On the union of $\kappa$-curved objects

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Communicated by J. Matoušek; submitted 1 June 1998; accepted 1 September 1999

Abstract

A (not necessarily convex) object $C$ in the plane is $\kappa$-curved for some constant $0 < \kappa < 1$, if it has constant description complexity, and for each point $p$ on the boundary of $C$, one can place a disk $B \subseteq C$ of radius $\kappa \cdot \text{diam}(C)$ whose boundary passes through $p$. We prove that the combinatorial complexity of the boundary of the union of a set $C$ of $n$ $\kappa$-curved objects (e.g., fat ellipses or rounded heart-shaped objects) is $O(s(n) \log n)$, for some constant $s$. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Fat objects; Union of objects; Combinatorial complexity; Davenport–Schinzel sequences

1. Introduction

Let $C$ be a not necessarily convex object in the plane, and let $\kappa$, $0 < \kappa < 1$, be a constant. We say that $C$ is $\kappa$-curved if:

(i) $C$ has constant description complexity. By constant description complexity we mean that $C$’s boundary consists of a constant number of algebraic arcs, each of degree at most $b$.

(ii) For each point $p$ on $\partial C$, we can place a disk $B \subseteq C$ of radius $\kappa \cdot \text{diam}(C)$ whose boundary passes through $p$ (see Fig. 1). We say that the radius $\kappa \cdot \text{diam}(C)$ is the critical radius of $C$, and that the disk $B$ is a critical disk of $C$ at $p$.

The first condition implies that there are at most some constant number of local minima or maxima on $C$’s boundary, and that $C$ has a tangent at each point $p$ on its boundary, except for at most some constant number of points where the tangent is not defined. At these points, though, $C$ does have a left and a right tangent.
The second condition is similar to bounding the curvature of the boundary of $C$, but is more general (see Fig. 1). It can be illustrated as follows: Imagine a car moving along the boundary of $C$ such that the interior of $C$ is to its left. Then the car is allowed to make very sharp right turns, but when turning left, the radius of the turn is bounded from below by some fraction of the diameter of $C$.

Let $C = \{C_1, \ldots, C_n\}$ be a set of $n$ $\kappa$-curved objects. By the constant description complexity condition, the boundaries of any pair of objects in $C$ cross each other in at most some constant number, $s_0$, of points. (We assume for simplicity of exposition that the intersection of a pair of boundaries of objects in $C$ is a collection of isolated points, at which the boundaries cross each other from side to side.) In this paper we prove that the number of vertices on the boundary of the union $U$ of the objects in $C$ is only $O(\lambda_s(n) \log n)$, for some constant $s$. ($\lambda_s(n)$ is the maximum length of an $(n, s)$ Davenport–Schinzel sequence; it is nearly linear in $n$ for any constant $s$ [12].)

$\kappa$-curved objects are fat: we say that an object $C$ is $\alpha$-fat, for a constant $\alpha > 1$, if $r_1 / r_2 \leq \alpha$, where $r_1$ is the radius of a smallest disk containing $C$, and $r_2$ is the radius of a largest disk contained in $C$. Obviously, a $\kappa$-curved object is $\alpha$-fat for an appropriate constant $\alpha$ (e.g., $\alpha = 1/\kappa$), but the opposite statement is false. Fat objects received much attention in recent years. One of the first papers on fat objects is by Matoušek et al. [10] who showed that a set of $n$ fat triangles determines only a linear number of “holes”, and that the combinatorial complexity of the boundary of its union (i.e., the number of vertices on the boundary) is only $O(n \log \log n)$. Since then many authors considered various definitions of fatness (which are all more or less equivalent—at least for convex objects), and obtained either interesting combinatorial results or efficient geometric algorithms (see, e.g., [1–3,5,6,8,9,16,11,13–17]). However, the question whether the number of vertices on the boundary of the union of a set of convex, fat, relatively-general objects is always subquadratic, remained open for quite a few years. Recently, Efrat and Sharir [7] showed that if the objects are convex, fat, and the boundaries of each pair intersect at most a constant number of times, then the boundary of their union consists of only $O(n^{1+\varepsilon})$ vertices, for any constant $\varepsilon > 0$.\footnote{Throughout the paper, $\varepsilon$ stands for a positive constant which can be chosen arbitrarily small with an appropriate choice of other constants of the big-O notation.} In a preliminary version of their paper, it was shown that if, in addition, the objects have bounded curvature and are more or less of the same size, then the number of vertices on the union’s boundary is only $O(\lambda_s(n))$, for some constant $s$.

Our result improves upon the result of [7] for convex $\kappa$-curved objects such as fat ellipses, and complements it for objects that are non-convex (but $\kappa$-curved). We prove the following theorem.

**Theorem 1.1.** The combinatorial complexity of the boundary of the union of $n$ $\kappa$-curved objects is $O(\lambda_s(n) \log n)$, for some constant $s$. 

\textbf{Fig. 1.} Three $\kappa$-curved objects.
In the proof we use a well known data structure, namely, a segment tree, and its properties. We project the input \( \kappa \)-curved objects on the \( y \)-axis, and construct a (skeleton of a) segment tree \( T \) for these projections. We then insert the objects into \( T \) according to their projection on the \( y \)-axis. As usual, we associate with a node \( \mu \) of \( T \) its \( y \)-interval, which we think of as a horizontal slab. Now, roughly speaking, the vertices on the boundary of the union of the input objects are distributed among the nodes of \( T \), so that, if a vertex \( w \) ends up at a node \( \mu \), then \( w \) lies in the canonical slab of \( \mu \) and is formed by a pair of objects that are stored at \( \mu \). (The objects that are stored at \( \mu \) consist of the objects in the canonical subset of \( \mu \) and the objects in the canonical subsets of all the descendants of \( \mu \).) By proving a connection between the number of vertices that end up at \( \mu \) and the number of objects that are stored at \( \mu \), and by summing over all nodes in \( T \), we obtain the claimed bound.

