STUDENTS' THINKING ABOUT FUNDAMENTAL REAL NUMBERS PROPERTIES

Eustathios Giannakoulias, Alkeos Souyoul, Theodossios Zachariades

Department of Mathematics, University of Athens

This paper presents a part of a research concerning school graduate students' difficulties in their understanding of real numbers and fundamental calculus concepts. Particularly, we focus on the difficulties concerning the identification of rational and irrational numbers, on the importance of the decimal and fraction representation in this process, as well as on the real numbers density. Based on these difficulties, the data analysis suggested a classification of the students into four groups with certain characteristics. Several difficulties seem to persist even after school graduation, while the students have developed interesting thinking strategies.

INTRODUCTION

There are numerus pieces of research that confirm several cognition problems about the real numbers (Zazkis and Sirotic 2004; Moseley, 2005). Particularly, many studies show that students face difficulties in identifying rational and irrational numbers. The distinction between the different categories of numbers remains fuzzy and strongly dependent on their semiotic representations (O' Connor, 2001; Munyazikwiye, 1995). The order and density of real numbers also cause cognitive problems (Merenluoto and Lehtinen 2006; Vamvakoussi and Vosniadou 2006). Most of the studies about the above mentioned difficulties concern elementary or junior high school students (ages 6-15). In this paper we examine school graduate students' comprehension of the structure and the representations of real numbers. Using a methodology, which is based on statistics, we have divided the whole set of students into four groups and we compare the structure of understanding among the groups. One of the main aims of the paper is to compare different levels-degrees of understandings of real numbers.

THEORETICAL BACKGROUND

Real numbers appear in school mathematics education through a process of enrichment of the set of natural numbers. The set of natural numbers expands to the set of the integers in order to include negative numbers. The integers extend to the set of rational numbers so as to provide ratios of integers. Finally irrational numbers join the set of rational numbers and construct the set of real numbers. Each of the sets mentioned above appears normally in a context of a necessary expansion of each set in order to solve problems that the subset cannot interfere with. Every bigger set preserves some of the properties of the subset (but not all) and it has its own new properties. Research on numbers education shows that the process described above hides a number of problems in students' understanding. It has been shown that students' knowledge of real numbers is often highly compartmentalized, and not linked to their broader mathematical knowledge (Moseley, 2005).

Some cognitive problems concerning the number concept arise from the fact that in school mathematics numbers are not (and cannot be) defined in a formal way. Pupils in school instead of a definition for real numbers, have some concept image, in the sense of Tall and Vinner (1981), acting as a definition. An essential component of the fundamental change from elementary to advanced mathematical thinking is described schematically in the following:

Concept image → Definition

Definition → Concept image

Sometimes the formal definition of a concept comes in a later step of the didactical process, after the students have already been familiar with the concept in an intuitive/informal context. In this case the concept image determines the formal definition. On the other hand, in formal mathematics the definition is used to prove the properties of the mathematical concept which it defines. In this case the definition determines the concept. This reversal is an epistemological obstacle which can cause great difficulty (Pinto, Tall 1996).

The number concept image in school mathematics involves multiple representations for numbers such as, points on what is called the "real line", decimals, fractions and some other numbers –the irrationals– that cannot be expressed as fractions. Problems involving the ability to move between different representations of the real numbers are discussed in Zazkis, Sirotic (2004) and Pinto, Tall (1996). In a conceptual change framework (Vosniadou, 1994; Vosniadou and Verschaffel 2004) students form synthetic models when they face problems of rational numbers. The natural numbers' discrete structure usually acts as a barrier when students have to cope with the rational numbers dense structure (Vamvakoussi and Vosniadou, 2004). The counter-intuitive nature of incommensurability and density, seems to cause some of the problems in real numbers' understanding (Fischbein et al. 1995). Incommensurability and density in the real numbers set are considered to have poor intuitive representations. This results to a counter-intuitive nature for the irrational numbers.

