GROUP THEORETIC DESCRIPTION OF JUST INTONATION

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ABSTRACT

In this paper we present a group theoretic description of just intonation. All possible musical intervals belonging to 5-limit just intonation can be represented by the mathematical group $\{2^p3^q5^r | p,q,r \in Z\}$. Considering only the intervals within one octave, an isomorphism with the 2-dimensional lattice Z^2 can be made. Plotting the intervals on the lattice according to the isomorphism, we can identify certain connected and convex sets of elements that represent the major and minor diatonic scale and the 12-tone chromatic scale. These sets of elements remain convex and the area each set spans remains invariant under lattice transformations. The fact that the major, minor and chromatic scale arise naturally from our representation of 5-limit just intonation triggers discussion about the origin of scales and might suggest that those scales have a mathematical origin. With this, we challenge the framework of the 12-tone system.

Our representation of 5-limit just intonation is compared with Balzano's `thirds-space' in which he builds the intervals of the 12-tone scale from major and minor thirds also using group theory. This comparison gives a new insight on the compromises regarding intonation that are made when introducing a 12-tone temperament.

1. INTRODUCTION

Any musical interval can be expressed in terms of a frequency ratio. Since any positive integer a can be written as a unique product $a = p_1^{e_1} p_1^{e_2} \dots p_n^{e_n}$ of positive integer powers e_i of primes $p_1 < p_2 < \dots < p_n$, any frequency ratio can be expressed as

$$2^p 3^q 5^r \dots \qquad p,q,r \in Z . \tag{1}$$

For example $2^{-1}3^1$=(3/2) represents a perfect fifth and $2^{-2}5^1$ (=5/4) a major third. Tuning according to whole number ratios is referred to as just-tuning. If the highest prime that is taken into account in describing a set of intervals is n, then we speak about just intonation to the n limit.

In this paper we will use mathematical group theory to describe just intonation and our main focus is on just intonation to the 5-limit. We will make a representation of all intervals in this intonation within one octave, from which we will derive the major and minor diatonic scale as well as the chromatic scale. We present this group theoretic description of just intonation systems as a realistic and natural framework of music, and with this we challenge the framework of the 12-tone system. Since this 12-tone system has become so familiar in Western music, it is sometimes even used by music theorists as a framework to explore the resources of a pitch system like the diatonic scale and modulation possibilities ([1], [8]). We compare our representation with Balzano's group theoretic description of a 12-fold system.

The paper is organized as follows. In section 2 the basics of mathematical group theory are explained and 5-limit just intonation is described by the group P_3. The intervals of 5-limit just intonation within one octave can be plotted on a 2-dimensional lattice from which several scales can be derived. In section 3 the lattice is plotted with respect to different basis vectors, and then compared to the note name system and Balzano's representation of a 12-tone system.

2. GROUP THEORETIC AND GEOMETRIC INTERPRETATION OF JUST INTONATION

Considering powers of the first two primes, we can create all positive numbers and ratios that are not multiples of 5 or a higher prime.

$$2^p 3^q = \{1,2,3,4,6,\dots,\frac{3}{2},\frac{4}{3},\frac{9}{8},\dots\} \qquad p,q \in Z \tag{2}$$

We can write:

$$2^p 3^q = 2^{p+q}(\frac{3}{2})^q = 2^u(\frac{3}{2})^v \tag{3}$$

with u=p+q and v=q . If we consider all numbers resulting from equation (2) to be frequency ratios, we see now that those intervals can all be built from a certain number of octaves (2/1) and fifths (3/2). This is called Pythagorean tuning, and is a special form of just intonation. This set of numbers (2) together with the operation of multiplication is a mathematical group. Definition: A group consists of a non-empty set G and a binary operation on G (usually written as composition with the symbol $\circ$) satisfying the following conditions:

- The binary operation is associative: $(x \circ y) \circ z = x \circ (y \circ z)$ for any $x, y, z \in G$.

- There is a unique element $e \in G$, called the identity element of G, such that $x \circ e = x$ and $e \circ x = x$ for any. $x \in G$

- For every $x \in G$ there is a unique element $x^{-1} \in G$, called the inverse of x, with the property that $x \circ x^{-1} = x^{-1} \circ x = e$.

Using group theory we can make an abstraction of several tuning systems. We will call (2) the group P_2 (since all elements are powers of the first two primes) and write:

$$P_2 = \{\, 2^p (\frac{3}{2})^q \mid p, q \in Z \,\}. \tag{4}$$

The group has unit element 1 and the inverse of an element a is a^{-1}. It is an abelian group (i.e. the elements commute) with an infinite number of elements.