The paper is organized as follows. In Section 2 we first decompose the plane, using the skeleton of the segment tree mentioned above, and then partition (some of) the objects that are stored at the nodes of the tree, on the basis of this decomposition. This section lays the ground for Section 3 in which we bound the number of vertices on the boundary of the union of the input objects, thus proving Theorem 1.1. Section 4 concludes the paper.

2. Partitioning the object boundaries

Let \( C \) be a set of \( n \) \( \kappa \)-curved objects, and let \( U \) denote the union of the objects in \( C \). Our goal is to prove that the combinatorial complexity of \( \partial U \) is \( O(\lambda_s(n) \log n) \), for some constant \( s \). In this section we first decompose the plane into horizontal strips, which are then partitioned into squares. The decomposition process is guided by (the skeleton of) a segment tree for the projections of the objects in \( C \) on the \( y \)-axis. Next, each node in this tree stores some objects of \( C \) (see below), each of which is either ‘large’ or ‘small’, and we partition the boundaries of the large objects into a constant number of smaller pieces, on the basis of the plane decomposition.

2.1. Decomposing the plane

Project the objects in \( C \) on the \( y \)-axis, and construct a (skeleton of a) segment tree \( T \) for these projections. Insert the objects of \( C \) into \( T \) according to their projection on the \( y \)-axis. For a node \( \mu \) of \( T \), let \( y_\mu \) denote the canonical \( y \)-interval that is associated with \( \mu \), and let \( C_\mu \subseteq C \) be the canonical subset that is stored at \( \mu \). (Recall that \( C_\mu \) consists of the objects in \( C \) whose projection on the \( y \)-axis contains \( y_\mu \) but not \( y_{\text{parent}(\mu)} \).) We think of \( y_\mu \) as the horizontal slab whose top (respectively bottom) defining line passes through the top (respectively bottom) endpoint of the \( y \)-interval denoted by \( y_\mu \). We also store at \( \mu \) a second subset \( D_\mu \subseteq C \) which is the union of all canonical subsets stored at (the proper) descendants of \( \mu \). It is well known (see, e.g., [4]) that

\[
\sum_{\mu} (|C_\mu| + |D_\mu|) = O(n \log n) .
\]  

(1)

Let \( U_\mu \) denote the union of the objects in \( C_\mu \cup D_\mu \) restricted to the slab \( y_\mu \). We first prove the following easy (and known) claim.

Claim 2.1. Let \( w \) be a vertex on \( \partial U \) that is an intersection point between the boundaries of two objects \( C_1, C_2 \in C \), then there exists a node \( \mu \) of \( T \) such that \( w \) lies in the slab \( y_\mu \), and either

\[
\sum_{\mu} (|C_\mu| + |D_\mu|) = O(n \log n) .
\]  

(1)
(1) both \( C_1 \) and \( C_2 \) are in \( C_\mu \), or 
(2) one of them is in \( C_\mu \) and the other is in \( D_\mu \).

Moreover, \( w \) is a vertex on \( \partial U_\mu \).

**Proof.** The first part (i.e., there exists such a node \( \mu \)) follows from basic properties of segment trees, since the projection of \( w \) on the \( y \)-axis lies in both the projection of \( C_1 \) and the projection of \( C_2 \). The second part (i.e., \( w \) is a vertex on \( \partial U_\mu \)) is also obvious, since \( w \) is a vertex on the boundary of the union of any subset of \( C \) that includes both \( C_1 \) and \( C_2 \). \( \square \)

We thus distinguish between two types of vertices on \( \partial U_\mu \). A vertex of type I is an intersection point between the boundaries of two objects in \( C_\mu \), and a vertex of type II is an intersection point between the boundaries of an object in \( C_\mu \) and an object in \( D_\mu \). (We ignore the vertices on \( \partial U_\mu \) that belong to a single object, since there are at most \( O(|C_\mu| + |D_\mu|) \) such vertices.) Let \( u_\mu \) be the number of vertices on \( \partial U_\mu \) of type I and type II. We prove that \( u_\mu = O(\lambda_s(|C_\mu| + |D_\mu|)) \), for some constant \( s \), and therefore (since the function \( \lambda_s(x)/x \) is increasing [12] and given Eq. (1))

\[
\sum_\mu u_\mu = \sum_\mu O(\lambda_s(|C_\mu| + |D_\mu|)) \\
= \sum_\mu (|C_\mu| + |D_\mu|) \cdot \frac{O(\lambda_s(|C_\mu| + |D_\mu|))}{|C_\mu| + |D_\mu|} \\
\leq \sum_\mu (|C_\mu| + |D_\mu|) \cdot O\left( \frac{\lambda_s(n)}{n} \right) \\
= O(\lambda_s(n) \log n).
\]

This together with Claim 2.1 yields the desired result, i.e., the number of vertices on \( \partial U \) is \( O(\lambda_s(n) \log n) \).