Understanding of the real numbers structure is a presupposed knowledge for university mathematics. Students should be familiar with the real numbers in order to face the fundamental calculus concepts. Most of the studies in mathematics education, about numbers understanding, concern primary or junior high school students (6-15 years old). The study presented in this paper focuses on two research questions:

- Do some of the above mentioned problems rersist after school graduation? In particular, we focus on the distinction between the elements of the basic subsets of the real numbers, on the role that the fraction and decimal representation plays in this distinction, as well as on the dense structure of the real numbers set.

- Do school graduates have certain thinking structures about the real numbers and in what extend do these appear?

METHODOLOGY

Data reported in this paper were collected by questionnaires administered to 215 first year students who studied mathematics and had mathematics as a major subject in school. The tests were administered during the students' Calculus course early in their first semester. They had not yet been taught in a university level, the structure of real numbers. So, it is assumed that they answered the questionnaire using their knowledge from school. The questionnaire is part of a larger diagnostic test that we have devised, in order to identify problems that first year mathematics students face in the fundamental calculus concepts.

Students' fully correct responses were marked with 1 and the incorrect responses with 0. The quantitative data analysis was made with the use of latent class analysis (LCA) with categorical variables (Barholomew et al. 2002, Kline, 1998). This analysis, which is part of mixture growth analysis, is a statistical method for finding subtypes of related cases (latent classes) from multivariate data. The results of LCA were used to classify cases to their most likely latent class. That is, given a sample of subsets measured on several variables, one wishes to know if there is a small number of basic groups into which cases fall. The statistical software used for the analysis was Muthen & Muthen Mplus, which is appropriate for discrete variables. More information on the statistical method used can be found in Bartholomew et al. (2002), Muthén (2001), and Muthén & Muthén (2006).

QUESTIONNAIRE

The questionnaire was divided into four parts. The questions are displayed providing also the percentage of correct answers in the parentheses. The first part consisted of four questions asking the students to distinguish the basic subsets of real numbers.

A1. Write a natural number (99.1%)

A2. Write an integer number that is not natural. (96.3%)

A3. Write a rational number that is not integer.(97.2%)

A4. Write an irrational number. (92.6%)

Questions in the second part are related to the order and the density of the real numbers.

Compare the following pairs of numbers.

B1.1 0.999… 0.999 (80,9%)

B1.2	1.888…	1.9	(95,8%)
B1.3	2.999…	3	(10,7%)

In each of the following pairs of numbers write a number lying between them (if such number exists). If there is no such number, write "there is not".

B2.1	0.1	0.11	(71,2%)
B2.2	1.888…	1.9	(58,1%)
B2.3	2.999…	3	(90,2%)
B2.4	1/3	2/3	(86,5%)

B3. Is there any rational number q being greater than 3/5, having the property: 'there is no number between q and 3/5'? If there is such a number, write it. If it does not exist, write 'there is no such number'. (65,5%)

B4. Can you find two real numbers such that there is no other number between them? If you can find a couple of numbers with this property, write the numbers. If you believe that there is not such a couple, write 'there are no such numbers'. (59,5%)

In the third part students have to characterize the following statements as 'true or false'. In this part of the questionnaire the students have to identify five different numbers as real, rational, irrational. For the numbers √2 and 2/3 which are not given in decimal representation the students have to answer three more questions about their decimal representation.

C1.1	√2 is a real number.	(86%)
C1.2	√2 is a rational number.	(95,8%)
C1.3	√2 is an irrational number.	(95,3%)
C1.4	√2 has a decimal representation with infinite decimal digits.	(78,1%)
C1.5	√2 has a decimal representation with finite decimal digits.	(80,9%)
C1.6	√2 does not have a decimal representation.	(84,7%)
C2.1	3.46 is a real number	(96,7%)
C2.2	3,46 is a rational number..	(85,1%)
C2.3	3,46 is an irrational number.	(85,1%)
C3.1	0.78634… is a real number.	(90,2%)
C3.2	0.78634… is a rational number.	(79,1%)
C3.3	0.78634… is an irrational number.	(78,1%)
C4.1	0.777… is a real number.	(89,8%)
C4.2	0.777… is a rational number.	(30,2%)
C4.3	0.777… is an irrational number.	(32,6%)

C5.1 2/3 is a real number. (95,3%)

C5.2 2/3 is a rational number. (75,8%)

C5.3 2/3 is an irrational number. (74,9%)

C5.4 2/3 has a decimal representation with infinite decimal digits. (79,5%)

C5.5 2/3 has a decimal representation with finite decimal digits. (76,3%)

C5.6 2/3 does not have a decimal representation. (93,5%)

The fourth part has two general questions about the decimal representation.