The group P_3 (taking into account the first three primes) can be defined by $\{2^p 3^q 5^r | p,q,r \in Z\}$ or, equivalently by

$$P_3 = \{\, 2^p (\frac{3}{2})^q (\frac{5}{4})^r \mid p, q, r \in Z \,\}. \tag{5}$$

so that the elements can be seen as all intervals built from octaves (2/1), perfect fifths (3/2) and major thirds (5/4). More groups like this can be defined, taking into account multiples of 7 and higher primes. All groups represent a form of just tuning. Usually, this is referred to as 'just tuning to the 5-limit' for P_3, 'just tuning to the 7-limit' for P_4 and so on (always referring to the highest prime used). Every defined group is a subgroup[1] of the group which takes into account higher primes.

$$\{2^p\} \subset \{2^p 3^q\} \subset \{2^p 3^q 5^r\} \subset \ldots \qquad p, q, r \in Z \tag{6}$$

$$P_1 \quad \subset P_2 \qquad \subset P_3 \qquad\qquad \subset \ldots \tag{7}$$

In this paper our focus is on P_3 which represents the intervals in 5-limit just-intonation. First of all, we notice that the group P_3 is isomorphic to the group Z^3. For the proof, see appendix A. Hence we can represent the elements of P_3 in a 3-dimensional lattice labeled by the elements of Z^3. For simplification however, we want to consider only the intervals lying within one octave. That means, considering the elements of P_3 lying within the interval [1,2). To accomplish this, we make a map $\varphi : P_3 \rightarrow Z^2$ that divides P_3 in equivalence classes of (the same) intervals over all octaves. This means for example that 3/2 is in the same equivalence class as 6/2, 1/2, 12/2 and so on. The map is given by

$$\varphi : 2^p (\frac{3}{2})^q (\frac{5}{4})^r \rightarrow (q, r) \tag{8}$$

This map is a group homomorphism since

$$\varphi(xy) = \varphi(x)\varphi(y) \quad \text{for all} \qquad x, y \in P_3 \tag{9}$$

The kernel of the map is $\{2^p | p \in Z\}$ which are the elements that are projected on the unit element (0,0) of Z^2. The quotient-group

$$\hat{P}_3 = \frac{P_3}{\{2^p \mid p \in Z\}} \tag{10}$$

is the group of cosets[2] of the subgroup $P_1 = \{2^p | p \in Z\}$. We write

$$\hat{P}_3 = \frac{P_3}{P_1} = \{ xP_1 \mid x \in P_3 \} \tag{11}$$

which means that every element of $\frac{P_3}{P_1}$ is an equivalence class of elements $\{2^p (\frac{3}{2})^\alpha (\frac{5}{4})^\beta \mid \alpha, \beta$ fixed, $p \in Z\}$. From every coset we can choose one representative that lies within the interval [1,2) (one octave). The group $\frac{P_3}{P_1}$ is isomorphic to Z^2:

$$\frac{P_3}{\{2^p \mid p \in Z\}} \cong Z^2 \tag{12}$$

since the map φ is a homomorphism and $\{2^p | p \in Z\}$ is its kernel. Figure 1 shows the representatives of the elements of $\hat{P}_3$ ordered according to the 2-D lattice of Z^2.

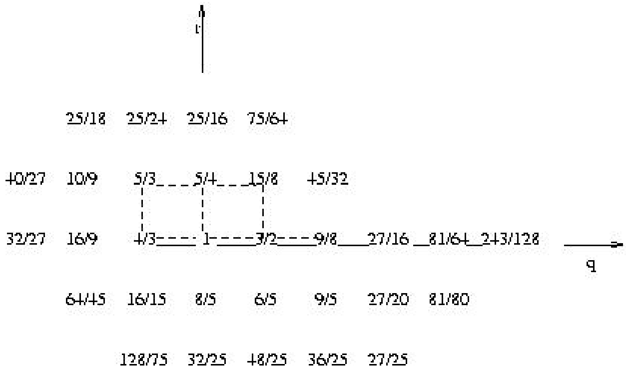

Figure 1: Space of thirds and fifths. Major scale in Pythagorian tuning indicated by lines. Major scale in just-intonation indicated by dashed lines.

[1] A subgroup H of G is a subset of G which itself forms a group under the composition law of G.

[2] Given a subgroup H of a group G, the (left) coset of an element g ∈ G, written gH is defined as the set of elements obtained by multiplying all the elements of H on the left by g.