Consider a node \( \mu \) of \( T \), and let \( d \) be the width of the slab \( y_\mu \). We partition the slab \( y_\mu \) into \( \lfloor \sqrt{2}/\kappa \rfloor \) horizontal strips each of width at most \( \kappa/Nd \). We partition each of these strips into disjoint squares \( \sigma_1, \sigma_2, \ldots \) of edge-length \( \kappa/Nd \), by adding vertical walls (see Fig. 2). Consider any one of the strips \( \rho \). We show in Section 3 that the number of vertices on \( \partial U_\mu \) of type I and type II that lie in \( \rho \) is \( O(\lambda_s(|C_\mu| + |D_\mu|)) \), and, therefore, that \( u_\mu = O(\lambda_s(|C_\mu| + |D_\mu|)) \) (since \( y_\mu \) was partitioned into a constant number of strips).

Clearly any object in \( C_\mu \) has diameter at least \( d \). Let \( C \) be an object in \( C_\mu \cup D_\mu \) whose diameter is at least \( d \). (The ratio between the critical radius of \( C \) and the width of \( \rho \) is therefore at least \( \sqrt{2} \).) In the next subsection we show how to partition the boundary of \( C \) into a constant number of pieces.

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Fig. 2. The slab \( y_\mu \) partitioned into 3 strips \( \rho_1, \rho_2, \rho_3 \).
2.2. Partitioning the boundaries on the basis of the decomposition

Let \( \rho \) be a horizontal strip of width \( \delta \) that is divided into squares \( \Sigma = \{ \sigma_1, \sigma_2, \ldots \} \) of edge-length \( \delta \). Let \( \ell_{\text{bottom}} \) and \( \ell_{\text{top}} \) denote the bottom and top horizontal lines defining \( \rho \). Let \( C \) be a \( \kappa \)-curved object whose diameter is at least \( (\sqrt{2}/\kappa)\delta \). (The radius of any critical disk of \( C \) is therefore at least \( \sqrt{2} \delta \).) In this section we show that it is possible to obtain from \( C \) a constant number of smaller objects (called parts), such that (i) each part is contained in \( C \) and has some desirable properties, and (ii) for each point \( p \) on \( \rho \cap \partial C \), there exists a part such that \( p \) lies on its boundary.

We need the following definition.

**Definition 2.2.** Consider an object \( C' \) in the plane, and a horizontal line \( l \). We say that \( C' \) is *function-defined* with respect to \( l \) (from above), if for any point \( p \) on \( l \), either the vertical ray emanating from \( p \) and directed upwards does not intersect \( C' \), or the intersection of this ray with \( C' \) is a closed segment whose bottom endpoint is \( p \) (see Fig. 3). In other words, the top endpoints of all these closed segments consist of the graph of some function defined over a subset of \( l \), and the region enclosed between this graph and the line \( l \) is exactly \( C' \setminus l^{+} \), where \( l^{+} \) is the closed halfplane lying above \( l \). Analogously, we define the statements: \( C' \) is *function-defined* with respect to \( l \) from below, or, for a vertical line \( l \), \( C' \) is *function-defined* with respect to \( l \) from the left (alternatively, from the right). Clearly, if both \( C' \) and \( C'' \) are function-defined with respect to the same line \( l \), then so is their union.

In other words, the top endpoints of all these closed segments consist of the graph of some function defined over a subset of \( l \), and the region enclosed between this graph and the line \( l \) is exactly \( C' \setminus l^{+} \), where \( l^{+} \) is the closed halfplane lying above \( l \). Analogously, we define the statements: \( C' \) is *function-defined* with respect to \( l \) from below, or, for a vertical line \( l \), \( C' \) is *function-defined* with respect to \( l \) from the left (alternatively, from the right). Clearly, if both \( C' \) and \( C'' \) are function-defined with respect to the same line \( l \), then so is their union.

For each point \( p \) on \( \partial C \), let \( \mathcal{B}(p) \) denote the set of all critical disks of \( C \) at \( p \). If \( C \) has a tangent at \( p \), then \( \mathcal{B}(p) \) consists of a single disk, whose tangent at \( p \) coincides with the tangent of \( C \) at \( p \). Therefore, by the constant description complexity condition, the number of points on \( \partial C \) for which \( |\mathcal{B}(p)| > 1 \) is bounded by some constant. For a disk \( B \in \mathcal{B}(p) \), let \( \bar{B} \subseteq B \) denote the disk of radius \( \sqrt{2} \delta \) obtained from \( B \) by moving its center towards \( p \) while maintaining the contact with \( p \). That is, \( \bar{B} \) is the disk of radius \( \sqrt{2} \delta \) whose center lies on the segment connecting \( B \)'s center and \( p \), and whose boundary passes through \( p \). Let \( h_{\text{bottom}}^{\bar{B}} \) (respectively \( h_{\text{top}}^{\bar{B}} \)) denote the halfplane bounded by \( \ell_{\text{bottom}} \) (respectively \( \ell_{\text{top}} \)) and containing \( \bar{B} \). Similarly, for a square \( \sigma \in \Sigma \), let \( \ell_{\sigma, \text{left}} \) and \( \ell_{\sigma, \text{right}} \) denote the lines containing the left and right edges of \( \sigma \), respectively, and let \( h_{\sigma, \text{left}}^{\bar{B}} \) (respectively \( h_{\sigma, \text{right}}^{\bar{B}} \)) denote the halfplane bounded by \( \ell_{\sigma, \text{left}} \) (respectively \( \ell_{\sigma, \text{right}} \)) and containing \( \sigma \). We use the following simple but important observation (see Fig. 4).
Claim 2.3. Let $B$ be a disk of radius at least $\sqrt{2}\delta$ that intersects $\rho$, and let $q$ be a point on $\rho \cap \partial B$. Let $\sigma$ be the cell of $\Sigma$ containing $q$. Then $B \cap h^+_{\sigma, \text{right}}$ is function-defined with respect to $\ell_{\text{right}}$, or $B \cap h^-_{\sigma, \text{right}}$ is function-defined with respect to $\ell_{\text{right}}$. We next define the set (i.e., the part) $C_{\text{bottom}} \subseteq C$ (see Fig. 5). Informally, $C_{\text{bottom}}$ is obtained by walking along the pieces of $\partial C$ that are contained in $\rho$. At any (regular) point $p$ on the way, if its corresponding disk $\tilde{B}$ is function-defined with respect to $\ell_{\text{bottom}}$ (or, in other words, if the center of $\tilde{B}$ does not lie above $\ell_{\text{bottom}}$), then we add the region $\tilde{B} \cap h^+_{\text{bottom}}$ to the set $C_{\text{bottom}}$ that is being constructed. More formally,

