially proposed in Tanaka et al. (1999).

Objectives

In this work we have considered two objectives, which should be kept in mind when choosing the best portfolio:

a) Maximization of the expected return of the portfolio $P$, $E_P = \sum_{i=1}^{n} E(R_i) x_i$.

b) Minimization of the fuzzy portfolios’ risk measured by portfolio fuzzy Beta, $\tilde{\beta}_P$ (see ref. [11]), $\tilde{\beta}_P = \sum_{i=1}^{n} \tilde{\beta}_i x_i$,

where $\tilde{\beta}_i = (h_1, b_2, b_3, b_4)$ is a subjective estimation of future Beta of each asset given by the analyst and described in fuzzy terms represented by a trapezoidal fuzzy number (Dubois and Prade (1985)). Both the statistical and the imprecise estimation of the Beta coefficients of each asset can be simply and clearly represented in the trapezoidal number.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Possibility distribution of fuzzy Beta $\tilde{\beta}_i$.}
\end{figure}

In the figure above, $\mu_{\tilde{\beta}}$, with $0 \leq \mu_{\tilde{\beta}} \leq 1$, represents the possibility degree of occurrence of $\tilde{\beta}_i$. Thus, the values with a higher possibility degree of occurrence, i.e. the central values of the fuzzy trapezoidal number, are contained in the interval $[b_2, b_3]$, which has $\mu_{\tilde{\beta}} = 1$. As well, we can observe that there are two possible values of Beta $h_1$ and $b_4$, lateral data in the fuzzy trapezoidal number which have possibility degree equal to zero.

In this paper, the design of $\tilde{\beta}_i$ has been carried out by bearing in mind the fact that its nucleus, $[b_2, b_3]$, should contain the estimated Beta by means of regression techniques, starting from the most recent historical series in the forecasted period, Beta obtained through adjusting previous ones and taking into account the information provided by the analyst based on his/her most reliance estimations (see techniques of Blume (1975), Vasicek (1973), Beaver, et al. (1970) and Rosenberg et al. (1976)).

Constraints

Short sales are not allowed, therefore a non-negative condition is verified for $x_i$.

a) Budget constraint: $\sum_{i=1}^{n} x_i = 1$.

b) Maximum investment in every asset (naïve diversification): Upper bounds for the proportions invested in every asset are established. The so-called naïve diversification may be wrong from the portfolio theory point of view, but it may reflect decision maker’s preferences and, on the other hand, should be borne in mind by the mutual fund manager due to legal restrictions on maximum asset or asset group proportions.
b) Diversification is attained establishing a maximum fuzzy amount \( \tilde{k}_i \% \) of the portfolio which can be invested in asset \( i \): \( x_i \leq \frac{\tilde{k}_i}{100} \).

c) Upper bounds in high-risk assets: The investor can classify assets according to their Beta and decide that an acceptable portfolio should, if possible, contain a maximum around \( \tilde{j} \% \) of assets whose expected value of Beta is greater or equal to a pre-set value higher than 1. We are going to denote these assets as high-risk assets \( (i \in hr) \):
\[
\sum_{i \in hr} x_i \leq \frac{\tilde{j}}{100}.
\]

d) Lower bounds in low-risk assets: Investor establishes a minimum around \( \tilde{l} \% \) of assets whose expected value of Beta is smaller or equal to a pre-set value lower than 1. We are going to denote these assets as low-risk assets \( (i \in lr) \):
\[
\sum_{i \in lr} x_i \geq \frac{\tilde{l}}{100}.
\]

3. Optimization technique: Fuzzy Compromise Programming

We shall consider the following multiobjective possibilistic programming problem:
\[
\begin{align*}
\min & \quad \tilde{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) = (\tilde{c}_1 x_1, \tilde{c}_2 x_2, \ldots, \tilde{c}_n x_n) \\
\text{s. t.} & \quad x \in \mathcal{X}(\tilde{A}, \tilde{b}) = \left\{ \tilde{a} x \leq \tilde{b}, \quad i = 1, \ldots, m \right\} \\
& \quad x \geq 0
\end{align*}
\]

where \( x' = (x_1, x_2, \ldots, x_n) \) is the crisp decision vector, \( \tilde{c}' = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n) \) are the fuzzy parameters of the \( k \) considered objectives, \( \tilde{A} = \left[ \tilde{a}_{ik} \right]_{m \times n} \) is the fuzzy technological matrix and \( \tilde{b}' = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m) \) are also fuzzy parameters.