The lattice is unbounded but only part of it is shown. Notice that (for r=0) on the q-axis the (representatives of the) group $\hat{P}_2 = \frac{P_2}{P_1}$ can be found. In this figure two important scales can be found. The major scale in Pythagorean tuning is represented by the seven elements connected by lines (figure 1). An often used definition for the major scale in just intonation to the 5-limit is the scale in which each of the major triads I, IV and V is taken to have frequency ratios 4:5:6 (see for example [3]). Table 1 shows the ratios of the just major scale with respect to the fundamental.

Note	do	re	mi	fa	so	la	ti	do
Ratio	1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Table 1: Frequency ratios between the different notes of the major scale and the fundamental 'do'.

In figure 1 these ratios are indicated by dashed lines connecting the points.

If we consider the area spanned up by these points we conclude that this is a convex area. We will use the following definition of a discrete convex set: Consider a lattice L. The set S is a convex subset of L if:

$$\forall x, y \in S \qquad \forall \alpha \in [0,1]$$

$$\alpha x + (1-\alpha) y \in L \Rightarrow \alpha x + (1-\alpha) y \in S \tag{13}$$

meaning that, if drawing a line between any two points of the set, the points laying on that line, should also be in the set. Therefore, the points in Z^2 labeled by the ratios from the major scale are a convex subset of Z^2, as well as the points labeled by the ratios of Pythagorean tuning. Since convexity of a set is a special property we might wonder if this could tell us anything about these scales. We will later come back to this.

We see from figure 1 that all intervals (in the 5-limit just-intonation) can be built from perfect fifths and major thirds (and transposing octaves back). The q-axis represents the number of perfect fifths, the r-axis the major thirds. From the figure we can for example see that two perfect fifths and one major third up (and one octave down) gives an augmented fourth of ratio 45/32. Interval addition can be seen here as vector addition (all vectors with their origin in $(0,0) \in Z^2$). In figure 2 is shown that a perfect fifth added to a major seventh results in an augmented fourth.

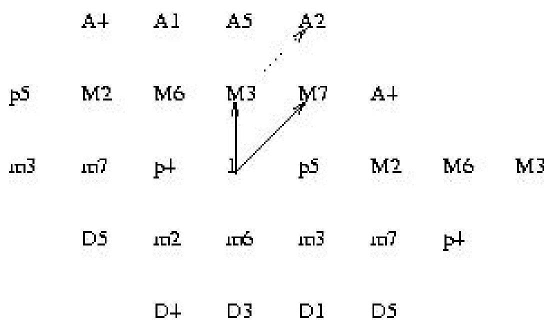

Figure 2: Space of intervals, the letters p, M, m, A and D mean perfect, major, minor, augmented and diminished respectively.

Putting figure 2 on top of figure 1, it becomes clear which frequency ratios belong to which intervals. Note that the inverses of the intervals lie exactly at the other side of the center 1 (point symmetry).

3. DIFFERENT REALIZATIONS OF 'THIRDS SPACE'

In this section we will show the representation of a different isomorphism between $\hat{P}_3 = \frac{P_3}{P_1}$ and Z^2 and concentrate on several scales which appear in this representation as special subsets. We will compare our representation with the one Balzano ([1]) made using equal temperament in a group theoretic description as well as with the familiar note name system. This will result in a new view on the compromises made when our note name system and when a 12 tone temperament was introduced.

There are several ways to make an isomorphism between $\hat{P}_3 = \frac{P_3}{P_1}$ and Z^2. The group Z^2 has generating subset {(1,0), (0,1)}. This means that all elements from Z^2 can be represented as linear combinations of basis vectors (1,0) and (0,1). In figure 1 these elements (or vectors) are associated with the intervals 3/2 and 5/4 meaning that all intervals in this figure can be built from perfect fifths and major thirds (and octaves, to get back to the representatives from the coset that lie in the interval [1,2)). These generating elements are (here) not unique. We can for example write:

$$2^u 3^v 5^w = 2^{u+v+2w} (\frac{5}{4})^{v+w} (\frac{6}{5})^v = 2^k (\frac{5}{4})^l (\frac{6}{5})^m \tag{14}$$