$$C_{\text{bottom}} = \bigcup \{ \tilde{B} \cap h^+_{\text{bottom}} \mid B \text{ is a disk of } B(p) \text{ for some } p \in \rho \cap \partial C, \text{ and }$$

$$\tilde{B} \cap h^+_{\text{bottom}} \text{ is function-defined for } \ell_{\text{bottom}} \}. \$$

The set (part) $C_{\text{top}}$ is defined analogously. It is easy to see that the sets $C_{\text{bottom}}$ and $C_{\text{top}}$ are function-defined with respect to $\ell_{\text{bottom}}$ and $\ell_{\text{top}}$, respectively, and that they have constant description complexity (since $C$ has constant description complexity). For each $\sigma \in \Sigma$ we define the set (part)

$$C_{\sigma, \text{left}} = \bigcup \{ \tilde{B} \cap h^+_{\sigma, \text{left}} \mid B \text{ is a disk of } B(p) \text{ for some } p \in \sigma \cap \partial C, \text{ and }$$

$$\tilde{B} \cap h^+_{\text{bottom}} \text{ is not function-defined for } \ell_{\text{bottom}}, \text{ and }$$

$$\tilde{B} \cap h^-_{\text{top}} \text{ is not function-defined for } \ell_{\text{top}}, \text{ and }$$

$$\tilde{B} \cap h^+_{\sigma, \text{left}} \text{ is function-defined for } \ell_{\sigma, \text{left}} \}. \$$
The set (part) \( C_{\sigma, \text{right}} \) is defined analogously. It is easy to see that the set \( C_{\sigma, \text{left}} \) (respectively \( C_{\sigma, \text{right}} \)) is function-defined with respect to \( \ell_{\sigma, \text{left}} \) (respectively \( \ell_{\sigma, \text{right}} \)), that it has constant description complexity, and that its projection on the \( x \)-axis is contained in the projection on the \( x \)-axis of \( \sigma \) and the cell immediately to its right (respectively left).

Define the (partial) function \( f_{\text{bottom}} \) on \( \ell_{\text{bottom}} \) as follows: For \( x \in \ell_{\text{bottom}} \), if there is a point in \( C_{\text{bottom}} \) lying vertically above \( x \), then \( f_{\text{bottom}}(x) = y \), where \( y \) is the \( y \)-coordinate of the highest such point; otherwise, \( f_{\text{bottom}}(x) \) is not defined. The graph of the function \( f_{\text{bottom}} \) is actually the upper envelope of \( C_{\text{bottom}} \). The functions \( f_{\text{top}}, f_{\sigma, \text{left}}, f_{\sigma, \text{right}} \) are defined analogously. From Claim 2.3 it follows that \( \rho \cap \partial C \) is contained in the union of the graph of \( f_{\text{bottom}} \), the graph of \( f_{\text{top}} \), and the graphs of \( f_{\sigma, \text{left}} \) and of \( f_{\sigma, \text{right}} \) for all \( \sigma \in \Sigma \).

Next we claim that \( C_{\sigma, \text{left}} \) and \( C_{\sigma, \text{right}} \) are not empty only for a constant number of cells \( \sigma \). Indeed, let \( p \) be a point on \( \sigma \cap \partial C \) for which there exists a disk \( B \in B(p) \) such that \( B \) is function-defined, say, for the left edge of \( \sigma \), but not for \( \ell_{\text{bottom}} \) nor \( \ell_{\text{top}} \). Then the center of \( B \) must lie in one of the two cells immediately to the left of \( \sigma \), and must contain at least one of the vertical edges of \( \sigma \), the cell immediately to the left of \( \sigma \). Therefore, either \( \partial C \) intersects \( \ell_{\text{bottom}} \) or \( \ell_{\text{top}} \) within \( \sigma \) or \( \sigma^\prime \), or \( \partial C \) has a (locally) leftmost point in \( \sigma \) or in \( \sigma^\prime \) (i.e., \( \partial C \) turns leftwards). Thus in both cases some event occurs either in \( \sigma \) or in \( \sigma^\prime \). However, the constant description complexity condition for \( C \) implies that both these types of events may occur only a constant number of times, and therefore the number of non-empty sets of the form \( C_{\sigma, \text{left}} \) or \( C_{\sigma, \text{right}} \) is bounded by some constant.

The following lemma summarizes the results of this subsection.

**Lemma 2.4.** It is possible to obtain from \( C \) a constant number of (not necessarily connected) parts, such that (i) each of the parts is function-defined with respect to either \( \ell_{\text{bottom}} \), \( \ell_{\text{top}} \), or a line containing a vertical wall in \( \rho \), (ii) each of the parts has constant description complexity, (iii) those parts that are function-defined with respect to a line containing a vertical wall \( e \), are contained in a vertical slab defined by a section of \( \rho \) that begins at \( e \) and is two squares wide, and (iv) if \( p \) is a point on \( \rho \cap \partial C \), then \( p \) lies on the (appropriate) envelope of one of the parts (i.e., either on the upper envelope of \( C_{\text{bottom}} \), or on the lower envelope of \( C_{\text{top}} \), etc.).