D1 Every real number has a decimal representation. (63,7%)

D2 Every number having a decimal representation is real. (67,9%)

RESULTS

In the modelling process we used a method of successive steps. That is, we tested the model under the assumption that there are two (BIC 6406.705), three (BIC: 6385.987), four (BIC: 6355.086) and five (BIC: 6477.895) groups of subjects. The best fitting model with the smallest BIC was the one involving four groups. The clarity of the classification was indicated by the Entropy summary measure which had its maximum value for the models tested. The average latent class probabilities for the groups are 0.964, 0.985, 0.992 and 0.967 respectively, which enable us to conclude that the four classes are quite distinct, thus indicating that each class has its own characteristics.

We should note that there are many questions in which students have a high percentage of success while there are some questions that ask similar things (for example whether a certain number is rational or irrational). This results to extremely highly correlated variables. In general, the use of such variables should be avoided in LCA as they can result to more classes having no real meaning rather than explain these high correlations. In the present analysis this did not happen and we have not excluded these questions for the following reasons: We do not want to lose some valuable information. Some students think that the rational and irrational numbers are not distinct sets, some other that a real number can be neither rational nor irrational, or finally that a number can have no decimal representation. Furthermore, in our study, LCA was used as an exploration tool and it provided us with a very interesting categorisation which enabled us to focus on certain similarities. We then went back to the individual questionnaires; we examined the similarities provided by the analysis more closely and confirmed their existence. Such a categorisation was not achieved by omitting some of the questions.

Table 1 displays a summary of the results for the class analysis. Each of the columns represents a group and each of the rows represents a question. The number of each cell displays the probability that a student in a particular group answers correctly in the corresponding question. We have to note that this is not equal to the fraction of the students in the group that have answered correctly the corresponding question although we will treat them as such. This happens because the model class for the latent class patterns is based on estimated posterior probabilities and does not result to integer counts for the groups. For example, based on the most likely latent class membership, the first group consists of 63 students, while the model class count for this group is 62.269. Low performance (0-50%) is displayed in grey, average performance (50%-75%) in normal and high performance (75%-100%) in bold.

We first present a description of the main characteristics of each group. The first group consists of 63 students and has the highest performance among the groups. They answer correctly in most of the questions of the first part, something that does not happen for the second part where a problem with questions B3 and B4, related to the density, is observed. Question B1.3 is the most difficult question in the whole questionnaire. In this question the first group has significantly higher score than the other groups, but it still remains very low. Most of the students in the first group identify as real all the numbers of the third part (C1.1, C2.1, C3.1, C4.1 and C5.1). They recognise that whenever a number can be turned into fraction of integers it is a rational number (C5.2 and C5.3). They also know that whenever a number is given in decimal representation it can be turned into fraction if the decimal part has a recurring pattern in its decimal part (C2.2, C2.3, C3.2, C3.3, C4.2 and C4.3). The lowest performance is observed in questions C3.2 and C3.3 where 9 of the 20 students who answered this question wrong, gave the response that they do not know whether 0.78634… is ra-