with $u,v,w,k,l,m \in Z$ and $k=u+v+2w$, $l=v+w$, $m=v$, such that $(u,v,w) \Rightarrow (k,l,m)$ is a bijective map (i.e. a one to one correspondence), proving that the intervals from P_3 can also be built from octaves (2/1), major thirds (5/4) and minor thirds (6/5). One could ask what all possibilities for 'building blocks' for P_3 would be. This problem is equivalent to finding all possibilities of generating subsets of Z^2. It turns out that the area of the parallelogram made by the two (alternative) basis vectors (or generating elements) should be one, in order to be able to represent every element of Z^2 as a linear combination of those vectors. This is equivalent to saying that the determinant of the matrix with the basis vectors as columns should be one. For the proof, see appendix B. There exists an infinite number of possibilities to choose a basis of Z^2. By choosing the map given by (8) we accomplished that the basis vectors corresponded to the perfect fifth and major third. This choice is motivated from the fact that these intervals (which together form a major triad) are seen as the building blocks of western music. Using the same argument but from the perspective that triads are built from major and minor thirds, we can just as well choose the map:

$$\varphi : 2^p (\frac{5}{4})^q (\frac{6}{5})^r \rightarrow (q, r) \tag{15}$$

In figure 3 the space, now constructed from major and minor thirds is shown. We will refer to this as the thirds space.

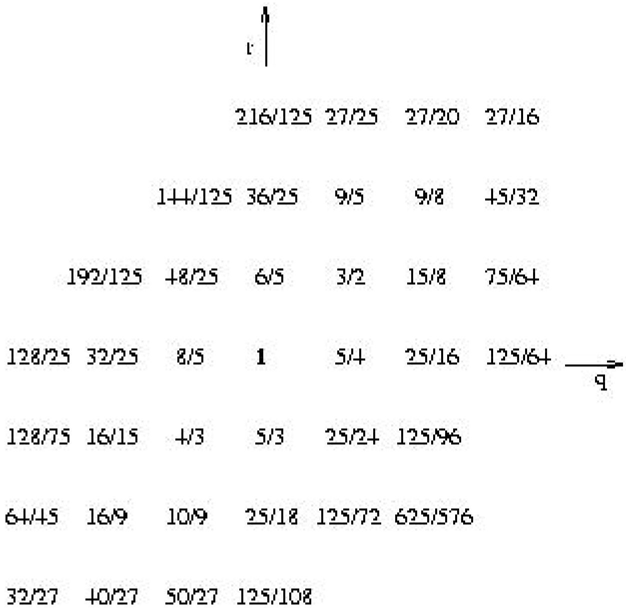

Figure 3: Space of major and minor thirds.

Now we want to compare this figure with the notes associated with it. Considering all intervals with respect to C (which is corresponding to $1 \in P_3$) we obtain figure 4. We can immediately see that our note name system is not sufficient to name every single interval. From figure 3 we see that, going three major thirds up and four minor thirds up (and two octaves down), we are not back on the note we started from - which is suggested by figure 4 - but one syntonic comma (81/80) higher (for information on commas, see for example [6]). Therefore, in figure 4 every note differs one or more syntonic commas from another note with the same name. For example, comparing the ratios 9/8 and 10/9, both corresponding to a D in figure 4 gives us a difference in pitch of 9/8x10/9=81/80 which is equal to 21.51 cents[3].

[3] An interval x/y has a width of 1200 $^2\log(x/y)$

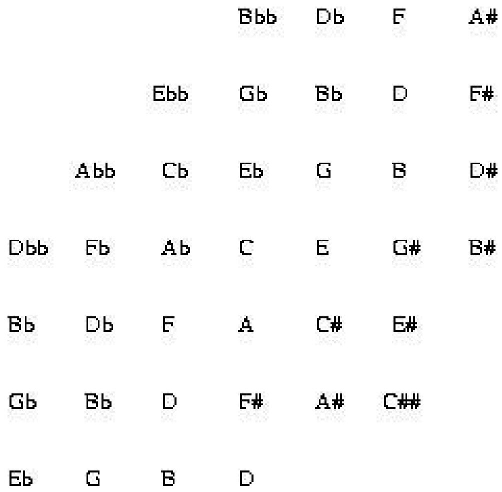

Figure 4: Note names in the space of major and minor thirds.

Let us compare figure 3 with the figure Balzano obtained by describing notes in 12-tone (equal) temperament in a cyclic group [1]. The octave divided into 12 semitones can be described by the set {0,1,2,3,4,5,6,7,8,9,10,11}. This is a group by defining the binary operation as addition, the identity element as 0 and the inverse of an element n as 12-n. This group is called C_{12} and is a cyclic group. Balzano showed that there exists an isomorphism between C_{12} and the direct product of its subgroups C_3 ={0,4,8} (augmented triad; built from major thirds) and C_4 ={0,3,6,9} (diminished seventh chord; built from minor thirds):

$$C_{12} \cong C_3 \times C_4 . \tag{16}$$

The resulting 'thirds-space' is shown in figure 5.