The uncertain and/or imprecise nature of the technological matrix and of the resource vector which define the set of constraints of the model leads us to compare fuzzy numbers. In this work we have handled fuzzy numbers through their expected intervals. Heilpern (1992) defined the expected interval and expected value of a fuzzy number \( \tilde{a} = (a^\ell, a^c, a^\sigma) \) as \( EI(\tilde{a}) = \left[ EI(\tilde{a})^\ell, EI(\tilde{a})^c \right] = \left[ \frac{a^\ell + a^c}{2}, \frac{a^c + a^\sigma}{2} \right] \) and \( EV(\tilde{a}) = \frac{EI(\tilde{a})^\ell + EI(\tilde{a})^c}{2} \) respectively.

In order to compare fuzzy number we have used the fuzzy relationship defined by Jiménez (1996) that leads us to the concept of \( \lambda \)-feasibility of a decision vector. A decision vector \( x \in IR^n \), is said to be \( \lambda \)-feasible for the problem FP-MOLP if \( x \) verifies the constraints at least in a degree \( \lambda \). That is, \( \tilde{a}_i x \leq \tilde{b}_i, \quad i = 1, \ldots, m \).

In accordance with the above considerations we shall solve the FP-MOLP through a family of \( \lambda \)-FP-MOLP problems, where \( 0 \leq \lambda \leq 1 \):
\[
\begin{align*}
\min & \quad \tilde{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) = (\tilde{c}_1 x_1, \tilde{c}_2 x_2, \ldots, \tilde{c}_n x_n) \\
\text{s. t.} & \quad \left[ (1-\lambda)E_i^\delta + \lambda E_i^\lambda \right] x \leq \lambda E_i^\delta + (1-\lambda)E_i^\lambda, \quad i = 1, \ldots, m \\
& \quad x \geq 0
\end{align*}
\]

The expected interval of the fuzzy vector \( \tilde{a}_i = (\tilde{a}_{i1}, \tilde{a}_{i2}, \ldots, \tilde{a}_{in}) \) is a vector whose components are the expected intervals of each fuzzy number of vector \( \tilde{a}_i \), that is: \( EI(\tilde{a}_i) = (EI(\tilde{a}_{i1}), EI(\tilde{a}_{i2}), \ldots, EI(\tilde{a}_{in})) \).

In order to apply the CP approach to solve the problem, we need to obtain the fuzzy ideal solution of the \( \lambda \)-FP-MOLP problem. For this we use the solving method proposed by Arenas et al. (1998). This method gives fuzzy solutions in the objectives space defined by their possibility distribution. The method is based on the extension principle and the
joint possibility distribution of the fuzzy parameters. It relies on the $\alpha$-cuts of the solution to generate its possibility distribution (for more details about the construction of this solution see Arenas et al. 1998, 1999). Arenas et al. (1998, theorem 5) have proved that solution $\hat{z}^*_r(\lambda)$ to problem ($\lambda$-FLP) is a fuzzy number.

In the CP framework, once the fuzzy ideal solution is found, we have to find the values of the decision variables, which determine a fuzzy solution, as accurately as possible to the fuzzy ideal solution.

We will solve the problem FP-MOLP by handling the fuzzy objectives, $\tilde{z} = \tilde{c}x$, and the $\lambda$-fuzzy ideal solution, $\hat{z}^*(\lambda)$, through their expected intervals. Therefore, the problem now is:

Find a $x \in X(\lambda)$ such that: $EI(\tilde{c}, x) \rightarrow EI(\hat{z}^*(\lambda))$, $r = 1, \ldots, k$ (4)

where $EI(\tilde{c}, x) = \left[ E^{\tilde{c}, x}_1, E^{\tilde{c}, x}_2 \right]$ and $EI(\hat{z}^*(\lambda)) = \left[ E^{\hat{z}^* (\lambda)}_1, E^{\hat{z}^* (\lambda)}_2 \right]$.

It should be considered desirable to obtain a fuzzy objective vector with less amplitude than the $\lambda$-fuzzy ideal solution, i.e., such that it should verify the following condition:

$$E^{\hat{z}^* (\lambda)}_2 - E^{\hat{z}^* (\lambda)}_1 \leq E^{\hat{z}^* (\lambda)}_2 - E^{\hat{z}^* (\lambda)}_1$$

It should be considered desirable to obtain a fuzzy objective vector with less amplitude than the $\lambda$-fuzzy ideal solution, i.e., such that it should verify the following condition:

$$E^{\hat{z}^* (\lambda)}_2 - E^{\hat{z}^* (\lambda)}_1 \leq E^{\hat{z}^* (\lambda)}_2 - E^{\hat{z}^* (\lambda)}_1$$