	group1	group2	group3	group4
A1	**1**	**0,986**	**1**	**0,972**
A2	**1**	**1**	**0,978**	**0,805**
A3	**1**	**1**	**0,955**	**0,889**
A4	**0,966**	**0,976**	**0,912**	**0,771**
B1.1	**0,781**	**0,916**	0,733	0,74
B1.2	**0,9**	**0,976**	**0,978**	**1**
B1.3	0,244	0,067	0,067	0
B2.1	**0,807**	0,738	0,671	0,544
B2.2	0,603	0,638	0,551	0,47
B2.3	**0,889**	**0,918**	**0,979**	**0,798**
B2.4	**0,878**	**0,898**	**0,841**	**0,808**
B3	0,694	0,697	0,65	0,515
B4	0,7	0,59	0,56	0,47
C1.1	**0,935**	**0,961**	**0,801**	0,604
C1.2	**1**	**1**	**1**	0,749
C1.3	**1**	**1**	**1**	0,722
C1.4	**0,827**	**0,825**	**0,934**	0,426
C1.5	**0,914**	**0,852**	**0,868**	0,47
C1.6	**0,796**	**0,916**	**0,893**	0,735
C2.1	**1**	**1**	**0,935**	**0,887**
C2.2	**0,903**	**0,878**	**0,87**	0,684
C2.3	**0,903**	**0,878**	**0,848**	0,712
C3.1	**0,983**	**1**	**0,846**	0,637
C3.2	0,716	**1**	**0,974**	0,272
C3.3	0,684	**1**	**0,974**	0,272
C4.1	**0,983**	**1**	**0,846**	0,609
C4.2	**0,848**	0,019	0	0,301
C4.3	**0,91**	0,007	0,023	0,328
C5.1	**1**	**1**	**0,89**	**0,859**
C5.2	**0,984**	**1**	0	**0,83**
C5.3	**0,984**	**1**	0	**0,774**
C5.4	**0,871**	0,707	**0,978**	0,613
C5.5	**0,84**	0,679	**0,916**	0,605
C5.6	**0,933**	**0,93**	**0,956**	**0,923**
D1	0,615	0,574	**0,8**	0,599
D2	**0,85**	**0,781**	0,516	0,383

Table 1

tional or irrational as there is not enough information provided in order to decide. The didactical contract suggests that if there was some recurring pattern, it should have shown before the dots "…". They however support that there could be a recursion in the digits that are not shown on the paper. Questions of the fourth part were posed in a theoretical way and in general they seem to confuse many of the students, even those of the first group. There is a subgroup of about 40% of the students in this group who support that real numbers which do not have a decimal representation exist.

The second group consists of 72 students. In the first two parts they have a performance similar to the first group. They too have difficulties with the questions regarding density. Students of the second group also identify as real the numbers of the third part (C1.1, C2.1, C3.1, C4.1 and C5.1). However they have a problem in rational-irrational identification. Although they recognise as rational the numbers that can be turned into fraction (C5.2 and C5.3), they additionally use another incorrect criterion when they have to decide whether a given number is rational or irrational from its decimal representation. For the students of this group a number is rational if it has a finite number of decimal digits different from 0. So, $\sqrt{2}$, 0.78634… and 0.777… are all irrational numbers as they have infinite decimal digits (C1.2, C1.3, C1.4, C1.5, C2.2, C2.3, C3.2, C3.3, C4.2, and C4.3). An interesting point is that about 30% of these students do not even make the division 2/3 in order to answer questions C5.4 and C5.5. They know that 2/3 is rational so they conclude that it has finite decimal digits. The rest of them consider 2/3 as rational although they can see that it has infinite decimal digits. This means that **the fraction criterion is stronger than the infinite digit criterion**. As we will see, this is the major difference from the third group. The performance of Group 2 in the fourth part is slightly poorer that that of group 1, where there were also many students who supported that real numbers which do not have a decimal representation exist.