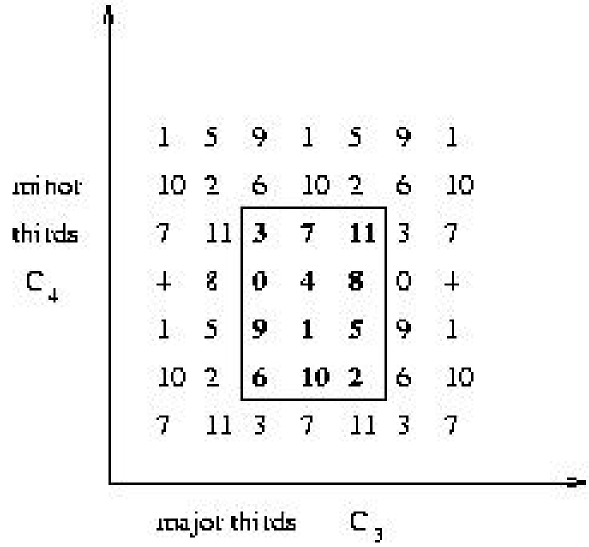

Figure 5: representation of Balzano's thirds-space: $C_3 \times C_4$.

3.1 Comparison and Interpretation

All three figures 3, 4 and 5 show different realizations of the same structure: one step to the right means a major third up, one step up means a minor third up. Our frequency ratios space is an infinite space to all directions; the note-name space can be rolled up in one direction by identifying corresponding note names with each other; the C_3 x C_4 space can be rolled up in two directions (along the sides of the square (see figure 5), to become a torus) because the 12-tone system treats enharmonically equivalent notes as the same element. The four 2's just inside and outside the square in figure 5 represent the same note in the C_3 x C_4 space, but four different frequency ratios in the frequency ratio space (figure 3). It surprisingly turns out that two of these frequencies differ more than one semitone from each other. The four frequency ratios corresponding to the four 2's are, from left to right, from top to bottom: 144/125, 9/8, 10/9, 625/576. The note names of these frequency ratios (compared to C) are Ebb, D, D, C## respectively. We just saw that the two D's differ one syntonic comma (=21.51 cents) from each other. The difference in cents between Ebb and D (10/9), and between D (9/8) and C## is 62.57 cents. The difference between Ebb and D (9/8), and between D (10/9) and C## is 41.06 cents. Finally the difference between Ebb and C## is 103.62 cents, which is more than one (equal tempered) semitone! That means two equal elements from Balzano's diagram can refer to two elements from 5-limit just intonation that differ 103.62 cents from each other.

Note that in the thirds-space (fig. 3, 4) the Ebb , D, D and C## lie relatively close to one another, while for example the familiar comma of Pythagoras, which measures only 24.07 cents though also audible, doesn't even fit in the figures. (For more reading on commas, see for example [6].) The syntonic comma shows here that the note name system is not as accurate as needed. Furthermore, the difference between enharmonically equivalent notes shows that the (equal) tempered 12-tone system is sometimes a pretty bad approximation to just-tuning. Of course, this is not new information, but these three realizations of ordered musical intervals (figures 3, 4, 5) can give a different insight on the compromises that are made introducing note names or equal temperament. In all temperaments compromises have been made, since not all intervals from just intonation can be included. Equal temperament represents the intervals from the just tuned major scale (table 1) by an approximation acceptable for the ear. However, it can happen in music that consecutive notes (not necessary in the same voice) are enharmonically equivalent. Using equal temperament, they are played as the same note, in just intonation the notes differ at most 103.62 cents from each other as is illustrated by figures 3 and 5. This places the acceptable equal temperament in another daylight.

Balzano found the diatonic major scale as connected region in figure 5 as a special property of the thirds-space. In our space of major and minor thirds we find a similar structure. Taking the ratios for the just major scale as written in table 1, one can see that these ratios form a connected region indicated by thick lines (figure 6), in the same position Balzano found the major scale in figure 5. Using definition (13), we see that the major scale is a convex set as well. Calculating also all intervals (in frequency ratios) with respect to every other note in table 1 (for example the interval between so and la in this table is 10/9), many more intervals are obtained. Precisely these intervals can be found in a larger connected and convex set in figure 6, indicated by lines.