From here, we assert that:

$$EI(\tilde{c}, x) \rightarrow EI(\hat{z}^*(\lambda)), \ r = 1, \ldots, k \quad \text{if and only if} \quad D_r \rightarrow 0$$

where: $D_r = \max \left\{ E^{\tilde{c}, x}_1 - E^{\hat{z}^* (\lambda)}_1, E^{\tilde{c}, x}_2 - E^{\hat{z}^* (\lambda)}_2 \right\}$, $r = 1, \ldots, k$ is the discrepancy between the $r$-th fuzzy objective $\tilde{c}, x$ and the $r$-th component of the $\lambda$-fuzzy ideal solution $\hat{z}^*(\lambda)$ (Arenas et al. 2004).

We shall solve now a new crisp CP problem where the objective is to minimize the discrepancy between the $\lambda$-fuzzy ideal solution and fuzzy objectives. Therefore, the ideal solution is the null vector and we define a $\lambda$-compromise solution of the FP-MOLP as a decision vector $x^*$ that is a compromise solution to the problem:

$$\min \left( D_1, D_2, \ldots, D_k \right)$$

s.t. $x \in X(\lambda)$

(7)

As problem (7) is crisp, the compromise programming approach to solve it is based on the $L_p$ family of distances:

$$\min \left( D_1, D_2, \ldots, D_k \right)$$

s.t. $x \in X(\lambda)$

(8)

where $w_j \geq 0$ can be regarded as a normalizing coefficient and also as a weighting one that measures the relative importance of the discrepancy between the $r$-th fuzzy objective and its fuzzy ideal value and may be established by the Decision Maker in an interactive process with the analyst.

The most commonly obtained compromise solutions are for metrics $p = 1$ and $p = \infty$ because for other metrics nonlinear mathematical programming algorithms are needed. Also, in the biobjective case (see Romero et al., 1999) they are the bounds of the whole compromise set.

In the framework above described, we shall solve the following portfolio selection problem:

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1 We shall denote the relation of “accurate” as ($\rightarrow$)

2 More details about these results can be found in Arenas et al. (2004).
max \ E_p = \sum_{i=1}^{n} E(R_i) \ x_i \\
min \ \beta_p = \sum_{i=1}^{n} \beta_i \ x_i \\
s.t. \ \sum_{i=1}^{n} x_i = 1 \\
x_i \leq \frac{k_i}{100} \\
\sum_{i \in b_r} x_i \leq \frac{j}{100} \\
\sum_{i \in b_l} x_i \geq \frac{j}{100} \\
x_i \geq 0 \ i = 1,...,n \tag{9}

where symbols with a tilde on top represent fuzzy parameters whose possibility distributions are given by fuzzy trapezoidal numbers. Using (4), (5) and (6) the problem can be reformulated as:

\[
\begin{align*}
\text{Find } & \ x \in \chi(\lambda) \text{ such that:} \\
E_p & \rightarrow E_p^* \\
\text{EI}(\beta_p) & \rightarrow \text{EI}(\tilde{\beta}_p^*) \\
\end{align*}
\tag{10}
\]

where \( E_p^* \) and \( \tilde{\beta}_p^* \) are the ideal values of the expected returns and fuzzy Beta respectively. In the following section we shall discuss a real application of the Fuzzy CP selection model to 26 Spanish mutual funds.

4. Numerical example

We have consider 26 Spanish mutual funds of the type called Fondos de Inversión Mobiliaria (FIM) whose quarterly returns data are referred to the period 1996-2000. Fuzzy Beta and expected returns are shown in Table 1.

We have considered a maximum fuzzy amount \( \tilde{k}_i \% \) of the portfolio which can be invested in each asset \( i \), \( \tilde{k}_i = (20, 30, 40, 50) \). The investor has classified assets according to their fuzzy Beta which have been modelled as trapezoidal fuzzy numbers handle by their expected values and expected intervals. In order to mimic the expert’s determination of \( \tilde{\beta} = (b_1, b_2, b_3, b_4) \), a statistical method has been used to construct Beta’s nucleus, but other forms of calculus may be done. Thus, in this paper, Beta’s nucleus contains the confidence interval of the minimum quadratic estimator at the level of 95%. The right and left hand side of \( \tilde{\beta} \) were obtained by means of Montecarlo Simulation. The investor decides that an acceptable portfolio should, if possible, contain a maximum around \( \tilde{j} = (0, 0, 40, 75) \% \) of assets whose expected value of Beta is greater or equal than 1.2, \( EV(\tilde{\beta}) \geq 1.2 \). He/she establishes too, a minimum around \( \tilde{l} = (20, 40, 100, 100) \% \) of assets whose expected value of Beta is smaller or equal than 1, \( EV(\tilde{\beta}) \leq 1 \).
Table 1. Beta and expected returns for 26 Spanish FIM