We now apply this claim to the large objects in \( C_\mu \cup D_\mu \). Let \( E_\mu \) be the set of all objects in \( C_\mu \cup D_\mu \) with diameter at least \( d \). We apply Lemma 2.4 to all objects in \( E_\mu \).

### 3. Bounding the number of vertices

In the section we show that the number of vertices on \( \partial U_\mu \) of type I and type II that lie in \( \rho \) is \( O(\lambda_\varepsilon(|C_\mu| + |D_\mu|)) \).

Let \( \gamma_1 \) (respectively \( \gamma_2 \)) denote the upper envelope (respectively lower envelope) of all parts that are function-defined with respect to \( \ell_{\text{bottom}} \) (respectively \( \ell_{\text{top}} \)). The combinatorial complexity of \( \gamma_i \) is \( O(\lambda_{s_0}(m_i)) \) [12], where \( m_i = O(|E_\mu|) = O(|C_\mu| + |D_\mu|) \) is the number of parts that are function-defined with respect to the appropriate bounding line of \( \rho \), \( i = 1, 2 \). (Recall that the boundaries of any pair of objects in \( C \) cross each other in at most \( s_0 \) points.) For each square \( \sigma \), let \( \gamma_\sigma^l \) (respectively \( \gamma_\sigma^r \)) denote the right envelope (respectively left envelope) of all parts that are function-defined with respect to the line containing the left (respectively right) edge of \( \sigma \). The combinatorial complexity of \( \gamma_\sigma^l \) (respectively
Proof.

Let $F$ be the set of all small objects in $C_\mu \cup D_\mu$. Clearly, $F \subseteq D_\mu$. In the remainder of this section we first show that in order to bound the number of vertices on $\partial U_\mu$ of type I and II that lie in $\rho$, it suffices to bound the number of bichromatic vertices on the boundaries of regions corresponding to various pairs of envelopes and various mixed pairs consisting of an envelope and a subset of $F$. Now, for pairs consisting of two opposite envelopes (i.e., upper and lower or right and left), it is easy to obtain such a bound. However, for mixed pairs or for pairs consisting of an upper/lower envelope and a left/right envelope, it is much more difficult and we do it in Section 3.2.

3.1. Pairs of opposite envelopes

Consider now the objects in $F_\mu$, i.e., the objects in $C_\mu \cup D_\mu$ with diameter less than $d$. Each such object intersects only a constant number of squares of $\rho$. For each square $\sigma$, let $F_\sigma \subseteq F_\mu$ be the subset of objects that intersect $\sigma$; we have $\sum_\sigma |F_\sigma| = O(|F_\mu|) = O(|D_\mu|)$. Recall that our goal is to bound the number of vertices on $\partial U_\mu$ of type I and II that lie in $\rho$. We bound the number of vertices that appear when considering various pairs of envelopes, and various pairs consisting of an envelope and a subset of $F$. That is, when considering a pair of envelopes we count the number of intersection points between the envelopes, or, in other words, if $X$ and $Y$ are the two underlying sets of parts, then we count the number of bichromatic vertices on the boundary of the union of the objects in $X \cup Y$, where a vertex is bichromatic if it lies both on the boundary of an object of $X$ and on the boundary of an object of $Y$. And when considering a pair consisting of an envelope and a subset $F'$ of $F_\mu$, we count the number of bichromatic vertices on the boundary of the union of the objects in $X \cup F'$, where $X$ is the set of parts underlying the envelope. More precisely, we bound the number of vertices that appear when considering the following pairs:

(a) $(\gamma_1, \gamma_2)$,
(b) for each square $\sigma$, $(\gamma_1, \sigma)$, $(\gamma_1, \gamma'_\sigma)$, $(\gamma_1, \gamma''_\sigma)$, $(\gamma_2, \gamma'_\sigma)$, $(\gamma_2, \gamma''_\sigma)$,
(c) for each square $\sigma$, $(\gamma'_\sigma, \gamma''_\sigma)$,
(d) $(\gamma_1, F_\mu)$, $(\gamma_2, F_\mu)$,
(e) for each square $\sigma$, $(\gamma'_\sigma, F_\sigma)$, $(\gamma''_\sigma, F_\sigma)$.

We now claim that all ‘interesting’ vertices of $\partial U_\mu$ appear in one of these cases.

Claim 3.1. If $w$ is a vertex on $\partial U_\mu$ of type I or II that lies in a square $\sigma$ of $\rho$, then either (i) $w$ is a vertex of one of the envelopes considered, or (ii) $w$ appears when one of the above pairs is considered.

Proof. Let $w$ be a vertex on $\partial U_\mu$ that lies in a square $\sigma$ of $\rho$. If $w$ is of type I, that is, $w$ is an intersection point of the boundaries of two objects in $C_\mu$, then clearly $w$ is either a vertex of one of the envelopes $\gamma_1, \gamma_2, \gamma'_\sigma, \gamma''_\sigma$, or a vertex that appears when considering one of the pairs in (a), (b) or (c) above.
If \( w \) is of type II, that is, \( w \) is an intersection point of the boundaries of an object in \( C \) and an object in \( D \), then we distinguish between two cases. If the object from \( D \) is large, i.e., it is in \( E \), then, as before, \( w \) is either a vertex of an envelope or appears when considering a pair of envelopes. Otherwise, the object from \( D \) is small (i.e., it is in \( F \)), and \( w \) is a vertex that appears when considering one of the pairs in (d) or (e).