The third group counts 45 students. In part A students do not have any particular problem. In part B, students answer in the same way with the students of the second group, facing difficulties with the density questions. Most of the students in this group identified the third part numbers as real (C1.1, C2.1, C3.1, C4.1 and C5.1). They identified as rational, numbers those that have finite decimal digits different than 0 (C2.2 and C2.3). They also identified as irrational numbers, those that have infinite decimal digits different than 0, without checking recurrence (C1.2, C1.3, C1.4, C1.5, C3.2, C3.3, C4.2 and C4.3). Finally they concluded that 2/3 equals 0.666… (C5.4, C5.5 and C5.6) and therefore 2/3 is an irrational number (C5.2 and C5.3). Students in this group believe that rational numbers can be written with finite decimal digits, while the decimal representation obscures the fraction representation. The difference from group 2 lies in the fact that in the third group **the digit criterion dominates the fraction criterion**. Students in the third group have the highest score in D1 and this is fully compatible with their answers in questions of the third part, as decimal representation plays a major role in their concept image for real numbers. On the contrary, half of these students believe that there are numbers having decimal representation which are not real numbers. (D2).

The fourth group has 35 students who in general have lower performance than the students in the other groups. This can be seen even in the easiest questions contained in part A. This group also has lower performance in most of the questions of part B. A high percent of the students of the fourth group have a different view of real numbers. They identify as real only the numbers which have a finite number of decimal digits different than 0 (C2.1, C5.1, C5.4 and C5.5). Numbers which have infinite decimal part are identified as not real (C1.1, C3.1 and C4.1). We can also remark that they identify the set of rational numbers with the set of real numbers (questions 2 and 3 in C1, C2, C3, C4 and C5). This is compatible to the group's low percentage of success in question A4. Students' answers in question C5 also show that 2/3 is considered rational with finite decimal digits. This resembles the second group's view, as the fraction criterion also prevails over the digit criterion. The fourth group has low scores in the fourth part. Their percentage of success in D2 is very low and this agrees with their answers in the third part of the questionnaire, as they do not identify irrational numbers as real (although they have a decimal representation). Students in the fourth group are also uncertain about whether real numbers having no decimal representation exist (D1).

DISCUSSION

The identification of rational-irrational numbers appears to be difficult for the majority of the study subjects. Only students of the first group check for recurring digits when they are asked to identify a number in decimal representation. Although all of the students have been taught a general way to convert a rational from decimal to fraction representation, this knowledge (if such exists) remains unconnected to the real numbers structure.

There is also an interesting remark for the whole sample. A 45% of all the students fall in a contradiction by giving the following combination of answers:

- 2.999… is less than 3 (an expected answer for question B1.3). So 2.999… is different than 3.
- In question B2.3 they answer correctly that there is no number between 2.999… and 3.
- Finally, in question B4 they also answer correctly that there is no pair of numbers having no number between them, instead of giving 2.999… and 3.

This contradiction is regardless of the students' distribution into each of the groups. The percent of students falling into contradiction for each group are 44.4%, 47.2%, 45.1% and 34.3% respectively.

Another interesting point comes from the identification criteria priority. Students in groups 2 and 3 use the finite digits criterion in order to identify rational numbers. In group 2 the fraction criterion dominates over the digits criterion while in group 3 the fraction representation is not adequate to guide the identification. A high percentage of the students in group 4 consider as real numbers only those with finite decimal digits. This leads them to exclude irrational numbers from the real numbers structure. Throughout school mathematics, there is no formal definition for the real numbers set. Taking into account Greek school textbooks, school graduates' concept image for the real numbers is expected to be the set of numbers in decimal representation. Questions D1 and D2 show that there is a general difficulty with decimal representation of real numbers. Only 39% of the sample has answered correctly in both D1 and D2. Question B3 is closely connected to the density of the rational numbers. Question B4 tests students' knowledge on real numbers density. A 42.3% of the students have answered to both of these questions correctly. Density is highly related to the understanding of the decimal representation's structure. However, only 18% of the students have answered correctly to all of the questions B3, B4, D1 and D2. The 48.7% of these students is in the first group, 30.8% in the second, 15.4% in the third and 5.1% in the fourth. This shows that students from the first group have a better understanding of the real numbers structure.

Concluding, we argue that several problems remain in graduate students concerning the rational-irrational numbers identification as well as the real numbers density. The importance of the decimal or fraction representation in the process of identification provided a classification of students into groups with several unique characteristics. On the other hand some other problems on real numbers structure appear unrelated to this classification.

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