<table>
<thead>
<tr>
<th>FIM</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
<th>$E(V(\beta))$</th>
<th>$E(R)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>0.0924</td>
<td>0.1560</td>
<td>0.3425</td>
<td>0.4460</td>
<td>0.259225</td>
<td>2.1650</td>
</tr>
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<td>AC</td>
<td>0.1660</td>
<td>0.7153</td>
<td>1.4682</td>
<td>2.4713</td>
<td>1.2052</td>
<td>5.7615</td>
</tr>
<tr>
<td>ARG</td>
<td>0.0214</td>
<td>0.0349</td>
<td>0.2842</td>
<td>0.3170</td>
<td>0.164375</td>
<td>5.2725</td>
</tr>
<tr>
<td>BBV</td>
<td>0.7834</td>
<td>0.8314</td>
<td>1.7044</td>
<td>1.8177</td>
<td>1.284225</td>
<td>5.2315</td>
</tr>
<tr>
<td>BK</td>
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<td>1.4167</td>
<td>1.7310</td>
<td>1.02155</td>
<td>5.4205</td>
</tr>
<tr>
<td>BM</td>
<td>0.9887</td>
<td>1.0292</td>
<td>1.8860</td>
<td>2.5797</td>
<td>1.6209</td>
<td>6.6535</td>
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<td>C2000</td>
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<td>0.5449</td>
<td>0.9737</td>
<td>1.6440</td>
<td>0.794125</td>
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</tr>
<tr>
<td>DB</td>
<td>0.4862</td>
<td>0.8141</td>
<td>1.6439</td>
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<td>EDM</td>
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<td>1.7474</td>
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<td>EUFND</td>
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</tr>
<tr>
<td>FONIND</td>
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</tr>
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<td>GNRBOL</td>
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<td>0.1447</td>
<td>0.5778</td>
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<td>0.8195</td>
</tr>
<tr>
<td>GREEN</td>
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<td>1.9980</td>
<td>1.03795</td>
<td>5.7970</td>
</tr>
<tr>
<td>IBEREU</td>
<td>0.1699</td>
<td>0.1730</td>
<td>0.3678</td>
<td>0.5599</td>
<td>0.31765</td>
<td>2.3435</td>
</tr>
<tr>
<td>IBEREQ</td>
<td>0.4112</td>
<td>0.4878</td>
<td>1.1184</td>
<td>1.6242</td>
<td>0.9104</td>
<td>3.9515</td>
</tr>
<tr>
<td>INDBOL</td>
<td>0.0399</td>
<td>0.7834</td>
<td>1.5536</td>
<td>2.1210</td>
<td>1.124475</td>
<td>4.9845</td>
</tr>
<tr>
<td>INVSAB</td>
<td>0.1832</td>
<td>0.2157</td>
<td>0.4124</td>
<td>0.7655</td>
<td>0.3942</td>
<td>2.6015</td>
</tr>
<tr>
<td>KUTXA</td>
<td>0.2589</td>
<td>0.3292</td>
<td>0.6423</td>
<td>0.7187</td>
<td>0.487275</td>
<td>2.5230</td>
</tr>
<tr>
<td>MARCH</td>
<td>0.0706</td>
<td>0.1540</td>
<td>0.3696</td>
<td>0.4877</td>
<td>0.270475</td>
<td>1.8465</td>
</tr>
<tr>
<td>NAVACC</td>
<td>0.6960</td>
<td>0.8101</td>
<td>1.5737</td>
<td>2.8075</td>
<td>1.471825</td>
<td>5.4870</td>
</tr>
<tr>
<td>SANFER</td>
<td>0.1747</td>
<td>0.7127</td>
<td>1.4621</td>
<td>2.0856</td>
<td>1.108775</td>
<td>5.5230</td>
</tr>
<tr>
<td>SANTDR</td>
<td>0.0959</td>
<td>0.6265</td>
<td>1.2047</td>
<td>1.6136</td>
<td>0.885175</td>
<td>4.9455</td>
</tr>
<tr>
<td>TLFVAR</td>
<td>0.0506</td>
<td>0.7890</td>
<td>1.4134</td>
<td>2.4792</td>
<td>1.18305</td>
<td>5.9260</td>
</tr>
<tr>
<td>URQCAP</td>
<td>0.0830</td>
<td>0.1691</td>
<td>0.3902</td>
<td>0.7090</td>
<td>0.337825</td>
<td>2.0975</td>
</tr>
<tr>
<td>URQCR</td>
<td>0.0032</td>
<td>0.6323</td>
<td>1.2929</td>
<td>2.2926</td>
<td>1.05525</td>
<td>4.6645</td>
</tr>
</tbody>
</table>