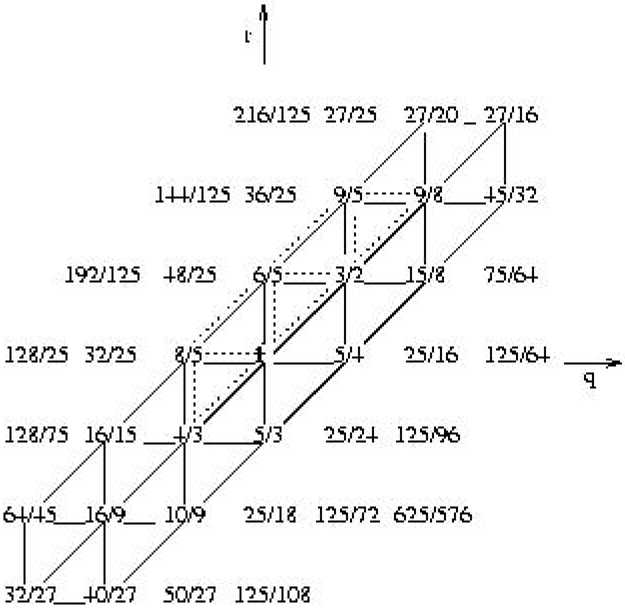

Figure 6: Space of major and minor thirds. See text for details.

The minor scale appears as a connected and convex set as well. Taking as a definition for the (natural) minor scale in just intonation the scale in which each of the minor triads I, IV and V have frequency ratios 10:12:15 (that is1:6/5:3/2), the intervals with respect to the fundamental are: 1, 9/8, 6/5, 4/3, 3/2, 8/5, 9/5, 2. We can find these intervals in figure 6 in a connected region indicated by dashed lines. Calculating all intervals (with respect to every note in the scale) of the minor scale, we can find those ratios as a connected region in figure 6, it is the same region as the one resulting from the major scale. The harmonic minor scale requires only a change of the interval 9/5 to 15/8, the melodic minor scale requires a change of 8/5 to 5/3 as well. Both scales stay convex sets in the thirds space. The chromatic scale in just intonation as defined according to [7] and found in most textbooks[4] can also be found as a connected and convex set in the thirds space, see figure 7.

[4] Sometimes the minor seventh in the chromatic scale is defined as 16/9 instead of 9/5. In both cases the resulting set is convex in our representation.

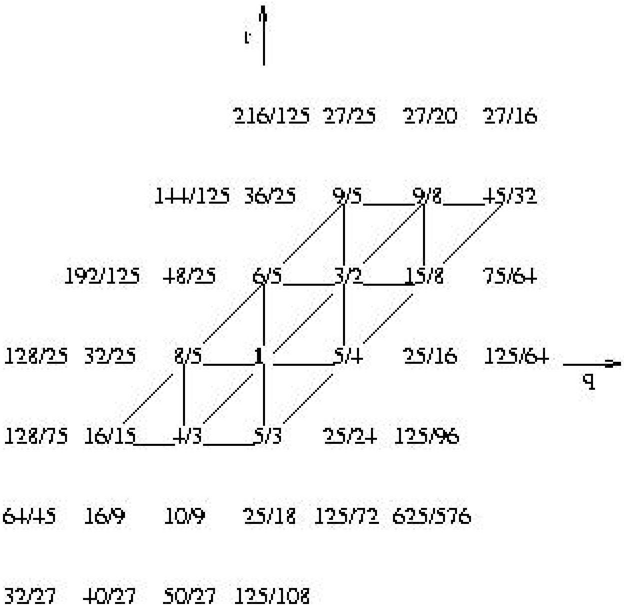

Figure 7: Space of major and minor thirds.

This might be a reason for many researchers and musicians to believe that 12 tones are a special set, and a possible explanation for the existence of the 12-tone chromatic scale.