Notice that we also count many ‘uninteresting’ vertices such as vertices that are formed by two objects in \( D \), or vertices that ‘do not make it’ to the boundary of the full union, or even artificial vertices that do not lie on the boundary of one of the input objects.

We can immediately bound the number of vertices that appear when considering the pair of opposite envelopes in (a) or the pairs of opposite envelopes in (c), and obtain in both cases a (total) bound of \( O(\lambda_s(|C| + |D|)) \).

In the next subsection we prove a key lemma stating that if \( \gamma \) is an envelope as those defined above, and \( \mathcal{A} \) is a set of \( \kappa \)-curved objects, then the number of visible bichromatic vertices on \( \gamma \) for which the larger object (of the two objects forming the vertex) comes from the set \( X \) underlying \( \mathcal{A} \) is \( O(\lambda_s(|X|) + \lambda_s(|A|)) \), for some constant \( s \), where a visible vertex is a vertex on the boundary of the union of \( X \cup \mathcal{A} \), and the size of a part in \( \mathcal{A} \) is the diameter of the object to which it belongs.

### 3.2. Other pairs

Let \( \rho \) be a strip of width \( \delta \) and let \( l \) denote its bottom defining line. Let \( S = \{ S_1, \ldots, S_m \} \subseteq \mathcal{C} \) be a set of \( m \) large input objects, that is, the diameter of \( S_i \) is at least \( \sqrt{2\delta} / \kappa \), \( i = 1, \ldots, m \) (so the ratio between the critical radius of \( S_i \) and the width of \( \rho \) is at least \( \sqrt{2} \)). Apply the process described in Section 2.2 to the objects \( S_1, \ldots, S_m \). Consider \( \gamma \), the upper envelope of the \( m \) bottom parts that are obtained, and denote by \( R \) the region enclosed by \( \gamma \) and \( l \). (In other words, \( R \) is the union of the \( m \) bottom parts.) Let \( \mathcal{A} \subseteq \mathcal{C} \) be a set of \( k \) input objects. We wish to bound the number of bichromatic vertices on the boundary of \( V = R \cup (\bigcup \mathcal{A}) \) that lie on \( \gamma \) and for which the larger of the two objects forming the vertex is a bottom part. (Recall that the size of a part of a primitive object in \( \mathcal{A} \) from which it was obtained.)

We divide each \( A \in \mathcal{A} \) into a constant number of primitive objects \( A_1, A_2, \ldots \) by vertically decomposing \( A \). That is, for each of the locally \( x \)-extreme points \( p \) on \( \partial A \), if we remain in \( A \) when moving slightly upwards (respectively downwards) from \( p \), then we draw a vertical segment beginning at \( p \) and directed upwards (respectively downwards), until it hits \( \partial A \). Denote by \( A' \) the set of primitive objects that is obtained; \( |A'| = O(k) \). (Notice that a primitive object is trapezoid-like, it is defined by (at most) two vertical walls and by two \( x \)-monotone curves, a top curve and a bottom curve.)

When walking along \( \gamma \) from left to right, let \( L_{\text{top}} \) (respectively \( L_{\text{bot}} \)) be the sequence of names of primitive objects in \( \mathcal{A}' \) corresponding to the bichromatic vertices on \( \partial V \) that lie on top (respectively bottom) boundaries of primitive objects in \( \mathcal{A}' \). In the remainder of this section we prove the following lemma.

**Lemma 3.2.** \[ |L_{\text{top}}| = |L_{\text{bot}}| = O(\lambda_s(m) + \lambda_s(k)), \] for some constant \( s \).

Notice that whenever there are more than \( s_0 \) consecutive occurrences of the same name, there must be a vertex of \( \gamma \) somewhere in between. Thus, if we replace in the sequence \( L_{\text{top}} \) (respectively \( L_{\text{bot}} \)) all consecutive occurrences of a name by a single representative occurrence of that name, we
The sequence $L'_{\text{top}}$ (alternatively $L'_{\text{bot}}$) is a Davenport–Schinzel sequence [12] of order $s = O(s_0 + 1/k)$.

**Proposition 3.3.**

**Proof.** Consider first the sequence $L'_{\text{top}}$. Assume that there are two primitive objects $\alpha, \beta \in A'$ with top boundaries $\overline{\alpha}$ and $\overline{\beta}$, respectively, for which there exists a long subsequence of $L'_{\text{top}}$ of the form $\alpha^1 \beta^1 \alpha^2 \beta^2 \ldots \alpha^l \beta^l$ (or $\alpha^1 \beta^1 \alpha^2 \beta^2 \ldots \alpha^l \beta^l \alpha^{l+1}$). We focus on the $x$-interval whose endpoints are $\alpha$ and $\beta$. We restrict our attention further to the triple $\alpha^i; \beta^i; \alpha^{i+1}$. (If $t$ is even then we disregard the last two representatives $\beta^t$ and $\alpha^t$.) We restrict our attention to the vertical slab whose left bounding line passes through the first occurrence in the sequence of occurrences represented by $\beta^1$ and whose right bounding line passes through the last occurrence in the sequence represented by $\alpha^{l+1}$. Let $A$ and $B$ be the objects of $A$ from which $\alpha$ and $\beta$ were obtained.

Consider four consecutive representatives $\beta^{2i-1}, \alpha^{2j}, \beta^{2i}, \alpha^{2j+1}$. (If $t$ is even then we disregard the last two representatives $\beta^t$ and $\alpha^t$.) We restrict our attention to the vertical slab $\psi$ whose left bounding line passes through the first occurrence in the sequence of occurrences represented by $\beta^{2i-1}$, and whose right bounding line passes through the last occurrence in the sequence represented by $\alpha^{2j+1}$. Let $A$ and $B$ be the objects of $A$ from which $\alpha$ and $\beta$ were obtained.