With these data and with the fixed weights $w_1 = \frac{2}{3}$ and $w_2 = \frac{1}{3}$, we have solved problem (9) applying the methodology developed by the authors and described in section 3, in an interactive way with the investor, who fixed too the feasibility degree he/she is willing to accept. In the following table we show the possibility distribution of the fuzzy ideal solution corresponding to the minimization of the portfolios’ risk for a feasibility degree $\lambda = 0.5$, obtained using PROMO software (Molina (2000)):

Table 2. Possibility distribution of fuzzy ideal solution of Beta.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\tilde{\beta}<em>p = \sum</em>{i=1}^{n} \tilde{\beta}_i x_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[0.0105915, 0.406556]</td>
</tr>
<tr>
<td>0.2</td>
<td>[0.0341394, 0.387338]</td>
</tr>
<tr>
<td>0.4</td>
<td>[0.0497983, 0.367957]</td>
</tr>
<tr>
<td>0.6</td>
<td>[0.0654572, 0.347946]</td>
</tr>
<tr>
<td>0.8</td>
<td>[0.0811161, 0.327934]</td>
</tr>
<tr>
<td>1</td>
<td>[0.096775, 0.307923]</td>
</tr>
</tbody>
</table>

With the single optimization of the expected return crisp objective we have obtained the expected returns’ ideal solution $E_p^* = 5.08134$. Once the fuzzy ideal solution has been obtained, we determine $\lambda$-compromise solutions $L_1$ and $L_\infty$.
Table 3. $\lambda$-compromise solutions

<table>
<thead>
<tr>
<th></th>
<th>$E_P$</th>
<th>$E[\hat{\beta}_P]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>2.45607</td>
<td>[0.222693, 0.561902]</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>2.67823</td>
<td>[0.171856, 0.511066]</td>
</tr>
</tbody>
</table>

If the investor chooses compromise solution $L_1$ he/she will obtain a higher level of expected returns than choosing compromise solution $L_\infty$, but that supposes assuming a higher risk level. On the other hand, if the investor prefers compromise solution $L_\infty$ he/she chooses obtaining a lower level of the portfolio return but a lower level too, of portfolio risk, that is, he/she obtains an equilibrated solution instead of a maximum efficiency one (solution $L_1$).

In table 4 and table 5 we show the obtained solutions related to the amount of investment in the selected asset:

Table 4. Compromise solution $L_1$ (investment in selected assets)

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_3$</th>
<th>$x_9$</th>
<th>$x_{15}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>0.186015</td>
<td>0.355</td>
<td>0.103985</td>
<td>0.355</td>
</tr>
</tbody>
</table>

Table 5. Compromise solution $L_\infty$ (investment in selected assets)

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_9$</th>
<th>$x_{15}$</th>
<th>$x_{19}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_\infty$</td>
<td>0.2255719</td>
<td>0.0342809</td>
<td>0.355</td>
<td>0.355</td>
</tr>
</tbody>
</table>

5. Conclusions

This paper has applied the Fuzzy Sets Theory to the problem of portfolio selection using Sharpe’s single-index model as a basis. Future Beta of each asset are estimated not only from historical data series, but reflecting expert’s subjective opinions in linguistic terms which suggests a fuzzy representation of Beta. We have represented the portfolio fuzzy Beta through a fuzzy trapezoidal number, which is able to bring together statistical knowledge and the relevant knowledge of financial analysts. The model works, therefore, with fuzzy data that contain more information than any classical model and does not involve a great deal of extra computational difficulty.

This model has been applied to deal with 26 Spanish mutual funds over the period 1996 – 2000. We have determined, using a Fuzzy Compromise Programming approach, two efficient portfolios, given the feasibility degree fixed by the investor, characterized by maximum efficiency (solution $L_1$) and by maximum equilibrium between risk and expected returns (solution $L_\infty$). Finally, it should be pointed out that a major feature of this model is its sensitivity to the analyst’s opinion as well as to the investors’ preferences. This allows interaction with both when it comes to designing the best portfolio.

6. References