We have shown that other bases can be chosen to represent the same set. It is important to find out whether a convex set in one basis is still a convex set after a basis transformation as, otherwise, convexity of a set would not be a special property. Using the definition for a convex set as given in equation (13), and applying a linear transformation T:

$$T(\alpha x + (1-\alpha) y) = \alpha T(x) + (1-\alpha) T(y), \qquad (17)$$

shows that the transformed set is still a convex set. This means that all the convex regions we found in our thirds space are convex regions in any space that is created by a basis transformation applied to the thirds space. The area of these convex sets is also invariant under basis transformations since every convex set can be split into a finite number of triangles and the area of an arbitrary triangle is invariant under basis transformations when the determinant of the transformation matrix is one[5]. These two special properties of a set, convexity and invariance of area put into question the origin of the scales described by these sets. A lot of research has been done on the origin of scales, see for example [2]. However, there is not a unique explanation as to how scales have developed which is widely accepted. Balzano ([1]) finds the diatonic scale as an unique pitch set emerging from 12 tone temperament. In the same way, we find here the diatonic scale now emerging from just intonation. More specifically, we find the diatonic major and minor scale, and all intervals appearing in these scales in connected and convex sets (figure 6). The chromatic 12 tone scale, which Balzano uses as a structure from which to derive the diatonic scales, also appears as a special set from within a system of just intonation (figure 7).

4. CONCLUSIONS

In this paper we presented a group theoretic description of just intonation. In studying 5-limit just intonation we considered only the intervals within one octave and could therefore create an isomorphism with Z^2. The intervals from 5-limit just intonation were plotted on a two dimensional lattice and from that representation we derived the major and minor diatonic scales and chromatic 12-tone scale in a connected and convex set. We proved that these sets would still be convex and would span up an equal area if a basis transformation of the lattice would be applied. This might suggest that both the diatonic scales and the chromatic scale originate from just intonation.

The lattice representation of 5-limit just intonation, with major and minor thirds as generating elements, was referred to as thirds space. We compared our thirds space with the thirds space Balzano created using 12 notes in an octave, and with the familiar note name system. Although the representation of note names is an unbounded region as well, it is not able to describe the notes of just intonation accurately. The comparison between the just intonation system and Balzano's system could give a new insight as to the compromises in intonation made by introducing a 12 tone system. One element in Balzano's thirds space refers to at least four[6] different frequency ratios from P_3. This indicates that two notes that are the same in a 12-tone system, can differ as much as 103.62 cents from each other.

APPENDIX A: ISOMORPHISM BETWEEN P_3 AND Z^3

Since Z is a group under addition, the set Z^3 can be represented as a three dimensional space of all points (p,q,r) where $p,q,r \in Z$, and is a group under vector addition with unit element (0,0,0). The group operation is vector addition, which means

$$(p,q,r) \circ (p',q',r') = (p+p', q+q', r+r') \qquad (19)$$

The group P_3 is isomorphic with Z^3. This means that there is a one to one correspondence between the elements of the groups. The isomorphism is given by the map ϕ:

[5] This can easily be verified by using the formula for the area A of a triangle given its coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3):

$$A = \frac{1}{2} Det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

[6] There are 12 elements in Balzano's group C_{12} whereas P_3 has an infinite number of elements. In the text we only pointed out four different ratios corresponding to one element in C_{12}, but there are many more.

$$\phi:(p,q,r)\in Z^3 \leftrightarrow (2^p(\frac{3}{2})^q(\frac{5}{4})^r)\in P_3 \qquad (20)$$

To prove that the two groups are isomorphic to one another we have to show that the map ϕ is a group homomorphism:

$$\begin{aligned}&\phi((p+p',q+q',r+r'))=\\&\phi((p,q,r))\bullet\phi((p',q',r'))\end{aligned} \qquad (21)$$

(where $\bullet$ is the group operation in P_3), and prove that ϕ is injective:

$$\begin{aligned}&(a,b,c),(d,e,f)\in Z^3:\\&\phi((a,b,c))=\phi((d,e,f))\Rightarrow\\&(a,b,c)=(d,e,f)\end{aligned} \qquad (22)$$

and surjective:

$$\forall y\in P_3,\exists(a,b,c)\in Z^3:\phi((a,b,c))=y \qquad (23)$$

First we prove that ϕ is a homomorphism:

$$\begin{aligned}&\phi((p+p',q+q',r+r'))\\&=2^{p+p'}(\frac{3}{2})^{q+q'}(\frac{5}{4})^{r+r'}\\&=2^{p'}(\frac{3}{2})^{q}(\frac{5}{4})^{r}\bullet 2^{p'}(\frac{3}{2})^{q'}(\frac{5}{4})^{r'}\\&=\phi((p,q,r))\bullet\phi((p',q',r'))\end{aligned} \qquad (24)$$

We prove injectivity of the map ϕ by the knowledge that every element from P_3 can be written as a unique product of the first three primes.