If $\alpha$ and $\beta$ intersect within $\psi$, then we ignore this quadruple, since this implies that the boundaries of $A$ and $B$ intersect within $\psi$, and therefore there are at most $s_0$ such quadruples. Thus we assume that either $\alpha$ is above $\beta$ in $\psi$, or vice versa. We show that the width of $\psi$ (under this assumption) is at least $2\kappa \cdot \text{diam}(C)$, where $C$ is the smaller object among $A$ and $B$, and therefore there can be at most $1/(2\kappa)$ such quadruples.

Assume first that $\beta$ is above $\alpha$ (see Fig. 6). We restrict our attention further to the triple $\alpha^{2i}, \beta^{2i}, \alpha^{2j+1}$. Consider $p$ the vertex corresponding to the first occurrence in the sequence represented by $\beta^{2i}$, and let $S \in \mathcal{S}$ be the object to which the arc of $\gamma$ that passes through $p$ belongs. Assume that $\partial S$ exits $\beta$ at $p$, and let $q$ be the first point to the right of $p$ on $\overline{\beta}$ where $\partial S$ enters $\beta$ (see Fig. 7, top). (If $\partial S$ enters $\beta$ at $p$, then we define $q$ to be the first point to the left of $p$ on $\overline{\beta}$ where $\partial S$ exists $\beta$, and proceed similarly.)
We now think of $\alpha^{2i}$ as the rightmost intersection point corresponding to it, and of $\alpha^{2i+1}$ as the leftmost intersection point corresponding to it.

We move $\gamma$ rigidly downwards, varying the points $\alpha^{2i}$, $p$, $q$ and $\alpha^{2i+1}$ accordingly, until $p$ and $q$ coincide at a point $x$ on $\partial B$ (see Fig. 7). In other words, during this process, $p$ is the (constantly moving rightwards) exit point of $\partial S$ and $q$ is the (constantly moving leftwards) entrance point of $\partial S$, $\alpha^{2i}$ is the rightmost intersection point of $\gamma$ and $\alpha$ to the left of $p$, and $\alpha^{2i+1}$ is the leftmost intersection point of $\gamma$ and $\alpha$ to the right of $q$. Notice that the path traced by $q$ (alternatively, $p$) on $\partial B$ is not necessarily connected (see Fig. 7). At the end of this process, $\partial S$ passes through $x$ and lies below $\partial B$ in a small neighborhood of $x$. Clearly, the final location of $\alpha^{2i}$ is more to the right than the initial location of $\alpha^{2i}$, and the final location of $\alpha^{2i+1}$ is more to the left than its initial location.

If $\partial B$ does not have a tangent at $x$, then we may ignore this quadruple, since there are at most some constant number of such points on $\partial B$. Therefore, we assume that $\partial B$ does have a tangent at $x$, and let $D$ be the critical disk of $B$ at $x$, i.e., $D$ is a disk of radius $\kappa \cdot \text{diam}(B)$ that is contained in $B$ and whose boundary passes through $x$. We now claim that the disk $D$ is also contained in $S$, and, therefore, it is contained in the region lying below $\gamma$.

Observe that if (as we assume) $\partial B$ has a tangent at $x$, then so does $\partial S$. Assume this is false, and let $r_1$ and $r_2$ be the two rays tangent from the left and from the right, respectively, to $\partial B$ at $x$ (see Fig. 8). $S$ lies locally below both of them. Let $\theta$ be the inward angle between $r_1$ and $r_2$, and let $\ell_x$ be the tangent to $\partial B$ at $x$. If $\theta < \pi$, then it is impossible to draw a disk that is contained in $S$ and whose boundary passes through $x$. If, on the other hand, $\theta > \pi$, then either $r_1$ or $r_2$, say $r_2$, is above $\ell_x$, but then all points of $\partial B$ to the right of $x$ and close enough to $x$, are below $\partial S$, which is impossible considering the way in
which $S$ was translated. Thus we conclude that $\theta = \pi$, i.e., $\partial S$ has a tangent at $x$. Moreover, this tangent is necessarily $\ell_x$. Since $S$ is larger than $B$, the disk $D$ is contained in the (unique) critical disk of $S$ at $x$.

The last claim implies that $D$ is contained in $\beta \cap \psi$, since if it is not, then the boundary of $D$ must intersect one of the bounding lines of $\psi$ at two points lying between the bottom and top boundaries of $\beta$. But if so, $\gamma$ cannot intersect the top boundary of $\alpha$ within the slab $\psi$ on both sides of $x$ (since $D \subseteq S$).

We now claim that (the current) $\alpha_{2i}$ lies completely to the left of $D$, and (the current) $\alpha_{2i+1}$ lies completely to the right of $D$ (and, therefore, this is surely true for the initial $\alpha_{2i}$ and $\alpha_{2i+1}$). Therefore, the horizontal distance between the initial $\alpha_{2i}$ and $\alpha_{2i+1}$ is at least $2\kappa \cdot \text{diam}(B)$. The claim is correct since $\alpha$ lies below $\beta$ in $\psi$, $D$ is contained in $\beta \cap \psi$ and $D$ is contained in the region below $\gamma$, and $\overline{\alpha}, \overline{\beta}$ and $\gamma$ are $x$-monotone. If $\alpha$ is above $\beta$, then we consider the triple $\beta_{2i-1}, \alpha_{2i}, \beta_{2i+1}$ and treat this case analogously.