$$\begin{aligned}&\phi((a,b,c))=\phi((d,e,f))\Rightarrow\\&2^a(\frac{3}{2})^b(\frac{5}{4})^c=2^d(\frac{3}{2})^e(\frac{5}{4})^f\Rightarrow\end{aligned}$$

$$2^{a-d}(\frac{3}{2})^{b-e}(\frac{5}{4})^{c-f}=1\Rightarrow(a,b,c)=(d,e,f) \qquad (25)$$

We prove surjectivity of the map ϕ by the definition of an element from P_3. The elements in P_3 are defined as $\{2^p(\frac{3}{2})^q(\frac{5}{4})^r \mid p,q,r\in Z\}$. So there is always an element $(p,q,r)\in Z^3$ such that

$$\phi((p,q,r))=2^p(\frac{3}{2})^q(\frac{5}{4})^r.$$

APPENDIX B: ALTERNATIVE BASES FOR Z^2

The lattice Z^2 is a subgroup and discrete subspace of the vector space R^2. They share the same basis: $e_1=(1,0)$, $e_2=(0,1)$. We can choose another basis for Z^2:

$$\{\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}\mid a,b,c,d\in Z, Det\begin{pmatrix}a&c\\b&d\end{pmatrix}\neq 0\} \qquad (26)$$

For simplicity we will use

$$A=\begin{pmatrix}a&c\\b&d\end{pmatrix}, \qquad (27)$$

and its inverse

$$A^{-1}=\begin{pmatrix}e&g\\f&h\end{pmatrix}. \qquad (28)$$

We want to prove the statement we made in the text: $\{(a,b),(c,d)\}$ is a basis of Z^2 $\Leftrightarrow$ $Det(A)=1$.

First assuming that $Det(A)=1$, we know that A^{-1} consists of integer elements[7]: $e,f,g,h\in Z$. Then,

$$AA^{-1}=\begin{pmatrix}a&c\\b&d\end{pmatrix}\begin{pmatrix}e&g\\f&h\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad (29)$$

implies that

$$e\begin{pmatrix}a\\b\end{pmatrix}+f\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}=e_{1,} \qquad (30)$$

$$g\begin{pmatrix}a\\b\end{pmatrix}+h\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}=e_{2.} \qquad (31)$$

And if e_1 and e_2 are elements of the space spanned up by (a,b) and (c,d), any element of Z^2 is in that space, so

$$\{\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}\}\text{ is a basis of } Z^2 \qquad (32)$$

To go the other way around, we assume $\{(a,b),(c,d)\mid a,b,c,d\in Z\}$ is a basis of Z^2, therefore:

$$e_1=e\begin{pmatrix}a\\b\end{pmatrix}+f\begin{pmatrix}c\\d\end{pmatrix}, \qquad (33)$$

$$e_2=g\begin{pmatrix}a\\b\end{pmatrix}+h\begin{pmatrix}c\\d\end{pmatrix}. \qquad (34)$$

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[7] Because for an 2x2 matrix $A=\begin{pmatrix}a&c\\b&d\end{pmatrix}$:

$$A^{-1}=\frac{1}{Det(A)}\begin{pmatrix}d&-c\\-b&a\end{pmatrix}$$

with $e, f, g, h \in Z$, which is equivalent to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & g \\ f & h \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad (35)$$

and therefore

$$Det(A^{-1})Det(A) = 1. \qquad (36)$$

Since all elements of A^{-1} and A are elements of Z, and therefore $Det(A^{-1})$ and $Det(A)$ should both be elements of Z there is no other possibility for $Det(A)$ than to be equal to 1 (or -1).

$$Det(A) = \pm 1 \qquad (37)$$

For more details on linear algebra, see for example [4].

5. REFERENCES

1. Gerald J. Balzano. (1980). The group theoretical description of 12-fold and microtonal pitch systems. *Computer Music Journal*, 4(4), pages 66-84.

2. Edward M. Burns. (1999). Intervals, scales, and tuning. In Diana Deutsch, editor, *The Psychology of Music*, chapter 7, pages 215-264. Academic Press, second edition.

3. Hermann Helmholtz. (1954). *On the Sensations of Tone*. Dover, second English edition.

4. Serge Lang. *Algebra*. (2002). Springer-Verlag, revised third edition.

5. Walter Piston and Mark DeVoto. (1989). *Harmony*. Victor Gollancz Ltd.

6. Martin Vogel. (1975). *Die Lehre von den Tonbeziehungen*. Verlag für systematische Musikwissenschaft, Bonn-Bad Godesberg.

7. Paul F. Zweifel. (1996). Generalized diatonic and pentatonic scales: A group-theoretic approach. *Perspectives of new music 34 (1)*, pages 140-161.