Consider now the sequence $L'_{\text{bot}}$. If $\beta$ is above $\alpha$ we consider the triple $\alpha_{2i}, \beta_{2i}, \alpha_{2i+1}$, and if $\alpha$ is above $\beta$ we consider the triple $\beta_{2i-1}, \alpha_{2i}, \beta_{2i}$. In both cases, we translate $\gamma$ downwards until $S$ just touches the top boundary of the lower object, and essentially continue as for the sequence $L'_{\text{top}}$. We describe in detail the case where $\beta$ is above $\alpha$, so we consider the triple $\alpha_{2i}, \beta_{2i}, \alpha_{2i+1}$ (see Fig. 9). Consider $p$ the vertex corresponding to the first occurrence in the sequence represented by $\beta_{2i}$, and let $S \in S$ be the object to which the arc of $\gamma$ that passes through $p$ belongs. Assume that $\partial S$ enters $\beta$ at $p$, and let $q$ be the first point to the right of $p$ on the bottom curve of $\beta$ where $\partial S$ exits $\beta$. (If $\partial S$ exits $\beta$ at $p$, then we define $q$ to be the first point to the left of $p$ on the bottom curve of $\beta$ where $\partial S$ enters $\beta$, and proceed similarly.)

We now think of $\alpha_{2i}$ as the rightmost intersection point corresponding to it, and of $\alpha_{2i+1}$ as the leftmost intersection point corresponding to it. We translate $\gamma$ rigidly downwards, until $\partial S$ touches $\overline{\gamma}$ at a point $x$, to the right of $p$ and to the left of $q$, and $\partial S$ lies below $\overline{\gamma}$ at a neighborhood of $x$. As above, if $A$ has a tangent at $x$, then so does $S$ and the two tangents coincide. We now distinguish between two cases. If $\text{diam}(B) < \text{diam}(A)$, then at $x$ we may draw a disk of radius $\kappa \cdot \text{diam}(B)$ which is surely contained in $A$ and in $S$. Again we claim that the points $\alpha_{2i}$ and $\alpha_{2i+1}$ are now closer to each other and that they are to the left and to the right of the disk we drew. This means that the horizontal distance between the initial $\alpha_{2i}$ and $\alpha_{2i+1}$ is at least $2\kappa \cdot \text{diam}(B)$. If however $\text{diam}(B) > \text{diam}(A)$, then at $x$ we draw a disk of radius $\kappa \cdot \text{diam}(A)$, which is also contained in $S$ since $\text{diam}(S) > \text{diam}(B)$, and the horizontal distance between the initial $\alpha_{2i}$ and $\alpha_{2i+1}$ is at least $2\kappa \cdot \text{diam}(A)$. □
We now employ Lemma 3.2 to bound the number of vertices that appear when considering the pairs in (c)–(e) above (see Section 3.1). We can immediately apply the lemma to the two pairs in (d), since each object in the set underlying $\gamma_i$ is larger than all objects in $F$, thus we obtain an $O(\lambda_s(|C_\mu| + |D_\mu|))$ bound for these two pairs. Similarly, we can apply the lemma to the pairs in (e). Recalling that the total complexity of the envelopes corresponding to vertical walls in $\rho$ is $O(\lambda_{\infty}(|C_\mu| + |D_\mu|))$, and that $\sum_{|F_\sigma|} = O(|D_\mu|)$, we obtain a bound of $O(\lambda_s(|C_\mu| + |D_\mu|))$ for all the pairs in (e) together.

In order to apply the lemma to the pairs in (b), we first observe that when a pair $(\gamma_i, \gamma_z)$, $i \in \{1, 2\}$, $z \in \{l, r\}$, is considered, we may restrict $\gamma_i$ to the square $\psi$. However, there is still a problem, since it is not true anymore that the larger object (of the two objects forming a countable vertex) always comes from the same underlying set. We thus consider a pair $(\gamma_i, \gamma_z^\sigma)$ twice. First we bound the number of vertices on $\gamma_i$ for which the smaller object (of the two objects forming it) comes from the set underlying $\gamma_z^\sigma$, by applying the lemma, and then we bound the number of vertices on $\gamma_z^\sigma$ for which the smaller object comes from the set underlying $\gamma_i$, again by applying the lemma. In this way we bound all vertices that appear when considering a pair $(\gamma_i, \gamma_z^\sigma)$, and obtain a bound of $O(\lambda_s(|C_\mu| + |D_\mu|))$ for all the pairs in (b) together.

We thus conclude that the number of vertices of $\partial U_\mu$ of type I or II that lie in $\rho$ is $O(\lambda_s(|C_\mu| + |D_\mu|))$, leading as detailed above to the main theorem.

**Theorem 3.1.** The combinatorial complexity of $\partial U$ is $O(\lambda_s(n) \log n)$, for some constant $s$.

4. Conclusion

We have proven that the combinatorial complexity of the boundary of the union of a set of $n$ $\kappa$-curved objects is $O(\lambda_s(n) \log n)$, for some constant $s$. This bound improves the recent bound of Efrat and Sharir [7] for the case of convex $\kappa$-curved objects (e.g., fat ellipses). (They obtained a bound of $O(n^{1+\varepsilon})$ for convex fat objects.) This bound is also the first non-trivial bound for the case of non-convex $\kappa$-curved objects (e.g., rounded heart-shaped objects). A natural question that arises is: is it possible to weaken
the assumption concerning the input objects so that it also holds for convex or non-convex fat polygons, without increasing the bound on the complexity of the boundary of their union.

Acknowledgements

We wish to thank the anonymous referees whose comments and suggestions have helped improve the presentation.

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